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Informational Rents in Bargaining with Serially Correlated Valuations

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Introduction

This paper extends the standard two-period bargaining model with one-sided private information to allow for transitions in the buyer's valuation between periods. The starting point is Fudenberg and Tirole (1983). An uninformed seller makes an offer in the first period, the informed buyer says yes or no, the seller makes another offer in the last period, the buyer says yes or no, and the game ends. The buyer's valuation is drawn from a two-point distribution at the beginning of the game. What is new in this paper is that the buyer draws a new valuation in the second period, and these two draws are correlated.

Introducing transitions in the buyer's valuation leads to some surprising results. The most interesting of these is the existence of pooling equilibria in which *both* types collect informational rents.

The paper characterizes the full set of sequential equilibria for the case in which the seller's prior belief is moderately optimistic: not optimistic enough to make a tough offer in the last period if it was revealed in the first period that the buyer's valuation was high, but optimistic enough to make a tough offer in the last period if nothing was revealed in the first period. Given a prior belief in this range, the set of equilibria can be roughly described as follows. (1) At the most optimistic end of the range, there is a unique equilibrium path which screens the buyer in the first period: either the price is the high valuation in both periods, with the high buyer randomizing in the first period and accepting in the last period, or else the price is such that the high buyer accepts for sure in the first period, and the low buyer rejects, and the price in the last period is the high valuation following acceptance, and the low valuation following rejection. (2) At the most pessimistic end of the range, there is a continuum of (first-period) pooling equilibria in which the price in the first period can be anywhere in an interval, where the upper end of the interval is the low valuation, and the length of the interval is the discount factor multiplied by the probability of making a transition from the low to the high valuation. (3) In the middle of the range, the screening and pooling equilibria coexist, and there are equilibria in which the seller randomizes in the first period.

Forward induction is then applied to the set of equilibria, to good effect. The multiplicity of equilibria arises because the seller is free to believe anything if a pooling offer is rejected in the first period. But rejection of a pooling offer generally yields a strictly lower expected payoff for the high buyer, regardless of what happens in the last period, while the low buyer gains if rejection convinces the seller that the valuation in the first period was indeed low. In this situation forward induction implies that the seller should indeed be convinced, and the beliefs in nearly all of the pooling equilibria do not have this property. In fact, only one pooling equilibrium survives this test, and it is the one where the pooling price is at the bottom end of the interval of equilibrium pooling prices. In effect, the low buyer has the option of rejecting a pooling offer in the first period, and if this induces the seller to switch to a soft offer in the last period, then the buyer gains if there is a transition to the high valuation in the last period. The informational rent that must be conceded to the low buyer is the value of this option to reject the pooling price.

The paper also provides some discussion of the case in which the seller's prior belief is moderately pessimistic: not pessimistic enough to make a soft offer in the last period if it was revealed in the first period that the buyer's valuation was high, but pessimistic enough to make a soft offer in the last period if nothing was revealed in the first period. There is again a continuum of pooling equilibria in this case, but now the application of forward induction does not shrink the set of equilibria, because rejection of a pooling offer would reduce the expected payoff for both buyer types. A surprising feature of these equilibria is that even though both buyer types strictly prefer to accept the seller's offer in the first period, there is no way for the seller to extract more surplus by raising the price.

2. The Bargaining Game

Consider a two-period bargaining game between a buyer and a seller where the surplus to be divided in each period is drawn from a two-point distribution. Denote the first period as t = 1, and the last period as t = 0, and let v_t be the surplus in period t. The conditional distribution of v_0 is specified by the with continuation probabilities ρ_L and ρ_H: if $v_1 = v_L$, then $v_0 = v_L$ with probability ρ_L, and if $v_1 = v_H$, then $v_0 = v_H$ with probability ρ_H. It is convenient to use the seller's opportunity cost as the origin and the difference between the high and low surplus as the unit, so let $\theta > 0$ be the low surplus, and let $1+\theta$ be the high surplus.[2] The realizations of the surplus are seen only by the buyer, and this is represented by introducing nature as a fictional player whose actions are seen only by the buyer. In each period, the seller makes an offer, and the buyer accepts or rejects. Both players maximize the present value of expected income, with a common discount factor δ. Thus the model is summarized by the four parameters $(\theta,\rho_L,\rho_H,\delta)$.

Actions

The actions $a_t = (n_t,p_t,q_t)$ taken in period t are as follows:

first, nature chooses a number n_t, and the surplus is $n_t + \theta$, where n_t is either 0 or 1;

then the seller chooses a price, p_t, from the real line;

[2]For example, if v is a firm's revenue during the term of a labor contract, net of all nonlabor costs, and if w_0 is the highest wage available to workers during a strike, the surplus is $v - w_0$. Then in the new units the low surplus is $\theta = (v_L-w_0)/(v_H-v_L)$, and the high surplus is $1+\theta = (v_H-w_0)/(v_H-v_L)$.

then the buyer chooses a quantity, q_t, which is either 0 or 1.

Payoffs

The seller's payoff in period t is $q_t\, p_t$.

The buyer's payoff in period t is $q_t\,(n_t + \theta - p_t)$. That is, the buyer's valuation is high ($v = 1+\theta$) when $n = 1$, and low ($v = \theta$) when $n = 0$.

Both the seller and the buyer maximize the expected present value of future payoffs, using the common discount factor δ. The seller's and the buyer's payoffs are

$$\tilde{U}(a) = p_1 q_1 + \delta p_0 q_0$$
$$\tilde{V}(a) = \left[n_1 + \theta - p_1\right] q_1 + \delta\left[n_0 + \theta - p_0\right] q_0$$

History

The history known to the seller when choosing p_1 is $h_1^0 = \varnothing$

The history known to the buyer when choosing q_1 is $h_1 = \{n_1, p_1\}$

The history known to the seller when choosing p_0 is $h_0^0 = \{p_1, q_1\}$

The history known to the buyer when choosing q_0 is $h_0 = \{n_1, p_1, q_1, n_0, p_0\}$

Strategies

Nature's strategy, σ^N, selects $n_1 = 0$ with probability ζ_1, $n_1 = 1$ with probability $1-\zeta_1$, $n_0 = 0$ with probability $\rho_L - \varphi n_1$, and $n_0 = 1$ with probability $1-\rho_L+\varphi n_1$, where $\varphi = \rho_L + \rho_H - 1$. It is assumed that φ is nonnegative.

A behavioral strategy for the seller is a pair (σ_1^S, σ_0^S), where σ_1^S is a probability distribution on R, and σ_0^S is a mapping from R $\times$ {0,1} to the set of probability distributions on R; $\sigma_0^S(p_1, q_1)$ is the probability distribution of p_0, given the actions p_1 and q_1 in the first period. The seller's strategy set Σ^S is the set of such pairs.

A behavioral strategy for the buyer is a pair (σ_1^B, σ_0^B), where

(i) σ_1^B is a mapping from {0,1} $\times$ R to the unit interval, and

(ii) σ_0^B is a mapping from {0,1} $\times$ R $\times$ {0,1} $\times$ {0,1} $\times$ R to the unit interval.

The buyer's strategy set Σ^B is the set of such pairs. The interpretation is that $q_1 = 1$ with probability $\sigma_1^B(n_1,p_1)$ when the buyer's valuation is $n_1+\theta$, and the price p_1 is offered in the first period, and similarly for $\sigma_0^B(n_1,p_1,q_1,n_0,p_0)$.

The notation $\sigma_1^L(p_1)$ will be used as an abbreviation for the buyer's first-period strategy when the valuation is low (i.e. $\sigma_1^L(p_1)$ is $\sigma_1^B(n_1,p_1)$ evaluated at $n_1 = 0$), and similarly σ_1^H is the buyer's first-period strategy when the valuation is high.

Each strategy profile $\sigma = (\sigma^N,\sigma^S,\sigma^B)$ determines a joint distribution $A(\sigma)$ for the random vector (a_1,a_0), taking values in $[\{0,1\}\times R\times\{0,1\}]^2$. The *path* of σ, denoted by $\underline{\sigma}$, is the support of this distribution – the set of sample paths having positive probability. A history is *on the path of σ* if the set of all sample paths beginning with this history has positive probability. Following any history h, σ determines a joint distribution $A(\sigma|h)$ for the remaining components of (a_1,a_0).

Beliefs

A belief-system Π is a pair (Π_1,Π_0), where Π_1 is a point in $\Delta(\{0,1\}^2)$, specifying a probability distribution over the four possible realizations of (n_1,n_0), and Π_0 is a mapping from $R\times\{0,1\}$ to $\Delta(\{0,1\}^2)$, specifying a probability distribution $\Pi_0(n_1,n_0|p_1,q_1)$ over (n_1,n_0), for each possible realization of (p_1,q_1). Since any belief system that might be part of an equilibrium must be consistent with σ^N, a belief system can be represented by the triple $[\zeta_1, \zeta_1^r(p_1), \zeta_1^a(p_1)]$, defined as follows:

ζ_1: the prior probability (according to Π_1) that n_1 is zero

$\zeta_1^r(p_1)$ the posterior probability (according to Π_0) that n_1 was zero, given that p_1 was rejected

$\zeta_1^a(p_1)$ the posterior probability (according to Π_0) that n_1 was zero, given that p_1 was accepted

Equilibrium

There is no general definition of sequential or perfect Bayesian equilibrium for games with infinite strategy sets, but a straightforward adaptation of the definitions in Kreps and Wilson (1982) and Fudenberg and Tirole (1991) yields a suitable definition of equilibrium for the game analyzed here.

Definition (Consistency)

The strategy-belief pair (σ,Π) is *consistent* if, for all prices p_1,

$$\left[1-\zeta_1\sigma_1^L(p_1)-(1-\zeta_1)\sigma_1^H(p_1)\right]\zeta^r(p_1) = \zeta_1\left[1-\sigma_1^L(p_1)\right]$$

$$\left[\zeta_1\sigma_1^L(p_1)+(1-\zeta_1)\sigma_1^H(p_1)\right]\zeta^a(p_1) = \zeta_1\sigma_1^L(p_1)$$

That is, the seller's posterior belief regarding the buyer's valuation in the first period, after seeing the buyer's response to any first-period price, satisfies Bayes' Rule.[3]

Sequential Optimality

For a given strategy profile σ, the buyer's expected payoff, conditional on the history h_t, is

$$V(\sigma|h_t) \ = \ E_{A(\sigma|h_t)}\tilde{V}(a)$$

where the expectation over future actions is computed according to the distribution $A(\sigma|h_t)$. For a given strategy-belief pair (σ,Π), the seller's expected payoff, conditional on h_t^0, is

$$U(\sigma,\Pi|h_t^0) \ = \ \sum_{h_t} \Pi_t(h_t|h_t^0)\, E_{A(\sigma|h_t)}\tilde{U}(a)$$

Definition (Sequential Optimality)

(a) The strategy profile σ is *sequentially optimal for the buyer* if, for all t, and for all histories h_t, and for all strategies $ς^B \in \Sigma^B$

$$V(\sigma^N,\sigma^S,\sigma^B|h_t) \ \geq \ V(\sigma^N,\sigma^S,ς^B|h_t)$$

(b) The strategy-belief pair (σ,Π) is *sequentially optimal for the seller* if, for all t, and for all public histories h_t^0, and for all strategies $ς^S \in \Sigma^S$.

$$U(\sigma^N,\sigma^S,\sigma^B,\Pi|h_t^0) \ \geq \ U(\sigma^N,ς^S,\sigma^B,\Pi|h_t^0)$$

The pair (σ,Π) is *sequentially optimal* if both (a) and (b) are true.

Definition A0: The strategy-belief pair (σ,Π) is a sequential equilibrium if it is consistent and sequentially optimal.

In a sequential equilibrium, after any first-period history, the seller uses Π_0 to determine the probability ζ_0 that the buyer's valuation is low in the last period, and the strategy components σ_0^S and σ_0^B are optimal.

[3]The Kreps-Wilson (1982) definition of consistency (for finite games) also requires that beliefs following probability-zero events can be rationalized as the limit of a sequence of strategy-belief pairs such that each strategy in the sequence is fully mixed, and so each belief system in the sequence is fully determined by Bayes rule. It is easy to check that this additional requirement is satisfied for all of the strategies considered below.

Thus the set of possible equilibria in the two-period game can be reduced by characterizing the set of equilibria in the one-period game parameterized by ζ_0. This is done in the next section. In fact, although the one-period game has a continuum of equilibria, the expected payoffs for the seller and the buyer are uniquely determined by ζ_0. Next, in the subgame initiated by any choice of p_1, the strategy component σ_1^B must be optimal, allowing for the relationship between ζ_0 and the buyer's first-period action. Finally, once the set of equilibria in the subgame is characterized, the seller's optimal choice of p_1 can be analyzed, and this completes the analysis of the two-period game.

3. The End Game

After any history, the last period is an ultimatum game parameterized by the seller's belief, ζ_0 (representing the probability that $n_0 = 1$, according to Π_0). There is a continuum of equilibria in this game, for any value of ζ_0, but the multiplicity of equilibria is not important – in particular (for any given value of ζ_0) the equilibrium path is unique.[4]

Define $\zeta_0^* = 1/(1+\theta)$: this is the threshold belief that determines whether screening or pooling is optimal for the seller..

Proposition 0:

I. If $\zeta_0 < \zeta_0^*$, then the strategy profile $\sigma_0 = (\sigma_0^S, \sigma_0^L, \sigma_0^H)$ is an equilibrium if and only if

a $\underline{\sigma}_0^S = \{1+\theta\}$

b $\sigma_0^L(p) = 1$ if $p < \theta$, $\sigma_0^L(p) = 0$ if $p > \theta$, $\sigma_0^L(\theta) = \alpha$, where $\alpha \in [0,1]$

c $\sigma_0^H(p) = 1$ if $p \leq 1+\theta$, $\sigma_0^H(p) = 0$ if $p > 1+\theta$

Thus there is a continuum of equilibrium strategy profiles, indexed by α.

II. If $\zeta_0 > \zeta_0^*$, then the strategy profile $\sigma_0 = (\sigma_0^S, \sigma_0^L, \sigma_0^H)$ is an equilibrium if and only if

a $\underline{\sigma}_0^S = \{\theta\}$

b $\sigma_0^L(p) = 1$ if $p \leq \theta$, $\sigma_0^L(p) = 0$ if $p > \theta$

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[4]According to Proposition 1 in Fudenberg and Tirole (1983), there exists a unique perfect Bayesian equilibrium for the two-period durable-good monopoly version of this game (i.e. the game in which the buyer's valuation does not change between periods, and the game ends when the buyer accepts an offer). Similarly Hart (1987, Proposition 1) asserts that there is a (generically) unique perfect Bayesian equilibrium for the T-period durable-good monopoly game. These results are not exactly right, since the buyer with valuation v_L is free to randomize at $p = v_L$ if the seller's equilibrium strategy does not offer $p = v_L$ with positive probability, and similarly for the buyer with valuation v_H if $p = v_H$ is not offered with positive probability.

c $\sigma_0^H(p) = 1$ if $p < 1+\theta$, $\sigma_0^H(p) = 0$ if $p > 1+\theta$, $\sigma_0^H(1+\theta) = \beta$, where $\beta \in [0,1]$

Thus there is a continuum of equilibrium strategy profiles, indexed by β.

III. If $\zeta_0 = \zeta_0^*$, then the strategy profile $\sigma_0 = (\sigma_0^S,\sigma_0^L,\sigma_0^H)$ is an equilibrium if and only if

a $\sigma_0^S(\{\theta\}) = \lambda$, $\sigma_0^S(\{1+\theta\}) = 1-\lambda$, where $\lambda \in [0,1]$

b $\sigma_0^L(p) = 1$ if $p < \theta$, $\sigma_0^L(p) = 0$ if $p > \theta$, $\sigma_0^L(\theta) = \alpha$, where $\alpha \in [0,1]$, with $\lambda(1-\alpha) = 0$

c $\sigma_0^H(p) = 1$ if $p \leq 1+\theta$, $\sigma_0^H(p) = 0$ if $p > 1+\theta$, $\sigma_0^H(1+\theta) = \beta$, where $\beta \in [0,1]$, with $(1-\lambda)(1-\beta) = 0$

Thus there is a continuum of equilibrium strategy profiles, indexed by λ (or by α, if $\lambda = 0$, or by β, if $\lambda = 1$).

Proof:

Optimality of the buyer's strategy requires that $p < \theta$ must be accepted and $p > 1+\theta$ must be rejected; also $p \in (\theta,1+\theta)$ must be accepted if the buyer's valuation is high, and rejected if the valuation is low. Thus the only prices that can be optimal for the seller are θ and $1+\theta$. Randomization is optimal for the buyer when the price matches the valuation. But in any equilibrium in which the seller's strategy specifies $p = \theta$, the low-valuation buyer must accept $p = \theta$ for sure, or else the seller could do better by shading the price. Similarly, in any equilibrium in which the seller is setting $p = 1+\theta$, the high-valuation buyer's strategy must accept $p = 1+\theta$ for sure.

For $p \in (\theta,1+\theta)$, the seller's expected payoff is $(1-\zeta_0)p$, and for $p < \theta$, the expected payoff is p. If $\zeta_0 < \zeta_0^*$, then the expected payoff when $p = 1+\theta-\varepsilon$ exceeds θ, so $p = \theta$ is not optimal for the seller. This means that $\underline{\sigma}_0^S = \{1+\theta\}$ and $\sigma_0^H(1+\theta) = 1$ in any equilibrium, but the low-valuation buyer is free to randomize at $p = \theta$. Similarly, if $\zeta_0 > \zeta_0^*$, then $\underline{\sigma}_0^S = \{\theta\}$ and $\sigma_0^L(\theta) = 1$, but the high-valuation buyer is free to randomize at $p = 1+\theta$.

Finally, if $\zeta_0 = \zeta_0^*$ then randomization over $\{\theta,1+\theta\}$ is optimal for the seller, and unless the seller's equilibrium strategy chooses one of these prices for sure, the buyer's equilibrium strategy must accept for sure when the price matches the valuation.

Corollary:

In the one-period game with belief ζ_0, the seller's expected payoff is

$$\hat{U}_0 = \max\left[\theta, (1-\zeta_0)(1+\theta)\right]$$

Proof:

Under case I of Proposition 0, the seller chooses $p_0 = 1+\theta$ and the probability of acceptance is $1-\zeta_0$, and the product of these is at least θ because $\zeta_0 < \zeta_0^*$, and conversely in case II, with equality in case III.

Informational Rents

From the buyer's point of view, the value of the one-period game depends on how pessimistic the seller is. If $\zeta_0 < \zeta_0^*$, the value is zero for both buyer types. But if $\zeta_0 > \zeta_0^*$ the high-valuation buyer gets an informational rent ($V_0^H = 1$) – a gain relative to the payoff in the full-information version of the game. This is a very well-known result, and it extends to the T-period game with permanent valuations. A natural conjecture in the two-period game with serially correlated valuations is that the value of the game is zero for the buyer who draws the low valuation in both periods, while the other three types of the buyer get informational rents if the seller is sufficiently pessimistic. Although this might seem obvious, it will be analyzed in detail in Section 5 below.

4. Equilibria in the Subgame Determined by p_1

Any price p_1 that might be offered by the seller in the first period starts a subgame: the buyer chooses q_1, the seller chooses p_0, and the buyer chooses q_0. The set of equilibria in the subgame depends on where $\zeta_0^r(p_1)$ and $\zeta_0^a(p_1)$ lie in relation to the threshold ζ_0^*. There are four cases to consider, but two are trivial:

Condition OO: $\rho_L < \zeta_0^*$
This means the seller is optimistic enough to screen in the last period, even if the buyer's action in the first period revealed that the first-period valuation was low.

Condition PP: $1-\rho_H > \zeta_0^*$
This means the seller is pessimistic enough to pool in the last period, even if the buyer's action revealed that the first-period valuation was high.

In each of these two cases, where the seller is either very optimistic or very pessimistic in the last period, regardless of what happened in the first period, the seller's action in the last period is predetermined, so the first period can be analyzed as a static game. The interesting cases arise when $1\text{-}\rho_H < \zeta_0^* < \rho_L$: this will be assumed from now on. Then the analysis depends on whether the seller is relatively optimistic or relatively pessimistic in the last period:

Condition O: $\varphi\zeta_1 + 1\text{-}\rho_H < \zeta_0^* < \rho_L$

This means the seller is optimistic enough to screen in the last period, if the buyer's action in the first period was uninformative, but not optimistic enough to screen if the buyer's action revealed the low valuation.

Condition P: $\varphi\zeta_1 + 1 - \rho_H > \zeta_0^* > 1 - \rho_H$

This means the seller is pessimistic enough to pool in the last period, if the buyer's action in the first period was uninformative, but not pessimistic enough to pool if the buyer's action revealed the high valuation.

Condition O is analogous to the "tough seller" condition in Fudenberg and Tirole (1983), and condition P is analogous to their "soft seller" condition.

Given that it is never optimal for the seller to charge prices other than θ or $1+\theta$ in the last period, let $\ell^r(p_1) = \sigma_0^S(p_1,0)(\theta)$ be the probability that $p_0 = \theta$ if the price p_1 is rejected in the first period, and let $\ell^a(p_1) = \sigma_0^S(p_1,1)(\theta)$ be the corresponding probability if the price p_1 is accepted. Then any strategy for the seller that is part of an equilibrium can be represented by the triple $[\sigma_1^S(p_1), \ell^r(p_1), \ell^a(p_1)]$, specifying a probability distribution for p_1, together with the probability of pooling in the last period if p_1 is accepted, and the probability of pooling in the last period if p_1 is rejected.

Proposition 0 immediately implies that in any sequential equilibrium, for any price p_1,

$$\ell^a(p_1) = 0 \quad \text{if } \zeta_0^a(p_1) < \zeta_0^*$$

$$\ell^a(p_1) = 1 \quad \text{if } \zeta_0^a(p_1) > \zeta_0^*$$

$$\ell^r(p_1) = 0 \quad \text{if } \zeta_0^r(p_1) < \zeta_0^*$$

$$\ell^r(p_1) = 1 \quad \text{if } \zeta_0^r(p_1) > \zeta_0^*$$

The following result provides the basic optimality condition governing the buyer's decision in the first period.

Lemma 1: In any equilibrium, the buyer's strategy has the following property:

If $p_1 < n_1 + \theta - \delta(1-\rho_L+\varphi n_1)[\ell^r(p_1) - \ell^a(p_1)]$ then $\sigma_1^B(n_1,p_1) = 1$

If $p_1 > n_1 + \theta - \delta(1-\rho_L+\varphi n_1)[\ell^r(p_1) - \ell^a(p_1)]$ then $\sigma_1^B(n_1,p_1) = 0$

Proof:

Suppose the seller offers a price p in the first period. Using the probabilities $\ell^r(p)$ and $\ell^a(p)$, the buyer compares the expected payoffs from $q_1 = 1$ and from $q_1 = 0$. These are as follows:

$q_1 = 1$: $\delta(1-\rho_L+\varphi n_1)\ell^a(p) + n_1 + \theta - p$

$q_1 = 0$: $\delta(1-\rho_L+\varphi n_1)\ell^r(p)$

This implies that any optimal strategy for the buyer must have the above property.

Corollary: in any equilibrium,

If $p_1 < \theta - \delta(1-\rho_L)$ then $\sigma_1^L(p_1) = 1$

If $p_1 < 1 + \theta - \delta\rho_H$ then $\sigma_1^H(p_1) = 1$

If $p_1 > \theta + \delta(1-\rho_L)$ then $\sigma_1^L(p_1) = 0$

Proof:

Since $-1 \le \ell^r(p_1) - \ell^a(p_1) \le 1$, these results follow immediately from Lemma 1.

Given $\ell^r(p_1) - \ell^a(p_1)$, Lemma 1 completely determines the buyer's strategy except for the choice of a (random) quantity when p_1 is equal to the reservation price. The analysis proceeds by studying the relationships between $\ell^r(p_1)$, $\ell^a(p_1)$, $\sigma_1^L(p_1)$ and $\sigma_1^H(p_1)$ that must hold in equilibrium.

Lemma 2: In any equilibrium, if $p_1 > 1+\theta$, then $\sigma_1^L(p_1) = \sigma_1^H(p_1) = 0$.

Proof:

Since $1+\theta > \theta + \delta(1-\rho_L)$ the Corollary to Lemma 1 implies $\sigma_1^L(p_1) = 0$. If $\sigma_1^H(p_1) > 0$, then $\zeta_0^a(p_1) = 1-\rho_H < \zeta_0^*$, so $\ell^a(p_1) = 0$. But in this case, from Lemma 1, the high buyer rejects any price above $1+\theta - \delta\rho_H\ell^r(p_1)$, which includes all prices above $1+\theta$, since $\ell^r(p_1) \ge 0$. Thus both buyer types surely reject all prices above $1+\theta$.

Lemma 3: In any equilibrium, if $p_1 > 1+\theta$, then $\ell^a(p_1) \le \overline{\ell}_H^a(p_1)$, where

$$\overline{\ell}_H^a(p_1) = \frac{p_1 - (1+\theta)}{\delta\rho_H}$$

Proof:

By Lemma 2, $\sigma_1^H(p_1) = 0$. By Lemma 1, the condition needed to ensure that this is optimal for the buyer is $p_1 \ge 1+\theta+\delta\rho_H\ell^a(p_1)$, which can be rearranged as $\ell^a(p_1) \le \overline{\ell}^a(p_1)$.

The next result says that any price that might be accepted by the low-valuation buyer in the first period is accepted for sure by the high-valuation buyer.

Lemma 4: In any equilibrium, if $\sigma_1^L(p_1) > 0$, then $\sigma_1^H(p_1) = 1$.

Proof:

If $\sigma_1^L(p_1) > 0$, then

$$p_1 \leq \theta - \delta(1-\rho_L)\,[\ell^r(p_1) - \ell^a(p_1)]$$

by Lemma 1. This implies

$$p_1 + 1 - \delta\varphi\,[\ell^r(p_1) - \ell^a(p_1)] \;\leq\; 1+\theta - \delta(1-\rho_L + \varphi)\,[\ell^r(p_1) - \ell^a(p_1)]$$

But $1 - \delta\varphi\,[\ell^r(p_1) - \ell^a(p_1)]$ is strictly positive (for $\delta\varphi$ strictly less than 1), because $\ell^a(p_1)$ is nonnegative, and $\ell^r(p_1) \leq 1$. So

$$p_1 < p_1 + 1 - \delta\varphi\,[\ell^r(p_1) - \ell^a(p_1)] \;\leq\; 1+\theta - \delta(1-\rho_L + \varphi)\,[\ell^r(p_1) - \ell^a(p_1)]$$

which implies $\sigma_1^H(p_1) = 1$ by Lemma 1.

Lemma 5:

In any equilibrium, if $0 < \sigma_1^L(p_1) < 1$, then $\ell^r(p_1) = 1$. Also, if $\sigma_1^L(p_1) = 0$ and if $\sigma_1^H(p_1) = 1$, then $\ell^r(p_1) = 1$.

Proof:

If $0 < \sigma_1^L(p_1) < 1$, then $\sigma_1^H(p_1) = 1$ by Lemma 4. If $\sigma_1^L(p_1) < 1$ and $\sigma_1^H(p_1) = 1$ consistency requires $\zeta_0^r(p_1) = \rho_L$, and then (under the maintained assumption that $\rho_L > \zeta_0^*$) optimality for the seller requires $\ell^r(p_1) = 1$.

Further analysis of equilibria in the subgame depends on whether the seller's prior belief satisfies Condition O or Condition P. In the remainder of this section, Condition O is assumed.

Lemma O6:

In any equilibrium, if Condition O holds, $\sigma_1^L(p_1) = 0$ if $p_1 > \theta$.

Proof:

Suppose $p_1 > \theta$ and $q_1^L = \sigma_1^L(p_1) \in (0,1)$. Then (by Lemma 4) the price p_1 is accepted for sure by the high buyer, so rejection must be interpreted as coming from the low buyer, by Bayes rule, which implies that $\ell^r(p_1) = 1$. Moreover, acceptance must make the seller more optimistic, so $\zeta_0 \geq \varphi\zeta_1 + 1\text{-}\rho_H$, and then Condition O implies that $\ell^a(p_1) = 0$. Thus acceptance of p_1 yields a negative payoff for the low buyer in the first period, and a zero payoff in the last period, while rejection yields zero now, and at least zero later.

Lemma O7: In any equilibrium, if Condition O holds, then $\sigma_1^H(p_1) > 0$ for all $p_1 < 1+\theta$.

Proof:

If $\sigma_1^H(p_1) = 0$, then $\sigma_1^L(p_1) = 0$ (by Lemma 4). So rejection carries no information, and the seller's belief in the last period must be in the screening region by Condition O. This implies that the high buyer's payoff is zero in each period. But if $p_1 < 1+\theta$, acceptance gives the high buyer a positive payoff in the first period, and the payoff is no worse than zero in the last period. So rejection can't be optimal for the high buyer.

Lemma O8: In any equilibrium, if Condition O holds, and if $\sigma_1^H(p_1) > 0$, then $\ell^a(p_1) = 0$.

Proof:

Any price that is accepted by the low buyer is accepted by the high buyer, so acceptance can't make the seller more pessimistic unless the probability of acceptance was zero. Since the probability is positive, it follows that the seller is at least as optimistic as if no information had been revealed in the first period, and this implies screening in the last period, by Condition O.

Lemma O9:

In any equilibrium, if Condition O holds, and if $p_1 > p_H$, then $\sigma_1^H(p_1) < 1$.

Proof:

Suppose $\sigma_1^H(p_1) = 1$. Since $p_H > \theta$, it follows that $\sigma_1^L(p_1) = 0$ by Lemma O6. This implies $\zeta_0^r(p_1) = \rho_L$, so $\ell^r(p_1) = 1$, and $\zeta_0^a(p_1) = 1 - \rho_H$, so $\ell^a(p_1) = 0$. But then Lemma 1 implies $\sigma_1^H(p_1) = 0$ for $p_1 > p_H$, a contradiction.

Lemma O10:

In any equilibrium, if Condition O holds, and if $p_1 = 1+\theta$, then $\sigma_1^H(p_1) \in [0, Q_1^H]$ and $\ell^a(p_1) = \ell^r(p_1) = 0$.

Proof:

Since $1+\theta > p_H$, Lemma O5 implies $\sigma_1^H(p_1) < 1$, so $\ell^a(p_1) \leq \ell^r(p_1)$ by Lemma 1. If $\sigma_1^H(p_1) = 0$, then consistency implies $\zeta_0^r(p_1) = \varphi\zeta_1+1-\rho_H$, and then Condition O implies $\ell^r(p_1) = 0$, so $\ell^a(p_1) = 0$. If $\sigma_1^H(p_1) > 0$, then $\ell^a(p_1) = \ell^r(p_1)$ by Lemma 1, while consistency implies $\zeta_0^a(p_1) = 1-\rho_H$ and then Condition O implies

$\ell^a(p_1) = 0$. Finally, $\ell^r(p_1) = 0$ requires $\zeta_0^r(p_1) \leq \zeta_0^*$. That is, the seller must be sufficiently optimistic to screen in the last period following rejection, and this requires the probability of rejection by the high-valuation buyer is not too small: specifically, $\sigma_1^H(p_1) \leq Q_1^H$.

Lemma O11:

In any equilibrium, if Condition O holds, and if $p_1 = \theta - \delta(1-p_L)$, then max $[\sigma_1^L(p_1), \ell^r(p_1)] = 1$.

Proof:

Suppose $\ell^r(p_1) < 1$. Then since $\ell^a(p_1) = 0$ by Lemma O8, $\sigma_1^L(p_1) = 1$ by Lemma 1.

These preliminary results can now be used to characterize the set of equilibria in the subgame starting from any price p_1. In each of the following results it is understood that $\sigma_0^L(p_0) = 1$ for all $p_0 < \theta$, $\sigma_0^L(p_0) = 0$ for all $p_0 > \theta$, $\sigma_0^H(p_0) = 1$ for all $p_0 \leq 1+\theta$, and $\sigma_0^H(p_0) = 0$ for all $p_0 > 1+\theta$. The first result is immediate but not important.

Proposition 1: If Condition O holds, then the subgame starting from any price $p_1 > 1+\theta$ has a unique equilibrium path. Moreover, the buyer's first-period strategy is uniquely determined. The set of equilibria can be characterized as follows

- $\sigma_1^L(p_1) = \sigma_1^H(p_1) = 0$
- $\ell^r(p_1) = 0$, $\ell^a(p_1)$ can be any number between 0 and 1
- $\sigma_0^L(\theta)$ can be any number between 0 and 1, and this number can depend in an arbitrary way on n_1, p_1 and q_1.
- $\sigma_0^H(p_0) = 1$ for all $p_0 = 1+\theta$
- $\zeta_0^r(p_1) = \varphi\zeta_1 + 1-\rho_H$, and $\zeta_0^a(p_1)$ can be any number between 0 and 1
  - with the restriction that if $\ell^a(p_1) = 0$ then $\zeta_0^a(p_1) \leq \zeta_0^*$
    - and if $\ell^a(p_1) = 1$ then $\zeta_0^r(p_1) \geq \zeta_0^*$.

The seller's expected payoff is

$$\hat{U}(p_1) = \delta(1+\theta)\left[\rho_H - \varphi\zeta_1\right]$$

This means that there is a multi-dimensional continuum of equilibria, but the equilibrium path is the same for each point in this set: $q_1 = 0$, $p_0 = 1+\theta$, $q_0 = n_0$.

The next result again deals with equilibria of the subgame for prices that are not on the equilibrium path of the two-period game.

Proposition 2: If Condition O holds, then the subgame starting from any price p_1 in the open interval $(\theta,1+\theta-\delta\rho_H)$ has a unique equilibrium path. Moreover, the seller's strategy, the buyer's first-period strategy and the beliefs are all uniquely determined. The set of equilibria can be characterized as follows

$\sigma_1^L(p_1) = 0$ and $\sigma_1^H(p_1) = 1$

$\zeta_0 = (1 - q_1)\,\rho_L + q_1(1 - \rho_H)$

$\underline{\sigma}_1^S(p_1,q_1) = \{q_1+\theta\}$

$\sigma_0^B(h_0) = 1$ for any history h_0 such that $p_0 = n_0 + \theta = q_1 + \theta$

$\sigma_0^B(h_0) \in [0,1]$ for any history h_0 such that $p_0 = n_0 + \theta \neq q_1 + \theta$

The seller's expected payoff is

$$\hat{U}(p_1) \;=\; \delta\theta + \left[1 - \zeta_1\right]\left[p_1 + \delta\rho_H - \delta(1-\rho_H)\theta\right]$$

Proof:

If there is an equilibrium then $\sigma_1^L(p_1) = 0$ by Lemma O6, and $\sigma_1^H(p_1) = 1$ by the Corollary to Lemma 1. Consistency then implies $\zeta_0^r(p_1) = \rho_L$, and $\zeta_0^a(p_1) = 1-\rho_H$, and Condition O requires $1 - \rho_H < \zeta_0^* < \rho_L$, so $\ell^r(p_1) = 1$ and $\ell^a(p_1) = 0$. The strategies and beliefs in the subgame are thus fully determined, except for the high buyer's response to an unexpected screening offer after $q_1 = 0$, and the low buyer's response to an unexpected pooling offer after $q_1 = 1$; these responses are arbitrary, and they do not affect the equilibrium path. Also σ_0^S and σ_0^B are clearly optimal, and σ_1^B is optimal because it satisfies the inequalities of Lemma 1.

Corollary:

In any equilibrium of the two-period game, $\sigma_1^S(p_1) = 0$ for $p_1 \in (\theta,1+\theta-\delta\rho_H)$.

Proof:

The seller's expected payoff is increasing in p_1, so no choice of p_1 in the open interval can be optimal for the seller.

Proposition 1 identifies a continuum of equilibria. If the seller unexpectedly chooses the high price in the last period, even though the buyer rejected p_1, and if the buyer's valuation is high in the last period, then the buyer's probability of acceptance is arbitrary; moreover, this probability can depend on whether the valuation was high or low in the first period, and it can depend on p_1. Similarly, if the seller unexpectedly chooses the low price in the last period, even though the buyer accepted p_1, the buyer's probability of acceptance is again arbitrary, and it can depend on n_1 and on p_1. Thus, for fixed p_1, the set of equilibria in the subgame can be represented by a unit cube in four dimensions. This might seem like a lot of equilibria, but of course all of these equilibria are virtually the same.

The main implication of Proposition 1 is that any price in the interval $(\theta,1+\theta-\delta\rho_H)$ is a screening price: the low buyer rejects it, and the high buyer accepts. Define $p_H = 1+\theta-\delta\rho_H$, the supremum of these prices. This is the first-period price in any equilibrium in which two buyer types separate in the first period, as will be shown below.

Proposition 3: If Condition O holds, then the subgame starting from $p_1 = p_H$ has a continuum of equilibria parameterized by $q_1^H = \sigma_1^H(p_H)$, where

$$Q_1^H = \frac{1}{1-\zeta_1}\left[1 - \frac{\varphi\zeta_1}{\zeta_0^* - (1-\rho_H)}\right] \leq q_1^H \leq 1$$

$\sigma_1^L(p_1) = 0$ and $\sigma_1^H(p_1) = 1$

$$\zeta_0 = \frac{\varphi\zeta_1(1-q_1)}{1-(1-\zeta_1)q_1^H} + 1 - \rho_H$$

$\underline{\sigma}_1^S(p_1,q_1) = \{q_1+\theta\}$

$\sigma_0^B(h_0) = 1$ for any history h_0 such that $q_1 = n_0$ and $p_0 = n_0 + \theta$

$\sigma_0^B(h_0) = \alpha(n_1) \in [0,1]$ for any history h_0 such that $q_1 = 1$ and $p_0 = \theta$ and $n_0 = 0$

$\sigma_0^B(h_0) = \beta(n_1) \in [0,1]$ for any history h_0 such that $q_1 = 0$ and $p_0 = 1+\theta$ and $n_0 = 1$

The seller's expected payoff is

$$\hat{U}(p_1) = \delta\theta + q_1^H \left[1 - \zeta_1\right]\left[1 + \theta - \delta(1-\rho_H)\theta\right]$$

Proof:

Since $p_H > \theta$, Lemma O6 implies $\sigma_1^L(p_H) = 0$. Suppose $\sigma_1^H(p_H) = 0$. Then rejection conveys no information, and Condition O implies $\ell^r(p_H) = 0$. But Lemma 1 then implies $\sigma_1^H(p_H) = 1$, a contradiction. Thus $\sigma_1^H(p_H) > 0$, and

$$\zeta_0 = \frac{\varphi\zeta_1(1-q_1)}{1-(1-\zeta_1)\sigma_1^H(p_H)} + 1 - \rho_H$$

The smallest value of $\sigma_1^H(p_H)$ consistent with $\zeta_0^r(p_H) \geq \zeta_0^*$ is Q_1^H, so if $\sigma_1^H(p_H) < Q_1^H$, Condition O implies $\ell^r(p_H) = 0$ and again Lemma 1 implies $\sigma_1^H(p_H) = 1$, a contradiction. So $\sigma_1^H(p_H) \geq Q_1^H$.

Proposition 4:

In any equilibrium, if Condition O holds, and if $p_1 \in (p_H, 1+\theta)$, then

$\sigma_1^L(p_1) = 0$ and

$$\sigma_1^H(p_1) = \frac{1}{1-\zeta_1}\left[\frac{\varphi\zeta_1}{\zeta_0^* - (1-\rho_H)} - 1\right] < 1$$

$\zeta_0 = (1 - q_1)\zeta_0^* + q_1(1 - \rho_H)$

$$\ell^r(p_1) = \frac{1+\theta-p_1}{\delta\rho_H} < 1 ; \quad \ell^a(p_1) = 0$$

$\sigma_0^B(h_0) = 1$ for any history h_0 such that $q_1 = 0$ and $p_0 = n_0 + \theta$

$\sigma_0^B(h_0) = 1$ for any history h_0 such that $q_1 = 1$ and $p_0 = 1 + \theta$ and $n_0 = 1$

$\sigma_0^B(h_0) \in [0,1]$ for any history h_0 such that $q_1 = 1$ and $p_0 = \theta$ and $n_0 = 0$

The seller's expected payoff is

$$\hat{U}(p_1) = p_1\left[1 - \frac{\varphi\zeta_1}{\zeta_0^* - (1-\rho_H)}\right] + \delta(1+\theta)\left[\rho_H - \varphi\zeta_1\right]$$

Proof:

[routine]

Corollary:

If Condition O holds, then in any equilibrium of the two-period game, $\sigma_1^S(p_1) = 0$ for $p_1 \in (1+\theta-\delta\rho_H, 1+\theta)$.

Proof:

For prices in this interval, the seller's expected payoff is

$$\hat{U}(p_1) = p_1\left[1 - \frac{\phi\zeta_1}{\zeta_0^* - (1-\rho_H)}\right] + \delta(1+\theta)\left[\rho_H - \phi\zeta_1\right]$$

Condition O holds if and only if the coefficient of p_1 in this expression is positive, so for any price $p_1 \in (1+\theta-\delta\rho_H, 1+\theta)$, a small increase in p_1 increases the seller's expected payoff, and thus p_1 cannot be optimal for the seller.

Proposition 5: In any equilibrium, if Condition O holds, and if $p_1 \in (\theta - \delta(1-p_L), \theta)$, then there is a continuum of equilibria with $\sigma_1^L(p_1) = 0$ and a continuum of equilibria with $\sigma_1^L(p_1) = 1$.

Proof:

First, by the Corollary to Lemma 1, $\sigma_1^H(p_1) = 1$ for all prices in this interval, and then acceptance cannot make the seller more pessimistic, so Condition O implies $\ell^a(p_1) = 0$. Thus the seller's strategy is summarized by $\ell^r(p_1)$.

If $\sigma_1^L(p_1) < 1$ then $\ell^r(p_1) = 1$ by Lemma 5, and if $\ell^r(p_1) = 1$, then Lemma 1 implies $\sigma_1^L(p_1) = 0$. So there are two cases: (A) $\sigma_1^L(p_1) = 0$, $\ell^r(p_1) = 1$, and (B) $\sigma_1^L(p_1) = 1$, $\ell^r(p_1) < 1$.

In case A:

- $\sigma_1^L(p_1) = 0$, $\sigma_1^H(p_1) = 1$
- $\ell^r(p_1) = 1$, $\ell^a(p_1) = 0$
- $\sigma_0^B(h_0) = 1$ for any history h_0 such that $q_1 = 0$ and $p_0 = \theta$ and $n_0 = 0$
- $\sigma_0^B(h_0) = 1$ for any history h_0 such that $q_1 = 1$ and $p_0 = 1 + \theta$ and $n_0 = 1$
- $\sigma_0^B(h_0) \in [0,1]$ for any history h_0 such that $q_1 = 0$ and $p_0 = 1 + \theta$ and $n_0 = 1$
- $\sigma_0^B(h_0) \in [0,1]$ for any history h_0 such that $q_1 = 1$ and $p_0 = \theta$ and $n_0 = 0$

- $\zeta_0^r(p_1) = \rho_L$ and $\zeta_0^a(p_1) = 1-\rho_H$

All of the equilibria in case A share the same path: the buyer accepts p_1 if and only if $n_1 = 1$, and the seller sets $p_0 = 1+\theta$ following acceptance of p_1, and $p_0 = \theta$ following rejection, and the buyer accepts p_0 if and only if $p_0 = n_0 + \theta$. Thus the seller's expected payoff in case A is

$$\hat{U}(p_1) = \left[1 - \zeta_1\right]\left[p_1 + \delta\rho_H(1+\theta)\right] + \zeta_1\delta\theta$$

In case B, $\ell^r(p_1)$ must be such that the low-valuation buyer weakly prefers to accept in the first period, so

$$0 \le \ell^r(p_1) \le \frac{\theta - p_1}{\delta(1-\rho_L)} < 1$$

If $\ell^r(p_1) > 0$, then $p_0 = \theta$ and $p_0 = 1+\theta$ must yield equal expected payoffs for the seller in the last period, with no possibility of increasing the expected payoff by charging $p_0 = \theta-\varepsilon$, which would surely be accepted, or $p_0 = 1+\theta-\varepsilon$, which would surely be accepted if $n_0 = 1$. This implies that the buyer must accept $p_0 = \theta$ for sure if $q_1 = 0$, and that $p_0 = 1+\theta$ must be accepted for sure if $q_1 = 0$ and $n_0 = 1$. Moreover, the seller's belief must satisfy $\zeta_0^r(p_1) = \zeta_0^*$.

On the other hand, if $\ell^r(p_1) = 0$, then $p_0 = 1+\theta$ must be accepted for sure if $q_1 = 0$ and $n_0 = 1$, but the probability of acceptance is arbitrary for $p_0 = \theta$, $q_1 = 0$ and $n_0 = 0$; moreover the seller's belief $\zeta_0^r(p_1)$ is arbitrary, subject to the condition $\zeta_0^r(p_1) \le \zeta_0^*$.

Thus for equilibria in case B:

- $\sigma_1^L(p_1) = 1$, $\sigma_1^H(p_1) = 1$
- $\ell^a(p_1) = 0$, and

$$0 \le \ell^r(p_1) \le \frac{\theta - p_1}{\delta(1-\rho_L)} < 1$$

- $\sigma_0^B(h_0) = 1$ for any history h_0 such that $q_1 = 0$ and $p_0 = \theta$ and $n_0 = 0$, if $\ell^r(p_1) > 0$
- $\sigma_0^B(h_0) \in [0,1]$ for any history h_0 such that $q_1 = 0$ and $p_0 = \theta$ and $n_0 = 0$, if $\ell^r(p_1) = 0$
- $\sigma_0^B(h_0) \in [0,1]$ for any history h_0 such that $q_1 = 1$ and $p_0 = \theta$ and $n_0 = 0$
- $\sigma_0^B(h_0) = 1$ for any history h_0 such that $p_0 = 1 + \theta$ and $n_0 = 1$

- $\zeta_0^a(p_1) = \varphi\zeta_1 + 1-\rho_H$ and

$$\zeta_0^* \le \zeta_0^r(p_1) \le 1 - \rho_H \quad \text{if } \ell^r(p_1) = 0$$
$$\zeta_0^r(p_1) = \zeta_0^* \quad \text{if } \ell^r(p_1) > 0$$

The set of equilibria in case B leaves the low buyer's response to an unexpected pooling offer in the last period undetermined; also either $\ell^r(p_1)$ or $\zeta_0^r(p_1)$ is undetermined. But all of these equilibria share the same path: the buyer's equilibrium strategy always accepts p_1, and the seller's equilibrium strategy sets $p_0 = 1+\theta$

following acceptance of p_1, and the buyer accepts this if and only if $n_0 = 1$. Thus the seller's expected payoff in case B is

$$\hat{U}(p_1) = p_1 + \delta\left[\rho_H - \varphi\zeta_1\right](1+\theta)$$

Summary of Equilibria in the Subgame under Condition O

The above results are summarized in Table 1, which characterizes the full set of sequential equilibria under Condition O. The table makes it clear that there are many tedious details that preclude a simple description of the set of equilibria. But there are also some interesting results. The most important of these is that there are two distinct equilibrium paths starting from prices between θ - $\delta(1-\rho_L)$ and θ. For prices p_1 in this interval, there are continuation equilibria in which p_1 separates the high and low buyers, and there are also equilibria in which p_1 acts as a pooling price. In particular, the natural intuition that both buyer types must surely accept any price below the low valuation is wrong.

Forward Induction

Given that the subgame has multiple equilibria, for $p_1 \in [\theta - \delta(1-\rho_L), \theta]$, an obvious question is whether some of these equilibria can be eliminated by standard refinement arguments. In particular, the logic of the Intuitive Criterion for signaling games (Cho and Kreps [1987]) suggests that some beliefs following rejection of a pooling offer can be ruled out. Cho (1987) extended the Intuitive Criterion from signaling games to general (finite) games, and although the game considered in this paper is not finite, the relevant parts of Cho's analysis are easily extended.

Consider a pooling equilibrium of the subgame starting from $p_1 \in [\theta - \delta(1-\rho_L), \theta]$. Since $\sigma_1^L(p_1) = \sigma_1^H(p_1) = 1$, the buyer's equilibrium payoffs are

$$V^L = \theta - p_1 + \delta(1-\rho_L)\ell^a(p_1)$$

$$V^H = 1+\theta - p_1 + \delta\rho_H\ell^a(p_1)$$

Let $\tilde{V}^H(p_1)$ be the buyer's expected payoff if $n_1 = 1$ and $q_1 = 0$ (contrary to $\sigma_1^H(p_1)$), and let $\tilde{V}^L(p_1)$ be the buyer's expected payoff if $n_1 = 0$ and $q_1 = 0$. Then

$$\tilde{V}^L = \delta(1-\rho_L)\tilde{\ell}^r(p_1)$$

$$\tilde{V}^H = \delta\rho_H\tilde{\ell}^r(p_1) < 1+\theta-p_1 + \delta\rho_H\ell^a(p_1)$$

If $\tilde{V}^H(p_1) < V^H(p_1)$ for *all* values of $\tilde{\ell}$, and if $\tilde{V}^L(p_1) > V^L(p_1)$ for *some* values of $\tilde{\ell}$, then Cho's "introspective consistency" condition requires $\zeta_1^r = 1$. That is, if rejection of p_1 is strictly worse than acceptance for the high buyer, given that the seller responds optimally with respect to some belief about the buyer's type, and if this is not true for the low buyer, then the seller should put zero probability on the high buyer type if p_1 is rejected. A forward induction equilibrium is then defined as a sequential equilibrium that satisfies introspective consistency.

Proposition 6: In any forward induction equilibrium, if Condition O holds, and if $\rho_L < 1$ and $p_1 \in (\theta - \delta(1-p_L), \theta]$, then $\sigma_1^L(p_1) = 0$ and $\sigma_1^H(p_1) = 1$.

Proof:

By Proposition 5, it need only be shown that the pooling equilibrium for any price in this interval fails to satisfy introspective consistency. Since $\tilde{\ell}^r(p_1) \leq 1$, and $\tilde{\ell}^a(p_1) \geq 0$, it follows that $\tilde{V}^H(p_1) < V^H(p_1)$ for $p_1 \leq \theta$, so introspective consistency requires $\zeta_1^r = 1$ and $\zeta_0^r = \rho_L$. But (according to Proposition 5) the pooling equilibrium specifies $\zeta_0^r \leq \zeta_0^* < \rho_L$.

This is one of the main results of the paper: forward induction implies that the seller cannot achieve pooling by offering to sell at the low valuation in the first period, unless the low valuation is permanent ($\rho_L = 1$).

Table 1: Summary of equilibria in the subgame from p_1, under Condition O

p_1	$(-\infty,p_L)$	p_L	(p_L,θ)		θ		(θ,p_H)	p_H	$(p_H,1+\theta)$	$1+\theta$	$(1+\theta,\infty)$
$\sigma_1^L(p_1)$	1	[0,1]	1	0	1	0	0	0	0	0	0
$\sigma_1^H(p_1)$	1	1	1		1		1	$q_1^H \in [Q_1^H,1]$	Q_1^H	$[0,Q_1^H]$	0
$\zeta_1^r(p_1)$	$[1-\rho_H, \zeta_0^*]$ if $\ell^r(p_1)=0$ ζ_0^* if $\ell^r(p_1)\in(0,1)$ $[\zeta_0^*,\rho_L]$ if $\ell^r(p_1)=1$	ρ_L if $\sigma_1^L(p_1)<1$ $[1-\rho_H, \zeta_0^*]$ if $\ell^r(p_1)=0$ ζ_0^* if $\ell^r(p_1)\in(0,1)$ $[\zeta_0^*,\rho_L]$ if $\ell^r(p_1)=1$	$[1-\rho_H, \zeta_0^*]$ if $\ell^r(p_1)=0$ ζ_0^* if $0<\ell^r(p_1)$	ρ_L	$[1-\rho_H, \zeta_0^*]$	ρ_L	ρ_L	$1-\rho_H+\varphi\tilde{\zeta}^r(p_H)$	ζ_0^*	$[1-\rho_H, \zeta_0^*]$	$\varphi\zeta_1+1-\rho_H$
$\zeta_1^a(p_1)$	$\varphi\zeta_1+1-\rho_H$	$1-\rho_H+\varphi\tilde{\zeta}^a(p_H)$	$\varphi\zeta_1+1-\rho_H$	$1-\rho_H$	ζ_0	$1-\rho_H$	$1-\rho_H$	$1-\rho_H$	$1-\rho_H$	$[1-\rho_H, \rho_L]$ if $\sigma_1^H(p_1)=0$ $1-\rho_H$ if $\sigma_1^H(p_1)>0$	$[1-\rho_H, \rho_L]$
$\ell^r(p_1)$	[0,1]	[0,1] $\ell^r(p_1)=1$ or $\sigma_1^L(p_1)=1$	$[0,\overline{\ell}_L^r(p_1)]$	1	0	1	1	1	$\overline{\ell}_H^r(p_1)$	0	0
$\ell^a(p_1)$	0	0	0		0		0	0	0	0	$[0,\overline{\ell}_H^a(p_1)]$
$\sigma_0^L(\theta\|q_1=0)$	1 if $\ell^r(p_1)>0$ $\alpha(n_1)$ else	1 if $\ell^r(p_1)>0$ $\alpha(n_1)$ else	1 if $\ell^r(p_1)>0$ $\alpha(n_1)$ else	1	$\alpha(n_1)$	1	1	1	1	$\alpha(n_1)$	$\alpha(n_1,p_1)$
$\sigma_0^L(\theta\|q_1=1)$	$\alpha(n_1)$	$\alpha(n_1)$	$\alpha(n_1)$		$\alpha(n_1)$		$\alpha(n_1)$	$\alpha(n_1)$	$\alpha(n_1)$	$\alpha(n_1)$	1 if $\ell^a(p_1)>0$ $\alpha(n_1,p_1)$ else
$\sigma_0^H(1+\theta\|q_1=0)$	1 if $\ell^r(p_1)<1$ $\beta(n_1)$ else	1 if $\ell^r(p_1)<1$ $\beta(n_1)$ else	1	$\beta(n_1)$	1	$\beta(n_1)$	$\beta(n_1)$	$\beta(n_1)$	1	1	1
$\sigma_0^H(1+\theta\|q_1=1)$	1	1	1		1		1	1	1	1	1 if $\ell^a(p_1)<1$ $\beta(n_1,p_1)$ else
			Prop 5				Prop 2	Prop 3	Prop 4		Prop 1

5. Equilibria in the Two-Period Game with an Optimistic Seller

For any price p_1 that the seller might choose in the first period, the analysis of the subgame yields a set of expected payoffs generated by the continuation equilibria of the subgame starting from p_1. The set of equilibria in the two-period game is then found by choosing prices that yield maximal expected payoffs for the seller.

This section assumes that Condition O holds; the alternative (Condition P) is considered in the next section. The first result says that there is no equilibrium in which the low-valuation buyer randomizes between acceptance and rejection of any first-period offer that is on the equilibrium path.

Lemma O12:

In any equilibrium, if Condition O holds, $\sigma_1^L(p_1) \in \{0,1\}$ for all $p_1 \in \underline{\sigma}_1^S$.

Proof:

First, if $p_1 > \theta$, then Condition O implies $\sigma_1^L(p_1) = 0$ by Lemma O6. Suppose $p_1 \leq \theta$ and $q_1^L = \sigma_1^L(p_1)\ \varepsilon\ (0,1)$. Then (by Lemma 4) the price p_1 is accepted for sure by the high buyer, so rejection must be interpreted as coming from the low buyer, by Bayes rule, which implies that $\ell^r(p_1) = 1$. Moreover, acceptance must make the seller more optimistic, so $\zeta_0^a \geq \zeta_1$, and then Condition O implies that $\ell^a(p_1) = 0$. The low buyer's indifference then implies that $p_1 = \theta - \delta(1-\rho_L)$. Any price below this is accepted for sure by the buyer (by the Corollary to Lemma 1). Thus by reducing p_1 slightly, the seller gets a sure acceptance, with a screening offer in the last period. The seller's expected payoff is a linear function of q_1^L. At one extreme ($q_1^L = 1$), the expected payoff is the value of a pure pooling offer, and at the other extreme it is the value of a screening offer with an inferior screening price ($p_1 \leq \theta < p_H$). Thus the seller can do better by choosing either $p_1 = p_H\text{-}\varepsilon$, or $p_1 = \theta\text{-}\delta(1\text{-}\rho_L)\text{-}\varepsilon$, for some small value of ε.

The next lemma proves the (obvious) result that there is no equilibrium price that would surely be rejected.

Lemma O13: In any equilibrium, if $p_1 \in \underline{\sigma}_1^S$ then $\sigma_1^H(p_1) > 0$.

Proof:

If $\sigma_1^H(p_1) = 0$ then $\sigma_1^L(p_1) = 0$, and $\zeta_0 = \varphi\zeta_1 + 1\text{-}\rho_H$, and the seller's expected payoff is the expected payoff from the one-period game with belief ζ_0, which is uniquely determined by the Corollary to Proposition 0. But if $0 < p < \theta\text{-}\delta(1\text{-}\rho_L)$ then $\sigma_1^H(p) = \sigma_1^L(p) = 1$, and again $\zeta_0 = \varphi\zeta_1 + 1\text{-}\rho_H$, so the seller's expected payoff is increased by the amount p, which implies that p_1 is not optimal for the seller.

The set of equilibrium screening prices can now be identified:

Lemma O14: In any equilibrium, if Condition O holds, and if $p_1 \in \underline{\sigma}_1^S$ and $\sigma_1^L(p_1) = 0$ then either $p_1 = p_H$ or $p_1 = 1+\theta$.

Proof:

Since $\sigma_1^L(p_1) = 0$, and $\sigma_1^H(p_1) > 0$ by Lemma O13, the Corollary to Lemma 1 implies $\theta - \delta(1\text{-}\rho_L) \leq p_1 \leq 1+\theta$. Prices in the interval (θ, p_H) are ruled out by the Corollary to Proposition 2, and when $\sigma_1^L(p_1) = 0$ the argument there extends immediately to $p_1 = \theta$. Prices in the interval $(p_H, 1+\theta)$ are ruled out by the Corollary to Proposition 4. Finally, prices in the interval $[\theta\text{-}\delta[1\text{-}\rho_L], \theta)$ can be ruled out by comparing the seller's expected payoff form Propositions 4 and 5: if $p_1 < p_H$ and $\sigma_1^L(p_1) = 0$ then p_1 is a screening price, and $p_H\text{-}\varepsilon$ is a better screening price.

Table 2 lists the expected payoffs for each price that the seller might choose in the first period, and the expected payoffs resulting in each equilibrium of the subgame. The last column of the table identifies that prices that are dominated.

Table 2: Equilibrium Paths and Expected Payoffs for the Seller in the Subgame, under Condition O

p_1	q_1	p_0	q_0	Seller's Expected Payoff (U)	Dominated by
$(-\infty,p_L)$	1	$1+\theta$	n_0	$p_1+\delta\left[\rho_H-\phi\zeta_1\right](1+\theta)$	$½p_1 + ½\,p_L$
p_L	1	$1+\theta$	n_0	$\left[1+\delta\rho_H\right]\theta+\delta\phi(1-\zeta_1[1+\theta])$	☺
	$n_1+(1-n_1)q_1^L$ $q_1^L\in(0,1)$	$q_1+\theta$	$1-q_1(1-n_0)$	$\delta\theta+\left[1-\zeta_1\right]\left[\theta+\delta\varphi-\delta(1-\rho_H)\theta\right]+\zeta_1\theta(1-\delta\rho_L)q_1^L$	$p_L-\varepsilon$
	n_1	$q_1+\theta$	$1-n_1(1-n_0)$	$\delta\theta+\left[1-\zeta_1\right]\left[\theta+\delta\varphi-\delta(1-\rho_H)\theta\right]$	$½\theta + ½\,p_H$
(p_L,θ)	1	$1+\theta$	n_0	$p_1+\delta\left[\rho_H-\phi\zeta_1\right](1+\theta)$	☺
	n_1	$q_1+\theta$	$1-n_1(1-n_0)$	$\delta\theta+\left[1-\zeta_1\right]\left[p_1+\delta\rho_H-\delta(1-\rho_H)\theta\right]$	$½\theta + ½\,p_H$
θ	1	$1+\theta$	n_0	$\theta+\delta\left[\rho_H-\phi\zeta_1\right](1+\theta)$	☺
	n_1	$q_1+\theta$	$1-n_1(1-n_0)$	$\delta\theta+\left[1-\zeta_1\right]\left[\theta+\delta\rho_H-\delta(1-\rho_H)\theta\right]$	$½\theta + ½\,p_H$
(θ,p_H)				$\delta\theta+\left[1-\zeta_1\right]\left[p_1+\delta\rho_H-\delta(1-\rho_H)\theta\right]$	$½p_1 + ½\,p_H$
p_H	n_1	$n_1+\theta$	$1-n_1(1-n_0)$	$\delta\theta+\left[1-\zeta_1\right]\left[1+\theta-\delta(1-\rho_H)\theta\right]$	☺
	$n_1 q_1^H$ $q_1^H\in[Q_1^H,1)$	$n_1+\theta$	$1-n_1(1-n_0)$	$\delta\theta+q_1^H\left[1-\zeta_1\right]\left[1+\theta-\delta(1-\rho_H)\theta\right]$	$p_H-\varepsilon$
$(p_H,1+\theta)$	$1-\dfrac{\varphi\zeta_1}{\zeta_0^*-(1-\rho_H)}$			$p_1\left[1-\dfrac{\varphi\zeta_1}{\zeta_0^*-(1-\rho_H)}\right]+\delta(1+\theta)\left[\rho_H-\varphi\zeta_1\right]$	$½p_1 + ½(1+\theta)$
	$n_1 Q_1^H$			$(1+\theta)\left[1-\dfrac{\varphi\zeta_1}{\zeta_0^*-(1-\rho_H)}\right]+\delta(1+\theta)\left[\rho_H-\varphi\zeta_1\right]$	☺
$1+\theta$	$n_1 q_1^H$ $q_1^H\in[0,Q_1^H)$			$(1+\theta)(1-\zeta_1)q_1^H+\delta(1+\theta)\left[\rho_H-\phi\zeta_1\right]$	$1+\theta-\varepsilon$
$(1+\theta,\infty)$	0	$1+\theta$	n_0	$\delta(1+\theta)\left[\rho_H-\phi\zeta_1\right]$	$p_1<p_L$

Characterization of the set of Sequential Equilibria [sketch]

Fix a first-period belief ζ_1 for the seller. Consider the seller's expected utility for each equilibrium (σ,Π). This defines a mapping $U = \upsilon(\sigma,\Pi;\zeta_1)$. The set of pairs (ζ_1,U) that are generated by some equilibrium is the outer envelope of a parallelogram (generated by first-period pooling strategies) and two lines (generated by first-period partial screening strategies and first-period full screening strategies). For any point (ζ_1,U) in the envelope, there is at least one equilibrium in which the seller's initial belief is ζ_1, and the seller's expected payoff is U.

There are three sets of potential equilibria, corresponding to pooling, full screening, and partial screening in the first period. Every sequential equilibrium must be in one of these three sets. Then the set of actual equilibria is found by using the envelope condition.

Define the following threshold beliefs

$$\bar{\zeta}_1^* \equiv \frac{1}{1 + \dfrac{\rho_L(1+\theta)-1}{\rho_H}\left[1 + \dfrac{1}{\delta\left[\zeta_0^* - (1-\rho_H)\right]}\right]}$$

$$\gamma_1(p_1) \equiv \frac{(1+\theta-p_1)[\zeta_0^* - (1-\rho_H)]}{\varphi(1+\theta)}$$

$$\gamma_1^*(p_1) \equiv \frac{1+\theta-\delta\rho_H-p_1}{1+\theta-\delta\varphi-\delta\rho_L\theta}$$

$\bar{\zeta}_1^*$	Screening ($p_1 = p_H$) vs. Partial Screening ($p_1 = 1+\theta$)
$\hat{\zeta}_1 = \gamma_1(\theta)$	Partial Screening ($p_1 = 1+\theta$) vs. Pooling at the highest price ($p_1 = \theta$)
$\hat{Z}_1 = \gamma_1(p_L)$	Partial Screening ($p_1 = 1+\theta$) vs. Pooling at the lowest price ($p_1 = p_L = \theta - \delta[1-\rho_L]$)
$\zeta_1^* = \gamma_1^*(\theta)$	Screening ($p_1 = p_H$) vs. Pooling at the highest price ($p_1 = \theta$)
$Z_1^* = \gamma_1^*(p_L)$	Screening ($p_1 = p_H$) vs. Pooling at the lowest price ($p_1 = \theta - \delta[1-\rho_L]$)

Note that $\hat{Z}_1 > \hat{\zeta}_1$, and $Z_1^* > \zeta_1^*$. Also, $\hat{\zeta}_1$ and ζ_1^* are on the same side of $\bar{\zeta}_1^*$ (either both above or both below) and $\hat{Z}_1$ and Z_1^* are on the same side of $\bar{\zeta}_1^*$ (either both above or both below).

Proposition 7:

In the two-period game, under Condition O, the set of equilibria is characterized as follows.

- If $\zeta_1 < \bar{\zeta}_1^*$, then screening is dominated by partial screening;
  - If ζ_1 is below $\hat{\zeta}_1$ then there is a unique equilibrium path (partial screening).
    - If ζ_1 is above $\hat{Z}_1$ then there is no screening equilibrium, and there is a continuum of pooling equilibria parameterized by p_1, where p_1 ranges over the interval $[\theta - \delta(1-\rho_L), \theta]$.
    - If ζ_1 is above $\hat{\zeta}_1$ and below $\hat{Z}_1$then there is one partial screening equilibrium, and there is a continuum of pooling equilibria parameterized by p_1, where p_1 ranges over a sub-interval of $[\theta - \delta(1-\rho_L), \theta]$. Also, there is a continuum of equilibria in which the seller randomizes between pooling and partial screening in the first period.
- If $\zeta_1 > \zeta^a$, then partial screening is dominated by screening.
  - If ζ_1 is below ζ_1^* then there is a unique equilibrium path (screening).
    - If ζ_1 is above Z_1^* then there is no screening equilibrium, and there is a continuum of pooling equilibria parameterized by p_1, where p_1 ranges over the interval $[\theta - \delta(1-\rho_L), \theta]$.
    - If ζ_1 is above ζ_1^* and below Z_1^* then there is one screening equilibrium, and there is a continuum of pooling equilibria parameterized by p_1, where p_1 ranges over a sub-interval of $[\theta - \delta(1-\rho_L), \theta]$. Also, there is a continuum of equilibria in which the seller randomizes between pooling and screening in the first period.

These results are sketched in Figure 1.

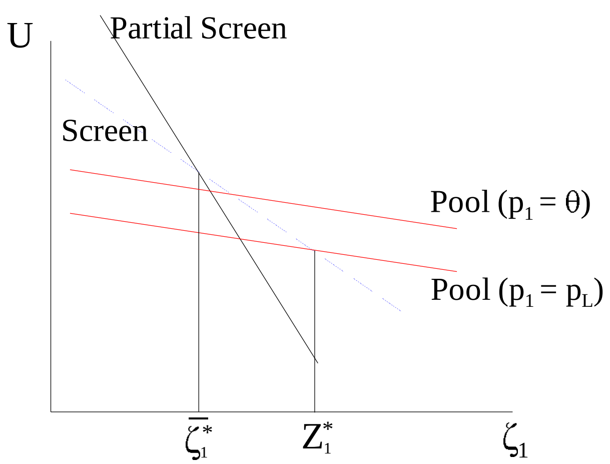

Figure 1a: expected payoffs for the seller

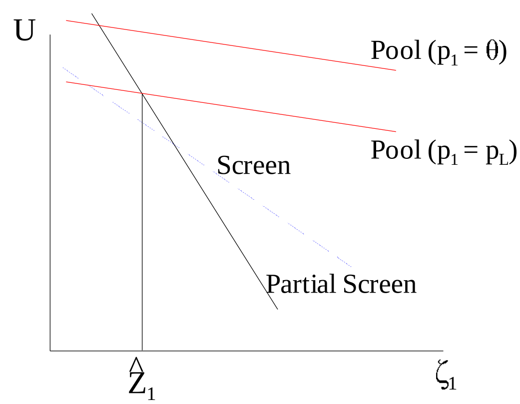

Figure 1b: expected payoffs for the seller

Characterization of the set of Sequential Equilibria that survive Forward Induction [sketch]

As was shown in Proposition 6 above, the highest pooling price in any forward induction equilibrium of the subgame is $\theta - \delta(1-\rho_L)$, when Condition O holds. The parallelogram representing the set of pooling equilibria is reduced to a line, which is the lower edge of the parallelogram. Then the set of pairs (ζ_1, U) that are generated by some sequential equilibrium that survives forward induction is the outer envelope of three lines. For any initial belief ζ_1, except for nongeneric cases where ζ_1 is such that two or three of these lines intersect, there is a unique equilibrium that survives forward induction.

Informational Rents in the Two-Period Game with an Optimistic Seller

Recall from Section 2 the conjecture that informational rents accrue to a high-valuation buyer who is mixed in with a relatively large proportion of low-valuation buyers, because the seller does not find it worthwhile to extract all of the high-valuation buyer's surplus, at the risk of losing all of the low-valuation buyer's surplus, which is available as a sure thing. It is now apparent that this conjecture is false. The only robust pooling equilibrium under Condition O concedes an informational rent to *all* types of the buyer. The reason is that the seller is unable to commit to making a pooling offer in the second period.

6. Equilibria in the Two-Period Game with a Pessimistic Seller

When the seller is relatively pessimistic at the beginning of the game, pooling in the first period implies a pooling offer in the last period. This makes a big difference. There is now a larger interval of prices that may act as pooling prices in the first period, including prices that are *above* the low valuation. That is, there are equilibria in which the informational rent collected by the lowest buyer type is negative. These equilibria are deleted by forward induction. But what remains is a set of pooling equilibria in which both types of the buyer *strictly* prefer to accept the seller's first-period offer, and yet the seller cannot extract more surplus by increasing the price.

Lemma 15: In any sequential equilibrium, if $p_1 > \theta + \delta(1-\rho_L)$ and $n_1 = 0$, then $\sigma_1^B(n_1,p_1) = 0$.

Proof:

If $n_1 = 0$, then the buyer's current payoff if $q_1 = 1$ is less than $-\delta(1-\rho_L)$, and the discounted payoff from the last period is no higher than $\delta(1-\rho_L)$.

Lemma P0: In any equilibrium, if Condition P holds, and if $\sigma_1^L(p_1) < 1$, then $\ell^r(p_1) = 1$.

Proof:

If $\sigma_1^L(p_1) = 0$ then $\zeta_0^r(p_1) \geq \varphi\zeta_1+1-\rho_H$, so $\zeta_0^r(p_1) > \zeta_0^*$ by Condition P, which implies $\ell^r(p_1) = 1$. If If $\sigma_1^L(p_1) > 0$ then $\sigma_1^H(p_1) = 1$, and $\sigma_1^L(p_1) < 1$ implies $\zeta_0^r(p_1) = \rho_L$ and $\ell^r(p_1) = 1$.

Lemma P1: In any equilibrium, if Condition P holds, and if $p_1 > \max[1+\theta-\delta\rho_H, \theta+\delta(1-\rho_L)]$, then $\sigma_1^L(p_1) = \sigma_1^H(p_1) = 0$.

Proof:

Since $p_1 > \theta+\delta(1-\rho_L)$, Lemma 1 implies $\sigma_1^L(p_1) = 0$, so $\ell^r(p_1) = 1$ by Lemma P0. Moreover $\sigma_1^H(p_1) > 0$ implies $\zeta_0^a(p_1) = 1-\rho_H$ and $\ell^a(p_1) = 0$, so $\sigma_1^H(p_1) = 0$ by Lemma 1, a contradiction. Thus $\sigma_1^H(p_1) = 0$.

Lemma P2: In any equilibrium, if Condition P holds, and if $p_1 \in (1+\theta-\delta\rho_H, \theta+\delta(1-\rho_L))$, then either $\sigma_1^L(p_1) = \sigma_1^H(p_1) = 0$, or $\sigma_1^L(p_1) = \sigma_1^H(p_1) = 1$.

Proof:

[to be added]

Define

$$\overline{\ell}_L^r(p_1) \equiv 1 - \frac{p_1 - \theta}{\delta(1-\rho_L)}$$

Lemma P3: If Condition P holds, and if $p_1 \in (\theta-\delta[1-\rho_L], \theta)$, then the set of equilibria of the subgame starting from p_1 can be characterized as follows:

$\sigma_1^H(p_1) = 1$, and

either $\sigma_1^L(p_1) = 0$, $\ell^r(p_1) = 1$, $\ell^a(p_1) = 0$

or $\sigma_1^L(p_1) = Q_1^L$, $\ell^r(p_1) = 1$, $\ell^a(p_1) = \overline{\ell}_L^a(p_1)$

or $\sigma_1^L(p_1) = 1$, $\ell^r(p_1) \in [0,1]$, $\ell^a(p_1) = 1$

Proof:

Since $p_1 < 1+\theta-\delta\rho_H$, it follows that $\sigma_1^H(p_1) = 1$. If $\sigma_1^L(p_1) = 1$, then $\ell^a(p_1) = 1$, so acceptance is (strictly) optimal according to Lemma 1, for any value of $\ell^r(p_1)$. If $\sigma_1^L(p_1) < 1$, then $\ell^r(p_1) = 1$. If $\sigma_1^L(p_1) = 0$, then $\zeta_0^a(p_1) = 1-\rho_H$, so $\ell^a(p_1) = 0$, and then rejection is optimal for the low buyer because $p_1 \geq \theta-\delta(1-\rho_L)$. If $0 < \sigma_1^L(p_1) < 1$, then the low buyer must be indifferent, and this requires

$$\ell^a(p_1) = 1 - \frac{\theta - p_1}{\delta(1-\rho_L)}$$

Also, if the low buyer is randomizing, the seller must be randomizing in the last period following a rejection, and this requires $\zeta = \zeta_0^*$, so the probability of acceptance in the first period is tied down at Q_1^L.

Notes

Suppose $p_1 = \theta-\delta[1-\rho_L]$ and $0 < \sigma_1^L(p_1) < 1$. Then $\ell^r(p_1) = 1$ and the low buyer's indifference requires $\ell^a(p_1) = 0$, so $\zeta_0^a(p_1) \leq \zeta_0^*$, which implies $\sigma_1^L(p_1) \leq Q_1^L$.

Suppose $p_1 = \theta$ and $0 < \sigma_1^L(p_1) < 1$. Then $\ell^r(p_1) = 1$ and the low buyer's indifference requires $\ell^a(p_1) = 1$, so $\zeta_0^a(p_1) \geq \zeta_0^*$, which implies $\sigma_1^L(p_1) \geq Q_1^L$.

Lemma P4: If Condition P holds, and if $p_1 \in (\theta,\theta+\delta[1-\rho_L])$, and if $p_1 < 1+\theta-\delta\rho_H$, then the set of equilibria of the subgame starting from p_1 can be characterized as follows:

$\sigma_1^H(p_1) = 1$, and

either $\sigma_1^L(p_1) = 0$, $\ell^r(p_1) = 1$, $\ell^a(p_1) = 0$

or $\sigma_1^L(p_1) = 1$, $\ell^a(p_1) = 1$, $\ell^r(p_1) \leq \overline{\ell}_L^r(p_1)$

Proof:

Since $p_1 < 1+\theta-\delta\rho_H$, it follows that $\sigma_1^H(p_1) = 1$. If $\sigma_1^L(p_1) = 1$, then $\ell^a(p_1) = 1$, so acceptance is optimal provided that $\ell^r(p_1) \leq \overline{\ell}_L^r(p_1)$.

If $\sigma_1^L(p_1) < 1$, then $\ell^r(p_1) = 1$, and then Lemma 1 requires $\sigma_1^L(p_1) = 0$, for any value of $\ell^a(p_1)$, because $p_1 > \theta$. That is, the low buyer can't be randomizing at prices above θ, because rejection would necessarily lead to a pooling offer, and acceptance would cost something now, and could not lead to anything better than a pooling offer. If $\sigma_1^L(p_1) = 0$, then $\zeta_0^a(p_1) = 1-\rho_H$, so $\ell^a(p_1) = 0$, and then rejection is optimal for the low buyer because $p_1 \geq \theta-\delta(1-\rho_L)$.

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