Output and Price Level Effects of Monetary Uncertainty in a Matching Model*

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Abstract

Monetary uncertainty and information lags are put into a random matching model so that the resulting setting has some meetings in which producers are relatively informed and others in which consumers are relatively informed. For that setting, the ex ante socially optimal way to conduct trade is characterized. The optimum can display a variety of relationships between money and total output and the price level. While the price level is always sticky, even the direction of its response and that of total output depend on the magnitude of the lag and on subtle features of the serial correlation properties of the money supply. JEL classification #'s: E30, E40, D82.

1. Introduction

We show that a rich theory of the relationships between money and aggregate output and the price level results from combining two by-now familiar ideas. One is that money is essential because of absence-of-double-coincidence difficulties. The other is that changes in the money supply affect output because of information lags. In this paper, the first idea takes the form of a fairly standard random-matching model of money (see Shi [1995] and Trejos and Wright [1995]). The second idea takes the form of lags in seeing monetary realizations that are drawn from a Markov process: some people see realizations when they occur while others see them only with a lag. In addition to being familiar, these two ideas fit well together. In order for money to be essential, it is necessary that individual histories are not common knowledge (see Kocherlakota [1998] and Wallace [1998]). Therefore, monetary transfers can plausibly be part of what is not common knowledge. Also, because trade occurs in informationally separated meetings, there are no commonly observed prices from which people can draw fully revealing inferences about monetary realizations even if those are the only shocks. We use a model that embodies these ideas to analyze the socially optimal way to conduct trade. The resulting theory is rich in that the optimum can display a variety of relationships between money and the endogenous aggregates. In particular, even the direction of the response of output depends sensitively on the serial correlation properties of the money supply process. Rare enough increases in the quantity of money can produce output declines, while other increases tend to be expansionary. Such a theory is desirable because, according to Backus and Kehoe (1992) and Jones and Manuelli (2000), data for different countries and different time periods display disparate responses.

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The effects of monetary shocks with information lags were first analyzed rigorously by Lucas (1972).[1] Although our specification of monetary shocks is similar, other features of our model are quite different. Differential information is implicit in the information-lag idea, if only because those who experience the monetary transfers tend to know more about realizations than others. In Lucas (1972) this played no role because of the special structure of the model: the relatively informed were the old in the last period of their lives who made no decisions. In our matching model context, this differential information comes to the fore. Moreover, except in Jones and Manuelli (2000), all subsequent work pursuing the information-lag approach has followed Lucas (1972) in assuming that consumers are relatively informed. We treat consumers and producers symmetrically with respect to information. Allowing producers to be relatively informed gives rise to additional possibilities for the relationships among aggregates.

On the surface, our model resembles Wallace (1997). He studied the effects of one-time uncertainty using the same background environment. However, Wallace evaded the explicit study of differential information by assuming that consumers are relatively informed and make take-it-or-leave-it offers. In addition, by studying one-time uncertainty, he avoided interaction between current trades and the future value of money. Relative to what is done in Wallace (1997), we (i) adopt an information structure which allows either producers or consumers to be relatively informed, (ii) assume that trades in meetings occur in a way that is best from an ex ante point of view (a mechanism design approach), and (iii) allow for interaction between current trades and the future value of money by deducing properties of the optimum Markov process for output and the price level given an assumed Markov process for the money supply. While Jones and Manuelli (2000) also allow for either producers or consumers to be relatively informed, they assume one-time uncertainty and a particular way of conducting trade.

Most previous work on matching models has assumed a bargaining rule that gives each side some fixed share of the "bargaining power." With differential information, there are any number of ways that trade could be conducted in meetings. Rather than adopt an arbitrary one, we analyze the best available mechanism—best for a society that takes as given its money supply process and the differential information about realizations and that chooses the optimal way to conduct trade in meetings subject to (sequential) individual rationality and truth-telling constraints. One interpretation is that we are studying the mechanism that would actually be chosen under ideal conditions; another is that we are studying a reference point against which any mechanism might be judged. Our approach ensures that the relationships between money and other aggregates that we find are not due to having imposed an inefficient way to conduct trade.

We analyze ongoing stochastic fluctuations in the money supply rather than one-time uncertainty because that permits us to deduce the times series implications of our model. Put differently, the analysis of one-time realizations, as is done in Wallace (1997) and Jones and Manuelli (2000), is at best a step toward the kind of analysis we do. From a methodological point of view, the distinction is like that between partial and general equilibrium analysis. If the money supply changes just once, then the value of money in the future may not be affected by the trading rule used before that change occurs, a simplification used in Wallace (1997) and Jones and Manuelli (2000). If, instead, there is ongoing uncertainty, which is the general situation, then alternative trading rules necessarily interact with the future value of money.[2]

[1] Models based on information lags are often viewed as inapplicable to the modern world economy in which there is rapid public dissemination of the actions of some central banks and governments. However, the availability of up-to-date information about such actions does not imply that there is complete information about all significant aggregate demand realizations.

[2] The mechanism design problem in standard static models of bilateral exchange with private information about valuations which are taken as fixed is concerned with the extent to which trade can be achieved when it is mutually beneficial (see e.g. Kennan and Wilson [1993]). In our model, on the other hand, the valuations depend on the

The main limitation of our model is that an individual's money holding is limited to be either zero or one unit. As a crude way to control for the effects of that restriction, we interpret the effects of information lags relative to a benchmark model in which everyone is informed about the current realization when it occurs. When everyone is informed, the price level and total output at a date depend only on the monetary realization for that date. The price level is increasing in the amount of money, but total output can be increasing or decreasing in the amount of money. We assume that the amount of money is never so large as to reduce the fraction of meetings in which trade can occur (because such crowding out of potential trade would not occur were it not for the upper bound on individual holdings). In other words, we assume that output is increasing in the amount of money at the extensive margin. However, because the amount produced in each meeting, the intensive margin, is governed partly by the probability of subsequently meeting people without money, it is decreasing in the amount of money. Either effect can dominate.[3]

The introduction of an information lag makes total output and the price level at a date dependent on both the current and last period's realization of the money supply. Total output relative to the benchmark behaves in a way that depends on what happens in meetings between the informed and the uninformed. The larger the amount of money, the larger the fraction of meetings between informed consumers and uninformed producers in which trade can occur. Moreover, because, as we show, there is trade in all those meetings and a level of production in each meeting which for higher than average realizations is higher than what would be produced if the producer were informed, such meetings are a source of positive impact effects relative to the benchmark. However, meetings between informed producers and uninformed consumers provide a potential offset. If a high monetary realization is sufficiently unlikely, then the optimum has no-trade in meetings between informed producers and uninformed consumers when such a realization occurs. When that happens and if there are enough such meetings, then there can be negative impact effects on output. The optimum need not have no-trade when a low probability monetary decrease occurs. Hence, impact effects can be asymmetric for increases and decreases.

A notable feature of the results is that price stickiness emerges as an optimal response to the incentive constraints: there is no need to introduce constraints on the frequency of price adjustments to explain why prices are slow to respond to changes in the money supply. When two uninformed people meet, their actions, of course, cannot depend on the current realization. In addition, when one trader is relatively informed, that information cannot affect the price without violating the truth-telling constraints.

We also find that a positive relationship between money and output can coexist with a negative relationship between money and the aggregate price level, even though the fluctuations in output and prices are driven entirely by shocks to the money supply, with no other disturbances. This may help shed some light on empirical findings that monetary expansions are often accompanied by reductions in inflation, and that the real effects of monetary surprises do not seem to be transmitted through prices.[4] Our result is related to our findings on price stickiness. An increase in the money supply tends to make money less valuable in the future, so that producers supply less in exchange for money. But this output decrease, which is a price increase, occurs only in meetings between informed people. Meanwhile, there are also meetings between informed and uninformed people, and

mechanism, and much of the analytical effort in the paper is devoted to the implications of that dependence.

[3] Nonneutrality when everyone is informed should not be disturbing. If money is divisible and if individual monetary transfers are not to fully reveal the aggregate state, then each person's transfer cannot be the same fraction of the aggregate change in the money supply except in very special settings. But, then, even if the aggregate state is known, there are distributional effects.

[4] See Lucas (1996) and Leeper, Sims and Zha (1996). Wallace (1992) shows that the introduction of an additional aggregate disturbance can generate a negative relationship between the price level and output, with a positive money-output correlation.

an increase in money makes it more likely that the informed person in such a meeting is the consumer; output is then relatively high, and prices are sticky in the sense just described. The upshot is that the price index may fall even though aggregate output rises.

The paper proceeds as follows. In section 2, we describe the environment we study. In section 3, we present a preview of the analysis and results. Section 4 describes the mechanisms we consider and the optimum problem. In section 5, we present a characterization of the optimum in an interesting region of the parameter space: the region in which producer participation constraints are binding, but consumer participation constraints are not. In section 6, we present examples which display the effects described above. Proofs of the main results appear in the Appendix; for other proofs, the reader is referred to Katzman, Kennan and Wallace (1999).

2. The environment

Aside from the uncertainty about the stock of money, the environment is that in Wallace (1997), which, in turn, is essentially that in Shi (1995) and Trejos and Wright (1995). Time is discrete and the horizon is infinite. There are N distinct, divisible, and perishable types of goods at each date and there is a $[0,1]$ continuum of each of N specialization types of people, where $N \geq 3$. A specialization type-n person consumes only good n and produces only good $n+1$ (modulo N), for $n = 1, 2, ..., N$. Each person maximizes expected discounted utility with discount factor $\beta \in (0,1)$. Utility in a period is given by $u(x) - y$, where x is the amount consumed and y is the amount produced. The function u is defined on $[0, \infty)$, is increasing, strictly concave, and satisfies $u(0) = 0$, $u'(0) = \infty$, and $u'(\infty) < 1$. In each period, people are randomly matched in pairs. Meetings are of two sorts: single-coincidence meetings, those between a type n person (the producer) and a type $n+1$ person (the consumer) for some n; and no-coincidence meetings, those in which neither person produces what the other consumes. (Because the number of types, N, exceeds two, there are no double-coincidence meetings.) We assume that people cannot commit to what they will do in future meetings. Money consists of perfectly durable and indivisible objects which cannot be produced and which do not yield utility directly. We assume that each person can carry from one date to the next at most one unit of money. We also assume that each trader in a meeting is able to see the trading partner's specialization type, money holdings, and whether the person is informed or not, but is otherwise ignorant about the trading partner's history.

The quantity of money follows an S-state Markov process. That is, there are S potential levels for the stock of money: $m_1, m_2, ..., m_S$ where $m_i \in (0,1)$ and $m_i < m_{i+1}$. Here m_i is the state i amount of money per specialization type. We let π_{ij} denote the probability that the current state is j given that the previous state is i and let Π denote the associated transition matrix. We assume that $\pi_{ij} > 0$ which implies that Π has a unique invariant distribution which assigns positive probability to each state. We want changes in the money supply to come about in a way that gives no immediate information to the uninformed. That is accomplished by having only informed people experience transfers of money in the following way. At the end of each date after meetings have dissolved, the current amount of money is publicly announced. (Consequently, at that time, people differ only in money holdings, not in information.) Then, a randomly chosen subset of each specialization type, of measure λ, is selected and is informed about the new state. If the previous state is i, then the measure of newly informed with money is $m_i\lambda$ and the measure without money is $(1 - m_i)\lambda$. If the new state is $j > i$, then a randomly chosen subset of the informed without money, a subset of measure $m_j - m_i$, is given a unit of money. If the new state is $j < i$, then a randomly chosen subset of the informed with money, a subset of measure $m_i - m_j$, loses a unit of money. Then meetings occur and the sequence is repeated.

The following table shows the fraction of each specialization type according to whether they are informed and whether they have money given the previous and current states.

Table 1. Distribution when the previous state is i and the current state is j.

	0 units of money	1 unit of money	sums
informed	$\lambda(1-m_i)-(m_j-m_i)\equiv\theta_{ji}$	$\lambda m_i+m_j-m_i\equiv\lambda-\theta_{ji}$	λ
uninformed	$(1-\lambda)(1-m_i)$	$(1-\lambda)m_i$	$1-\lambda$
sums	$1-m_j$	m_j	1

In order to be able to have all changes in the amount of money be experienced by those who are informed, we need to assume that λ is large enough relative to the monetary changes. The following assumptions accomplish that: $m_S \leq \frac{1}{2}$ and $\lambda \in [\frac{m_S-m_1}{m_S}, 1]$. It is then easily seen that θ_{ji} and $\lambda-\theta_{ji}$ are non-negative (see Katzman, Kennan and Wallace [1999] for a proof).

The monetary uncertainty has obvious incentive effects in our model. It makes those without money, the potential producers, less willing to produce in order acquire to money because (i) they may lose the money acquired before they get to spend it; and (ii) if they do not produce, then they may be given a unit. It also makes those with money, the potential consumers, more willing to spend money because (i) if they do not spend it, then they may lose it; and (ii) if they do spend it, then they may be given a unit.

3. Preview of the analysis and results

We study a class of deterministic mechanisms which can be described as follows. At each meeting, there is a computer that is programmed at date 0 before people go off to their meetings, but is able to receive messages. At the beginning of a date, when the previous state is publicly announced, that information is received by each computer. In addition, the computer at a meeting is able to see who is informed and who is a producer and who is a consumer. If both are informed, then both simultaneously announce to the computer a possible current state. If one person is informed, then that person announces to the computer a possible state. The uninformed person in the meeting does not see the announcement. In all cases, the computer then proposes a trade, which may be no trade. Then each person's choice is either to accept or reject, where rejection by either person implies no trade. Acceptance by both implies that the proposed trade is carried out. Whether trade occurs or not, the meeting ends. A mechanism is incentive feasible if it induces truth-telling and if the proposed trade satisfies individual rationality, and, therefore, induces acceptance.[5]

Because the mechanisms are deterministic, output in a meeting is positive if and only if money is transferred. It follows that the only meetings in which trade can occur are single-coincidence meetings in which the consumer has money and the producer does not. We call these *trade meetings*. We begin by pointing out the distinct potential roles of the current money supply. The amount of money has a direct effect on the number of trade meetings; in particular, because the amount of money is no greater than $\frac{1}{2}$, the number of trade meetings, which is proportional to $(1-m_s)m_s$, is increasing in m_s. This is the extensive-margin effect of the current money supply. In contrast, the current money supply has only an indirect effect on what happens in trade meetings, the intensive margins. Because, by assumption, there is one unit of money in each trade meeting, any effect on what happens in trade meetings is due to the effect of the current money supply on expectations: potential producers think about how valuable any acquired money will be in the future, while potential consumers compare

[5] Note that the ex post stage here is taken to be the situation at the end of a meeting, rather than the subsequent stage at which the current money supply is publicly announced (which is the point at which ex post welfare comparisons would conventionally be made). If it is assumed that all information available to the participants in each meeting is revealed in that meeting, then there are additional restrictions on the set of mechanisms. We are indebted to Nobuhiro Kiyotaki and Narayana Kocherlakota for conversations that helped steer us away from such a more restrictive specification.

the current state and future states as alternative times to spend money. It follows that the serial correlation properties of Π are critical; in particular, there would be no intensive margin effects if the money supply process was i.i.d. However, serial correlation is not sufficient for intensive-margin effects. To see why, it is helpful to examine the optimum problem for the case of alternative constant money supplies.

Suppose the money supply m_s is known and constant. Then our class of mechanisms reduces to ones with a constant amount y produced in every trade meeting. With no differential information, the only constraints are individual rationality (ir) constraints, one for the producer and one for the consumer. Let V_k be the discounted expected utility at the start of a period associated with holding k units of money, where $k = 0, 1$. Then $V_0 = \beta V_0 + \frac{m_s}{N}(-y + \beta[V_1 - V_0])$ and $V_1 = \beta V_1 + \frac{1-m_s}{N}(u(y) - \beta[V_1 - V_0])$. These two equations are linear in the V_k and can be solved uniquely for them in terms of y. Let $\Delta(y)$ denote the implied solution for $V_1 - V_0$. The producer's ir constraint is $y \leq \beta\Delta(y)$, while the consumer's is $\beta\Delta(y) \leq u(y)$. Two facts are easily derived. First, satisfaction of the the producer's ir constraint implies satisfaction of the consumer's. (After all, the producer experiences current disutility for a probability of consuming the same amount in the future. Hence, $u(y)$ must exceed y by a sufficient margin. The consumer, though, is in the opposite situation, so that $u(y) \geq y$ is sufficient for the consumer's ir constraint.) Second, the producer's ir constraint is satisfied for all y that satisfy $y \leq y^s_{\max}$, where $y^s_{\max}$ is the unique positive solution to $y = \beta\Delta(y)$; namely, the unique positive solution to

$$R_s u(y) = y. \tag{3.1}$$

where R_s, the discount factor for a claim that will be redeemed at the next consumption opportunity, is given by $\frac{1}{R_s} = 1 + \frac{(1-\beta)N}{\beta(1-m_s)}$.

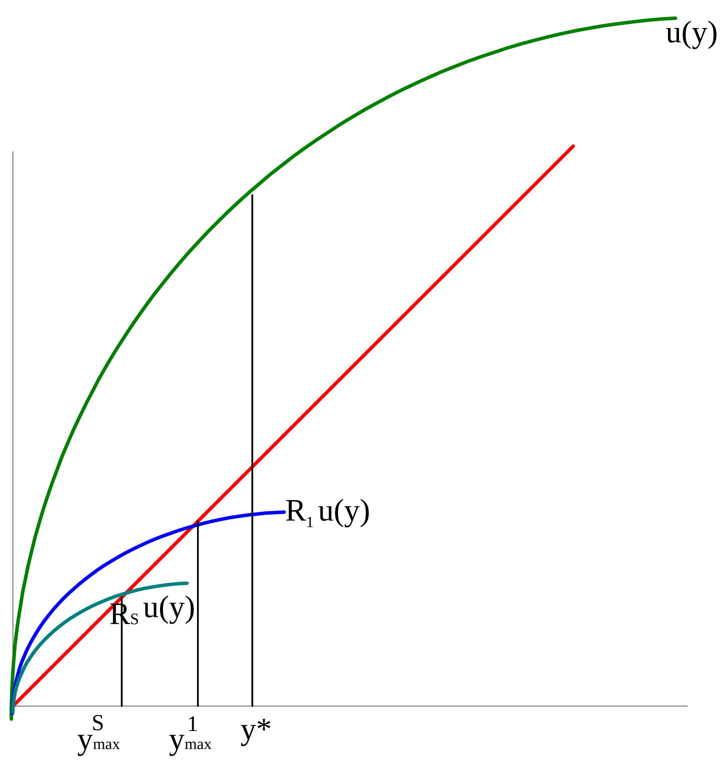

Figure 3.1: The Producer's IR Constraint

This situation is illustrated in Figure 3.1. Evidently, $y^s_{\max}$ is decreasing in m_s, N, and $\frac{1}{\beta}$, and $y^s_{\max}$ approaches 0 as either N or $\frac{1}{\beta}$ goes to infinity. Ex ante utility, our objective, is $(1 - m_s)V_0 + m_s V_1$, which is equal to $\frac{m_s(1-m_s)}{(1-\beta)N} z(y)$, where $z(y) \equiv u(y) - y$. Therefore, the optimum problem in this case is to choose y to maximize $z(y)$ subject to $y \leq y^s_{\max}$. Let y^* denote the unconstrained maximum of z; namely, the unique solution to $u'(y^*) = 1$. Then the solution to the optimum problem is $\min(y^s_{\max}, y^*)$. Notice that if $y^s_{\max} > y^*$, then the solution is y^* and does not depend on m_s. If, instead, $y^s_{\max} < y^*$, then the solution is $y^s_{\max}$. To avoid the uninteresting solution in which output in every trade meeting is y^* in the complete model, we assume throughout that $y^1_{\max} \leq y^*$ (a condition that can always be met by making β small or by making N large).[6] As we show below, this is sufficient to ensure that output in any meeting is less than y^*.

As we also show below, that upper bound on output implies that the objective for the complete model is increasing in outputs. Therefore, the bound tends to produce an optimum with binding producer *ir* constraints. For the case of a constant money supply, when that constraint is binding, it follows that the consumer *ir* constraint is slack. We impose parameter restrictions so that that slackness carries over to the complete model. That is achieved, as we show, by assuming that the range of the support for the money supply is not too large. [7]

It is under the above two assumptions ($y^1_{\max} \leq y^*$ and slack consumer *ir* constraints) that we characterize the optimum. The conjectures about the optimum in the general case come from thinking about the partial equilibrium problem in which current trades do not affect the future value of money. Because ex-ante welfare is increasing in output in every kind of meeting, if consumer *ir* constraints never bind and if there were no truth-telling constraints, then an optimum would satisfy producer *ir* constraints with equality (that is, all of the gains from trade would accrue to the consumer). However, there are truth-telling constraints for meetings in which one person is relatively informed. Because the amount of money that changes hands is always one unit, the amount produced in exchange for money in such meetings cannot depend on the current state. (If it did, then it would not be consistent with truth-telling by the informed person.) Therefore, for such meetings, the mechanism has only to describe the amount produced and a partition of the set of current states into trade and no-trade sets. Moreover, those sets have to be ordered: if the producer is informed, then the set of states in which trade occurs has to be composed of those in which acquiring a unit of money is most valuable; if the consumer is informed, then it has to be composed of those in which acquiring a unit of money is least valuable.

If the consumer is relatively informed and there is trade in all states, then the largest possible output is that consistent with the producer's participation constraint, given the information available to the producer (which includes the previous state and the observation that the consumer has money). Could it, instead, be worthwhile to have zero output in some states so as to have higher output in other states? Since the consumer is informed, truth-telling implies that the value of money would have to be higher where output is zero. But excluding such high-value states reduces the expected value of the money received by the producer, leading to lower output. Thus, it is undesirable to exclude states in this way.

If the producer is relatively informed and there is trade in all states, then the largest possible output is that consistent with the producer's participation constraint for the minimal value of acquiring

[6] The possibility that output is constant at y^* whenever trade occurs is an artifact of the unit upper bound on holdings. If money is divisible or even if individuals can hold many units of indivisible money and if expected discounted utility is strictly increasing in money holdings, then output equal to y^* would fail to satisfy producer *ir* constraints in meetings between producers with sufficiently high money holdings and consumers with sufficiently low money holdings.

[7] This assumption rules out an interesting possibility; namely, that a sufficiently large and rare decrease in the amount of money produces a large decline in output because in such states it is optimal for there to be no trade when informed consumers meet uninformed producers.

money. If, instead, output is zero in states where the value of money is low, then it is possible to increase output in the other states without violating the producer's participation constraint. Hence, it is desirable to set output to zero in the low-value states if they are unlikely enough given the previous state.

The above considerations led us to the following conjecture about the optimum for the complete model. It is optimal to have trade occur in every trade meeting except possibly in some of the meetings between informed producers and uninformed consumers. Moreover, in all meetings, there are binding producer individual-rationality constraints. We substantiate this claim in the next two sections.

4. Mechanisms and the optimum problem

The mechanism design problem is presented in three steps. First, we describe a set of deterministic Markov mechanisms. Second, we describe the subset of mechanisms that we call incentive-feasible. Lastly, we describe our selection from the set of incentive-feasible mechanisms.

Let $\mathbb{S} \equiv \{0, 1, 2, ..., S\}$ be the set of individual information states. For a generic element $s \in \mathbb{S}$, $s = 0$ means uninformed about the current state and $s > 0$ means informed and that the current amount of money is m_s. We also let $\mathbb{S}_+ \equiv \{1, 2, ..., S\}$. A Markov mechanism is a pair of functions that describes output and the transfer of money in trade meetings. That is, we let $y(s^c, s^p, i)$: $\mathbb{S} \times \mathbb{S} \times \mathbb{S}_+ \to R_+$ denote output in a trade meeting when the consumer announces information state s^c, the producer announces s^p, and the previous money-supply realization is i. We also let $a(s^c, s^p, i)$: $\mathbb{S} \times \mathbb{S} \times \mathbb{S}_+ \to \{0, 1\}$ denote the money transfer (0 means no transfer, 1 means transfer) in such a meeting. We record this notion of a mechanism as a definition.

Definition 1. *A mechanism is a pair* (y, a) *where* $y : \mathbb{S} \times \mathbb{S} \times \mathbb{S}_+ \to R_+$ *and* $a : \mathbb{S} \times \mathbb{S} \times \mathbb{S}_+ \to \{0, 1\}$.

In order to express the constraints on mechanisms implied by our specification of incentive-feasibility, it is helpful to have a notation for expected discounted utilities. We let $V_k(i)$ denote the expected utility of someone who has k units money just after the previous state i is announced (and before the determination of the new state and the new set of informed people). We let V_k denote the S-element vector with generic component $V_k(i)$. We also let $\Delta_i \equiv V_1(i) - V_0(i)$ and $\Delta \equiv V_1 - V_0$. We show in the Appendix that each mechanism (y, a) implies unique expected utilities $\{V_0, V_1\}$ and, hence, a unique Δ. Therefore, when convenient, we write $\Delta(y, a)$ to express the dependence of Δ on (y, a).

It is also helpful to let $G_k(\iota^c, \iota^p, s, i)$ be the gain, relative to not trading, in a trade meeting of someone with k units of money when the current realization is s and the previous realization is i, and when both parties announce truthfully. Here, $\iota^c \in \{0, 1\}$ indicates whether the consumer is informed ($\iota^c = 1$) or uninformed ($\iota^c = 0$), and $\iota^p \in \{0, 1\}$ indicates, in the same way, whether the producer is informed. Then,

$$\begin{aligned} G_0(\iota^c, \iota^p, s, i) &= -y(\iota^c s, \iota^p s, i) + a(\iota^c s, \iota^p s, i)\beta\Delta_s \\ G_1(\iota^c, \iota^p, s, i) &= u(y(\iota^c s, \iota^p s, i)) - a(\iota^c s, \iota^p s, i)\beta\Delta_s. \end{aligned} \tag{4.1}$$

Notice that

$$G_0(\iota^c, \iota^p, s, i) + G_1(\iota^c, \iota^p, s, i) = z(y(\iota^c s, \iota^p s, i)) \tag{4.2}$$

where $z(y) \equiv u(y) - y$.[8]

[8] The expression for G_0 follows from $G_0 = -y + a\beta V_1 + (1 - a)\beta V_0 - \beta V_0$ and similarly for G_1.

We can use these gain definitions to express the truth-telling and *ir* constraints. For trade meetings with symmetric information, the *ir* constraints are

$$G_0(1,1,s,i) \geq 0 \text{ and } \sum_{j=1}^{S} \pi_{ij} G_0(0,0,j,i) \geq 0. \tag{4.3}$$

$$G_1(1,1,s,i) \geq 0 \text{ and } \sum_{j=1}^{S} \pi_{ij} G_1(0,0,j,i) \geq 0. \tag{4.4}$$

where (4.3) refers to producers, and (4.4) refers to consumers. In each case, the first inequality pertains to an informed person who meets an informed person, while the second pertains to an uninformed person who meets an uninformed person. (Of course, for uninformed people, the condition is that there be *expected* gains from trade because the trade occurs before the participants know the current state.) When both are informed, truth-telling is always a best response (provided the mechanism proposes no-trade when it receives mutually inconsistent reports). Hence, truth-telling in meetings without differential information does not imply additional constraints.

We next consider truth-telling constraints in trade meetings between the informed and the uninformed. For all $(s, s') \in \mathbb{S}_+ \times \mathbb{S}_+$, we require

$$a(0,s,i)\beta\Delta_s - y(0,s,i) \geq a(0,s',i)\beta\Delta_s - y(0,s',i), \tag{4.5}$$

$$u(y(s,0,i)) - a(s,0,i)\beta\Delta_s \geq u(y(s',0,i)) - a(s',0,i)\beta\Delta_s. \tag{4.6}$$

where (4.5) refers to informed producers, and (4.6) refers to informed consumers. Thus, as noted above, if $a(0,s,i) = a(0,s',i)$, then the producer's constraint implies $y(0,s,i) = y(0,s',i)$; while if $a(s,0,i) = a(s',0,i)$, then the consumer's constraint implies $y(s,0,i) = y(s',0,i)$. Therefore, we let

$$S^p(i) = \{j \in \mathbb{S}_+ : a(0,j,i) = 1\} \text{ and } S^c(i) = \{j \in \mathbb{S}_+ : a(j,0,i) = 1\}, \tag{4.7}$$

and let $y(0,j,i) = Y^p(i)$ for $j \in S^p(i)$ and $y(j,0,i) = Y^c(i)$ for $j \in S^c(i)$. Then the truth-telling constraints are equivalent to

$$\max_{j \in \mathbb{S}_+ - S^p(i)} \{\beta\Delta_j\} \leq Y^p(i) \leq \min_{j \in S^p(i)} \{\beta\Delta_j\} \tag{4.8}$$

$$\max_{j \in S^c(i)} \{\beta\Delta_j\} \leq u(Y^c(i)) \leq \min_{j \in \mathbb{S}_+ - S^c(i)} \{\beta\Delta_j\}, \tag{4.9}$$

where we interpret (4.8) to be vacuous if $S^p(i)$ or $\mathbb{S}_+ - S^p(i)$ is empty and (4.9) to be vacuous if $S^c(i)$ or $\mathbb{S}_+ - S^c(i)$ is empty. These are the truth-telling constraints with regard to the partition of current states between the no-trade states and the trade states. They describe the sense in which the gain from acquiring money has to be ordered across trade and no-trade states.

Finally, we have *ir* constraints for participants in these meetings. For an informed producer, the *ir* constraint is the second inequality in (4.8), while for an uninformed producer, it is

$$Y^c(i) \leq \frac{\sum_{j \in S^c(i)} \pi_{ij}(\lambda - \theta_{ji})\beta\Delta_j}{\sum_{k \in S^c(i)} \pi_{ik}(\lambda - \theta_{ki})} \tag{4.10}$$

For an informed consumer, the *ir* constraint is the first inequality in (4.9), while for an uninformed consumer, it is

$$\frac{\sum_{j \in S^p(i)} \pi_{ij}\theta_{ji}\beta\Delta_j}{\sum_{k \in S^p(i)} \pi_{ik}\theta_{ki}} \leq u(Y^p(i)) \tag{4.11}$$

The probability weights in these expressions represent the beliefs of an uninformed person, given the information implied by truth-telling, which reveals the set of trade states, and the knowledge about the current state implied by observing the money holding of someone who could have gained or lost money.

We can now record the definition of an incentive-feasible mechanism.

Definition 2. *A mechanism (y, a) is incentive feasible if there exists (V_0, V_1) that satisfies (4.1)-(4.11).*

An immediate implication is that if (y, a) is incentive-feasible (in particular, satisfies the *ir* constraints), then V_0 and V are non-negative.

We now describe our selection from the above set of incentive-feasible mechanisms. Our selection maximizes ex ante utility, denoted Z, which is defined as

$$Z = \sum_{s=1}^{S} p_s[m_s V_1(s) + (1 - m_s)V_0(s)]. \tag{4.12}$$

Here p_i denotes the invariant probability, implied by Π, that the amount of money is m_i. This objective corresponds to starting up the economy by drawing the previous state from that invariant distribution and considering expected utility prior to such a draw and to initial assignments of money holdings. The following lemma expresses Z in terms of the parameters of the environment and the output levels in an arbitrary mechanism (see Katzman, Kennan and Wallace [1999] for a proof).

Lemma 1. *For any mechanism,*

$$\begin{aligned} Z(1-\beta)N &= (1-\lambda)^2 \sum_{i=1}^{S} p_i m_i (1-m_i) z[y(0,0,i)] + \sum_{i=1}^{S} p_i \sum_{s=1}^{S} \pi_{is}\theta_{si}(\lambda - \theta_{si}) z[y(s,s,i)] \\ & \qquad (4.13) \\ & \quad +(1-\lambda)\{\sum_{i=1}^{S} p_i m_i \sum_{s=1}^{S} \pi_{is}\theta_{si} z[y(0,s,i)] + \sum_{i=1}^{S} p_i (1-m_i) \sum_{s=1}^{S} \pi_{is}(\lambda - \theta_{si}) z[y(s,0,i)]\}. \end{aligned}$$

Notice that output levels, the components of y, appear in Z only by way of the function z and that a, the money transfer variable, does not appear. Thus Z is an average of the joint gains from trade in the various kinds of meetings, weighted by the relative frequencies of these meetings.

It is not difficult to show that an optimal mechanism exists. Recall that a mechanism (y, a) is a pair of functions with finite domain: in particular, y can be regarded as a finite dimensional vector. Because $z(y^*)$, where $u'(y^*) = 1$, is an upper bound on $z((y(s, s', i))$ and because the weights in (4.13) are non-negative, an upper bound on Z is obtained by having y^* produced and consumed in all trade meetings. Because one-quarter of all single-coincidence meetings, the maximum of the function $m(1-m)$ for $m \in [0, \frac{1}{2}]$, is an upper bound on the number of trade meetings, it follows that an upper bound on Z is $\frac{z(y^*)}{4N(1-\beta)} \equiv Z^*$.This implies a bound on the set of incentive-feasible outputs. Let $q = \min_s(p_s m_s)$. By (4.12), if (y, a) is incentive-feasible, then $V_1(s) \leq Z^*/q$ and $\Delta_s \leq Z^*/q$, by the non-negativity of the $V_k(s)$. By the *ir* constraints, it follows that each component of an incentive-feasible y is bounded above by $\beta Z^*/q$. Moreover, because the $V_k(s)$ and, therefore, Δ_s are continuous in y, and because the constraints are expressed as weak inequalities, it follows that the optimum problem amounts to maximizing a continuous function over a non empty compact set.[9] Therefore, a maximum exists.

5. A Partial Characterization of the Optimum

As noted above, we concentrate our study of optima in a region of the parameter space that is both interesting and tractable; namely, that for which $y^* \geq y^1_{\max}$ and $u(\beta \min_s \Delta_s) \geq y^1_{\max}$.[10] The following lemma, which uses the first assumption, shows that satisfaction of producer *ir* constraints implies that $y^1_{\max}$ is an upper bound on $\beta\Delta_i(y, a)$ and on output.

Lemma 2. *If (y, a) satisfies producer ir constraints, then $y(s, s', i) \leq \beta \max_i \Delta_i(y, a) \leq y^1_{\max}$.*

One consequence is that the objective Z is increasing in each component of an incentive-feasible y. Another is that our second assumption, that $u(\beta \min_s \Delta_s) \geq y^1_{\max}$, implies $u(\beta \min_s \Delta_s) \geq \beta \max_s \Delta_s$, which, in turn, implies that consumer *ir* constraints are not binding at an optimum.

Our main result is Proposition 1, which gives necessary conditions for an optimum.

Proposition 1. *If (y, a) is optimal and satisfies $u(\beta \min_s \Delta_s(y, a)) \geq y^1_{\max}$, then (i) $S^c(i) \equiv \mathbb{S}_+$,*

$$Y^c(i) = \frac{\sum\limits_{j \in \mathbb{S}_+} \pi_{ij}(\lambda - \theta_{ji})\beta\Delta_j(y, a)}{\sum\limits_{j \in \mathbb{S}_+} \pi_{ij}(\lambda - \theta_{ji})} \text{ for all } i \in \mathbb{S}_+, \tag{5.1}$$

$$a(0, 0, i) = 1 \text{ and } y(0, 0, i) = \sum_{j=1}^{S} \pi_{ij}\beta\Delta_j(y, a) \text{ for all } i \in \mathbb{S}_+, \tag{5.2}$$

$$a(j, j, i) = 1 \text{ and } y(j, j, i) = \beta\Delta_j(y, a) \text{ for all } (i, j) \in \mathbb{S}_+ \times \mathbb{S}_+, \tag{5.3}$$

and

$$Y^p(i) = \min_{j \in S^p(i)} \{\beta\Delta_j(y, a)\} \text{ for all } i \in \mathbb{S}_+; \tag{5.4}$$

(ii) for each i, $S^p(i)$ is not empty; and (iii) if $\Delta_j(y, a) = \Delta_k(y, a)$, then for each i either $j, k \in S^p(i)$ or $j, k \in \mathbb{S}_+ - S^p(i)$.

The rather lengthy proof proceeds by contradiction. We first suppose that (y, a) is optimal, but does not satisfy (i). Then we consider (y', a') given by (i), but with $\Delta(y, a)$ inserted on the right-hand sides of (5.1)-(5.4). Then incentive feasibility of y implies that $y' \geq y$. The main part of the proof involves showing that $\Delta(y', a') \geq \Delta(y, a)$. Given that inequality and lemma 2, it follows that (y', a') satisfies all the constraints except possibly truth-telling for producers. If it does, then we have a contradiction because $y^1_{\max} \geq y' \geq y$ and we have either increased output or replaced no-trade by trade, both of which increase Z. If not, then there is some previous state i and some current state r such that the producer would like to produce and acquire money; namely, $\beta\Delta_r(y', a') > Y^p(i)'$ $\geq \beta\Delta_r(y, a)$. We then consider (y', a''), where a'' differs from a' only in adding state r to $S^p(i)$, the set of trade states. We can show that $\Delta(y', a'') \geq \Delta(y, a)$. Repetition of this argument at most S

[9] Non-emptiness is implied by the fact that no trade satisfies all the constraints. Note that the Theorem of the Maximum does not apply because the constraint set is not lower hemicontinuous.

[10] Later, we present a sufficient condition in terms of parameters—essentially, a limitation on the range of the support for the money supply—for satisfaction of $u(\beta \min_s \Delta_s) \geq y^1_{\max}$, the assumption that implies that consumer *ir* constraints are slack.

times leads to trade in every state, which renders truth-telling vacuous. Parts (ii) and (iii) are proved using simple versions of the same argument.

The next lemma shows that for a given specification of the $S^p(i)$ sets there is a unique monetary mechanism (i.e., a mechanism with positive trade) that satisfies condition (i) of proposition 1.

Lemma 3. *Given $S^p(i)$ for each $i \in \mathbb{S}_+$, there exists a unique monetary mechanism that satisfies (5.1)-(5.4).*

The proof shows that a monotone and concave mapping whose fixed points coincide with solutions to (5.1)-(5.4) has a unique positive fixed point.

Proposition 1 and lemma 3 give us a simple procedure for finding the optimum for a given environment (that satisfies the hypotheses of proposition 1). For each specification of the $S^p(i)$ sets, obtain the unique monetary mechanism that satisfies (5.1)-(5.4). Discard those that are inconsistent with truth-telling for the informed producer. Among those that remain, the optimum is the one that gives the highest Z. Because $S^p(i) \equiv \mathbb{S}_+$ satisfies truth-telling, there is at least one such monetary mechanism which is incentive-feasible.

We now show that the maximized objective is continuous in Π provided that π_{ii} is bounded away from 0 for each i. We establish such continuity in two steps. First, we show that for given $S^p(i)$ sets, the unique monetary mechanism that satisfies (5.1)-(5.4) is continuous in Π (see Katzman, Kennan and Wallace [1999] for a proof).

Lemma 4. *Let $S^p(i)$ for $i \in \mathbb{S}_+$ be given. Let $\mathbb{P} = \{\Pi : \pi_{ii} \geq \alpha > 0$ for all $i\}$. Then the unique monetary mechanism that satisfies (5.1)-(5.4) is continuous in Π for $\Pi \in \mathbb{P}$.*

Now we give the main continuity result (see Katzman, Kennan and Wallace [1999] for a proof).

Proposition 2. *If the optimum satisfies $u(\beta \min_s \Delta_s) \geq y^1_{\max}$, then the maximized objective is continuous in Π for Π such that π_{ij} is positive and π_{ii} is bounded away from 0.*

While the proof uses lemma 4, it must do more than appeal to the fact that the maximum of continuous functions is continuous. The unique monetary mechanism that satisfies (5.1)-(5.4) for given $S^p(i)$ sets may satisfy truth-telling for informed producers for some $\Pi's$ and not others. We can show that if truth-telling holds at Π_0 but not in the neighborhood of Π_0, then condition (iii) in proposition 1 fails at Π_0. That, in turn, implies that the maximum cannot be at such a point. That and lemma 4 imply that the maximized objective is continuous.

There is a sense in which optima are uninteresting if they always satisfy $S^p(i) \equiv \mathbb{S}_+$. As our preview discussion suggests, that does not always happen. The following proposition says that if there is sufficient persistence, then the optimum has no trade whenever the money supply increases.

Proposition 3. *Let $\{\Pi_k\} \to I$ (the identity matrix) and be such that the corresponding sequence of invariant probability vectors $\{p_k\}$ converges to a strictly positive vector. Assume that the optimum for any Π in the neighborhood of $\Pi = I$ satisfies $u(\beta \min_s \Delta_s) \geq y^1_{\max}$. There exists K such that if $k > K$, then the optimum for Π_k has no-trade between informed producers and uninformed consumers whenever the current state, j, exceeds the previous state, i.*

Now that we have characterized the optimum, two comments are in order. The first is about implementing the optimum. The optimum we have described is easy to implement, and without using the fiction of a computer at each meeting that receives reports and is updated when the previous state is publicly announced. At each trade meeting and conditional on the knowledge that is common to the two people in the meeting, the optimum has a single positive trade proposal. Indeed, this is a feature of any incentive-feasible mechanism. Therefore, we can let the two people play a simultaneous move game in which the strategies are simply {trade, no-trade}. If both say trade, then the positive trade proposal is carried out. If either says no-trade, then there is no trade and each goes on to the next date. In part because the optimum has positive trade in every state when the consumer is informed and the producer is not, it is a dominant strategy for the consumer in every trade meeting to play trade. Given that play by the consumer, it is a best response for the producer to play the strategy that implements the optimum.

The other comment is about the relationship between the optimum and a mechanism which has consumers making take-it-or-leave-it offers. It turns out that the two may differ because an uninformed consumer facing an informed producer and making a take-it-or-leave-it offer tends to shut down trade in more states than the optimum dictates.[11] This is easy to see if, for the moment, we ignore the dependence of Δ on the $S^p(i)$ sets. Then, if $\Delta_j > \Delta_{j+1}$, the ex ante welfare gain from having positive output in one additional state, r, when the previous state is i is proportional to

$$z(\beta\Delta_r)\sum_{j=1}^{r}\pi_{ij}\theta_{ji} - z(\beta\Delta_{r-1})\sum_{j=1}^{r-1}\pi_{ij}\theta_{ji} \equiv B_p(r,i). \tag{5.5}$$

The difference here represents a trade-off between the output $\beta\Delta_{r-1}$ with some probability and the lower output $\beta\Delta_r$ with a higher probability. In contrast, the net gain to the consumer from demanding the smaller output $\beta\Delta_r$ rather than the output $\beta\Delta_{r-1}$ is proportional to

$$\sum_{j=1}^{r}\pi_{ij}\theta_{ji}[u(\beta\Delta_r) - \beta\Delta_j] - \sum_{j=1}^{r-1}\pi_{ij}\theta_{ji}[u(\beta\Delta_{r-1}) - \beta\Delta_j] = B_p(r,i) - \beta(\Delta_{r-1} - \Delta_r)\sum_{j=1}^{r-1}\pi_{ij}\theta_{ji}$$

Thus, the consumer sees an additional benefit to shutting down trade; by making the more demanding offer, the consumer avoids conceding an informational rent to the producer, a rent that is counted as part of ex ante welfare. Although this comparison is not correct because it ignores the dependence of Δ on the $S^p(i)$ sets, the comparison correctly suggests that if the optimum has always-trade by a small enough margin, then the optimum is not an equilibrium under take-it-or-leave-it offers by consumers (see Katzman, Kennan and Wallace [1999] for a proof).

Finally, we provide a sufficient condition for assuring that any optimum satisfies $u(\beta \min_s \Delta_s) \geq y^1_{\max}$, the hypothesis of propositions 1-3. The sufficient condition uses a lower bound on the optimal magnitude of Z, a bound we call $Z_{\min}$. As part of the proof that shows that a small enough range for the support of the money supply is sufficient to ensure $u(\beta \min_s \Delta_s) \geq y^1_{\max}$ (lemma 6 below), we provide one such $Z_{\min}$. More generally, $Z_{\min}$ could be obtained from any lemma 3 mechanism that is incentive feasible.

We begin by constructing an upper bound on ex ante utility implied by any allocation that violates $u(\beta \min_s \Delta_s) \geq y^1_{\max}$. The idea is that violating this inequality implies that at least one component of $\beta\Delta$ is less than $u^{-1}(y^1_{\max})$ and that this implies upper bounds on outputs and, hence, on ex ante utility. For each $k \in \mathbb{S}_+$, let $\Delta^k \in R^S$ be defined by $\Delta^k_j = y^1_{\max}/\beta$ for $j \neq k$ and $\beta\Delta^k_k = u^{-1}(y^1_{\max})$. Then let y^k be defined by

$$\begin{aligned} y^k(j,0,i) &= \frac{\sum_{s=1}^{S}\pi_{is}(\lambda-\theta_{si})\beta\Delta^k_s}{\sum_{s=1}^{S}\pi_{is}(\lambda-\theta_{si})}, \\ y^k(j,j,i) &= y^k(0,j,i) = \beta\Delta^k_j,\ y^k(0,0,i) = \beta\sum_{s=1}^{S}\pi_{is}\Delta^k_s, \end{aligned} \tag{5.6}$$

[11] The only reason the two may coincide is because we have imposed restrictions to keep output less than y^*. Someone who simply assumes such bargaining would not impose those restrictions.

for all $i, j \in \mathbb{S}_+$. The following lemma says that if the ex ante utility implied by y^k is less than that implied by some incentive-feasible mechanism, then the optimum satisfies $u(\beta \min_s \Delta_s) \geq y^1_{\max}$ (see Katzman, Kennan and Wallace [1999] for a proof).

Lemma 5. *Let $Z(y^k)$ be ex ante welfare implied by y^k as given by (5.6). If there exists $Z_{\min}$ such $Z_{\min}$ is the ex ante welfare implied by some incentive-feasible mechanism, with $Z(y^k) \leq Z_{\min}$ for each $k \in \mathbb{S}_+$, then any optimum satisfies $u(\beta \min_s \Delta_s) \geq y^1_{\max}$.*

Although lemma 5 is a crude result because the specification in (5.6) is a crude upper bound on an incentive feasible y given that $\beta\Delta_k < u^{-1}(y^1_{\max})$ for some k, the next result says that its hypothesis can always be met if the range of the support for the money supply is sufficiently small.

Lemma 6. *There exists $\varepsilon > 0$ such that if $m_S - m_1 < \varepsilon$, then the hypothesis of lemma 5 holds.*

To summarize, we can determine in two steps whether a given environment satisfies conditions sufficient to ensure that any optimum satisfies $y^1_{\max} \leq y^*$ and $u(\beta \min \Delta_i) \geq y^1_{\max}$. First, compute $y^1_{\max}$ and y^* and check whether $y^1_{\max} \leq y^*$. (Then lemma 2 implies that any optimum satisfies $y^1_{\max} \geq \beta \max \Delta_i$.) Second, attempt to find some $Z_{\min}$ (the optimum from lemma 3 and the construction in the proof of lemma 6 are possibilities) for which the hypothesis of lemma 5 holds.

6. Examples

We present two examples designed mainly to emphasize the consequences for aggregates of whether trade is shut down in some meetings between informed producers and uninformed consumers. As discussed in the introduction, we expect any such shutting down to affect the magnitude of the total output response to changes in the amount of money relative to what happens in the benchmark case of everyone informed ($\lambda = 1$).

The aggregates we study are total output and the price level. For each example, we present the optimal Markov process for each aggregate. Let Y_{ij} and P_{ij} denote total output and the price level, respectively, when the previous state is i and the current state is j. For a given mechanism, total output is the appropriate sum of outputs over meetings:

$$\begin{aligned} NY_{ij} &= m_i(1-m_i)(1-\lambda)^2 y(0,0,i) + (\lambda - \theta_{ji})\theta_{ji} y(j,j,i) + \\ &\quad (\lambda - \theta_{ji})(1-m_i)(1-\lambda) y(j,0,i) + m_i \theta_{ji}(1-\lambda) y(0,j,i). \end{aligned} \tag{6.1}$$

We take P_{ij} to be the total output deflator; namely, total nominal output, denoted X_{ij}, divided by Y_{ij}.[12] Because one unit of money is traded in each meeting in which trade occurs, for a given mechanism total nominal output is a weighted sum of the number of trades. That is, NX_{ij} is the right-hand side of (6.1) with $y(s, s', i)$ replaced by $a(s, s', i)$. The pair (Y_{ij}, P_{ij}) occurs with probability $p_i \pi_{ij}$, where, as above, p_i denotes the invariant probability that the state is i. Therefore, the pair (Y_{ij}, P_{ij}) and the exogenous probability $p_i \pi_{ij}$ for each (i, j) completely describe the Markov process for total output and the price level.

The two examples share all but one feature. Both have $S = 2$ and a symmetric Π with $\pi_{ii} = 1 - \varepsilon$. We let u be the square-root function—a simple function that satisfies our general assumptions. For that choice of u, $y^* = \frac{1}{4}$. We also let $N = 3$, the minimum consistent with no double-coincidence

[12] We must use a price index because, in general, the price of output depends on the kind of meeting. The price in a meeting is $1/y(s, s', i)$ for s and $s' \in \mathbb{S}$. Therefore, for given past and current states, the price varies with who in the meeting is informed. An outside observer collecting such prices must use an index to compute *a* price level.

meetings. For both examples, the support for the money supply is $(m_1, m_2) = (\frac{4}{9}, \frac{1}{2})$. We choose β so that $y^1_{\max} = y^*$, which gives $\beta = \frac{27}{32}$. The two examples differ regarding ε. In accord with proposition 3, we expect to have trade occur all the time if ε is not too close to zero and we expect to have no-trade occur when $(i, j) = (1, 2)$ if ε is small enough. Example 1 has $\varepsilon = 0.1$, which implies that trade always occurs. Example 2 has $\varepsilon = .0025$, which turns out to be small enough to get no-trade in meetings between informed producers and uninformed consumers when $(i, j) = (1, 2)$.[13] For each ε, we report results for three values of $\lambda : \lambda = \frac{1}{9}$, the minimum consistent with our assumptions; $\lambda = \frac{11}{20}$, the magnitude which maximizes the probability of meetings between informed producers and uninformed consumers; and $\lambda = 1$, the benchmark.

Before we present the results for aggregates, we present for one case the meeting-specific outputs from which the aggregates are deduced. When everyone is informed, there is only one kind of meeting per date and the output in that meeting depends only on the current state. In example 1 when $\lambda = 1$, $y(1, 1, i) = .2270$ and $y(2, 2, i) = .2160$. Because there is only one kind of meeting, the price level is simply the inverse of meeting-specific output. Notice that even in the low money supply state, output is less than $y^1_{\max} = \frac{1}{4}$. This happens for two reasons: the money supply may increase, which would reduce the probability of meeting someone without money, and the producer may be given money without producing. In all our examples, the persistence in the money supply process is sufficient to imply that $\Delta_1 > \Delta_2$; moreover, when $\lambda = 1$ the ordering for aggregate output is the same as that for meeting-specific output. This happens in part because our money supplies are near $\frac{1}{2}$, where the function $m(1 - m)$ is flat.

Table 2. Meeting specific outputs for example 1 with $\lambda = \frac{1}{9}$.

$y(j,j,i)$		$y(0,0,i)$		$y(j,0,i), j > 0$		$y(0,j,i), j > 0$
$j = 1$	$j = 2$	$i = 1$	$i = 2$	$i = 1$	$i = 2$	
.22559	.21541	.22458	.21642	.22365	.21541	.21541

Table 2 contains meeting-specific outputs for example 1 for $\lambda = \frac{1}{9}$. The first two columns contain output in meetings between informed people. These differ from those in the benchmark because λ affects Δ. The next two columns contain output in meetings between uninformed people. Here, there is no dependence on the current state and the dependence on the previous state is weaker than the dependence on the current state for meetings between informed people because the Markov process gives rise to less two-period persistence than one-period persistence. The next two columns contain output in meetings between informed consumers and uninformed producers. Here, output does not depend on the current state because of the truth-telling requirement. Notice also that for each i output is lower than when two uninformed people meet. This happens because uninformed producers who meet informed people with money place more weight than π_{12} on the possibility that the money supply has increased. (In fact, in this example, when the money supply decreases, no informed person has money, so when an uninformed producer meets an informed consumer with money, the producer is able to deduce that the current state is state 2 which implies the same output level as for an informed producer.) The last column contains output in meetings between uninformed consumers and informed producers. Here output depends on neither the current state nor the previous state and is the same as in a meeting between informed people when the current state is the high money supply state.

Complete results for examples 1 and 2 are given in Appendix B. Here we present in Figures 2 the effects of a "persistent" increase in the amount of money on output and the price level. That is, we present aggregates corresponding to a sequence of states $(1, 1, 2, 2)$ starting with the second

[13] The sufficient condition given in lemma 5 is satisfied for examples 1 and 2. The value of $Z(y^k)$, which is the same for each k by the symmetry of the example, never rises above .12. This is below the values of Z shown in tables 3 and 4 below, the values for the optima, which, as noted above, can play the role of $Z_{\min}$ in lemma 5.

term so that the label $t = 1$ in the figures is a date at which both the current and previous states are state 1. (While we could have presented the effects of an increase followed by a decrease, given the high degree of persistence in both examples, such an event is extremely rare.) As suggested above, impact effects on aggregate output depend on whether trade is ever shut down and on the probability of meetings between informed producers and uninformed consumers. For $\varepsilon = 0.1$, for which there is always-trade, there are positive impact effects on output relative to the benchmark. (When the money supply increases, meeting-specific output is higher in meetings between informed consumers and uninformed producers than it is between informed people when $\lambda = 1$.) For $\varepsilon = 0.0025$, for which there is no-trade when informed producers meet uninformed consumers and the money supply has increased, the direction of effect depends on λ. When $\lambda = \frac{11}{20}$, which maximizes the probability of meetings between informed producers and uninformed consumers, the increase produces a large decline in output relative to what happens in the benchmark. When $\lambda = \frac{1}{9}$, that does not happen because most informed people have money when the amount of money increases. [14]

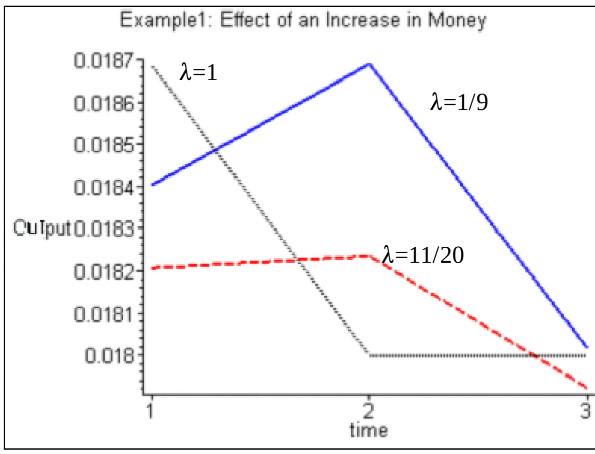

Figure 2a: Y_{11}, Y_{12}, Y_{22}; $\varepsilon = \frac{1}{10}$

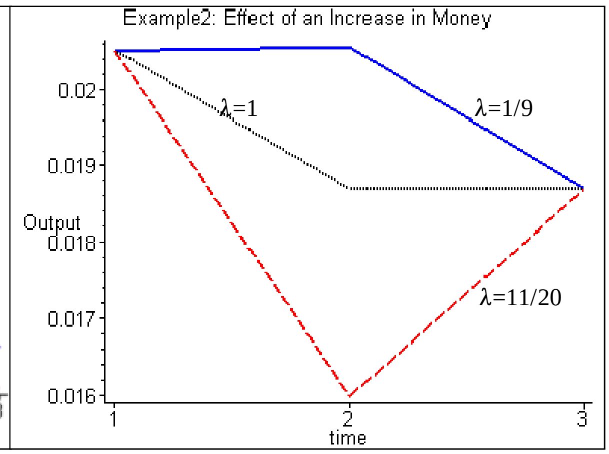

Figure 2b: Y_{11}, Y_{12}, Y_{22}; $\varepsilon = \frac{1}{400}$

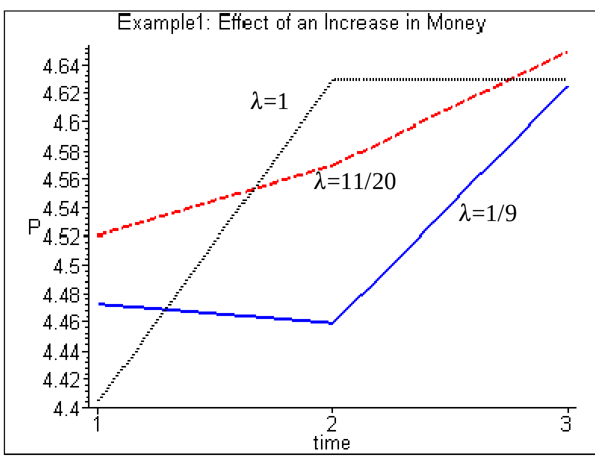

Figure 2c: P_{11}, P_{12}, P_{22}; $\varepsilon = \frac{1}{10}$

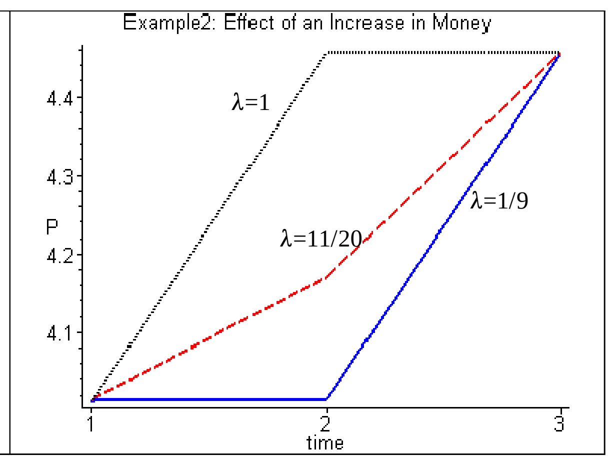

Figure 2d: P_{11}, P_{12}, P_{22}; $\varepsilon = \frac{1}{400}$

Recall that the price in each kind of meeting is simply the inverse of output in the meeting and that the only meeting-specific price that changes in example 1 when the money supply changes

[14] It is not difficult to construct an example which has non-monotone impact effects of increases in the amount of money on output: large and rare increases in the amount of money produce a shutting down of trade, but small and less rare increases do not. An example of this with $S = 3$ is given in Katzman, Kennan and Wallace (1999) .

is that in meetings between the informed. That effect tends to make the (aggregate) price level change in the same direction as the money supply. But, because it is the only price that changes, the impact effects on the price level tend to be weaker the smaller is λ. Moreover, that effect can even be offset by compositional effects, which happens in example 1 when $\lambda = \frac{1}{9}$. When $\lambda = \frac{1}{9}$ and the money supply increases, only $\frac{1}{18}$ of the informed are without money. Hence, there are very few trade-meetings among the informed. Also, there are fewer meetings between informed producers and uninformed consumers and more meetings between informed consumers and uninformed producers. Under always-trade, the meeting-specific price is higher when producers are relatively informed than when consumers are relatively informed. Hence, the altered composition produced by the monetary increase tends to lower the price index.

As Tables 3 and 4 show, in these examples the highest level of ex ante welfare, Z, is achieved when everyone is informed. However, ex ante welfare is not monotone in λ. The prevalence of meetings between informed producers and uninformed consumers, which depends on λ in a non-monotone way, is one source of the non-monotonicity of welfare. In example 1, when there is always-trade, output is minimal in such meetings because it satisfies the producer's individual rationality constraint for the smaller component of Δ no matter what is the current state. In example 2, this low output is partly overcome, but only at the cost of shutting down trade when the money supply increases.

7. Concluding Remarks

We have worked with a minimal model and a minimal class of mechanisms. In addition to considering randomization and other regions of the parameter space, it would be desirable to check the robustness of our findings to some of our more extreme assumptions. We have made substantial use of the indivisibility of money and the unit upper bound. Obviously, truth-telling constraints would not have such simple implications if the money trades could be other than surrendering 0 or 1 unit. Settings with richer individual holdings of money would be much harder to study because in them the distribution of money holdings depends on the mechanism. Another extension to consider is an information structure in which the identity of the informed is not common knowledge. Such a change gives rise to additional truth-telling constraints and to two-sided differential information in meetings. A third extension would consider lengthening the lag with which there is a public announcement of the state. Such a version would permit the study of whether it is optimal for relatively informed people to tell their trading partners what they know as they leave meetings.

In an important sense, though, the simplicity of our model is a virtue. Our background environment was designed by others to depict absence-of-double-coincidence problems so severe that the use of money is the only way to depart from autarky. We have put into that environment monetary uncertainty and differential information about realizations in a way that does not tie a person's information status to the person's money holdings. Then, despite assuming that money holdings are either 0 or 1 unit, that public revelation of realizations occurs after one period, that there is common knowledge about who is informed and that the society finds the best way of conducting trade in the presence of the uncertainty and the implied differential information, we find that the model delivers a rich theory of output and price level responses. Relative to the benchmark of everyone informed, there can be positive or negative impact effects on output and the price level tends to be sticky. In particular, the model suggests that data from different countries and time periods should be stratified according to the volatility and persistence of the money supply process before attempts are made to find stable relationships among the quantity of money, output, and the price level.

8. Appendix A: Proofs

To begin, we express the dependence of the V_k on the mechanism. It is convenient to do this in two steps. We let $v_k(\iota, s, i) : \{0,1\} \times \mathbb{S}_+ \times \mathbb{S}_+ \to R_+$ denote the expected discounted utility of someone who (i) holds $k \in \{0,1\}$ units of money, and (ii) is either informed ($\iota = 1$) or uninformed ($\iota = 0$) when the current state is s and the previous state is i—all of this prior to meetings but after determination of the set of newly informed people and after additions or subtractions of money. These continuation values satisfy

$$\begin{aligned} Nv_0(\iota, j, i) &= N\beta V_0(j) + (\lambda - \theta_{ji})\, G_0(1, \iota, j, i) + (1-\lambda)\, m_i G_0(0, \iota, j, i) \\ Nv_1(\iota, j, i) &= N\beta V_1(j) + \theta_{ji} G_1(\iota, 1, j, i) + (1-\lambda)\,(1-m_i)\, G_1(\iota, 0, j, i). \end{aligned} \tag{8.1}$$

In terms of them, the $V_k(i)$ are defined by

$$\begin{aligned} V_0(i) &= \sum_{j=1}^{S} \pi_{ij}[(1-\lambda)\, v_0(0,j,i) + \lambda v_0(1,j,i)] + \sum_{j=i}^{S} \pi_{ij}\eta_{ij}\,(v_{\ 1}(1,j,i) - v_0(1,j,i)) \\ V_1(i) &= \sum_{j=1}^{S} \pi_{ij}[(1-\lambda)\, v_1(0,j,i) + \lambda v_1(1,j,i)] - \sum_{j=1}^{i-1} \pi_{ij}\eta_{ij}\,(v_1(1,j,i) - v_0(1,j,i)), \end{aligned} \tag{8.2}$$

where

$$\eta_{ij} = \{ \begin{matrix} 1 - \frac{m_j}{m_i} \text{ if } j < i \\ \\ 1 - \frac{1-m_j}{1-m_i} \text{ if } j \geq i \end{matrix} . \tag{8.3}$$

It follows from the assumptions given immediately after Table 1 that $\eta_{ij} \leq \lambda$. Equations (8.2) with the $v_k(\iota, s, i)$ replaced by their expressions in (8.1) consist of $2S$ equations in V_0 and V_1 that are linear for a given (y, a). Moreover, with that substitution, for a given (y, a) the right sides of equations (8.2) can be viewed as a mapping from R^{2S} to R^{2S}. It is immediate that that mapping satisfies Blackwell's sufficient conditions for contraction: monotonicity and discounting. Therefore, equations (8.2) have a unique solution for V_0 and V_1 for a given (y, a). The implied unique $\Delta \equiv V_1 - V_0$ is denoted $\Delta(y, a)$.

We next establish some facts about Δ. The first result is about the weighted average of Δ that appears in the *ir* constraint for uninformed producers in meetings with informed consumers, (4.10). Define $\gamma_i^1(S^c(i) = \sum\limits_{j \in S^c(i)} \pi_{ij}(\lambda - \theta_{ji})$.

Lemma 7. *If (y, a) is incentive feasible, then $\Delta(y, a)$ satisfies,*

$$\frac{1}{\gamma_i^1(S^c(i))} \sum_{j \in S^c(i)} \pi_{ij}(\lambda - \theta_{ji})\beta\Delta_j \leq \frac{1}{\gamma_i^1(\mathbb{S}_+)} \sum_{j \in \mathbb{S}_+} \pi_{ij}(\lambda - \theta_{ji})\beta\Delta_j. \tag{8.4}$$

Proof. We have

$$\begin{aligned} D &\equiv \frac{\beta}{\gamma_i^1(\mathbb{S}_+)} \sum_{j \in \mathbb{S}_+} \pi_{ij}(\lambda - \theta_{ji})\Delta_j - \frac{\beta}{\gamma_i^1(S^c(i))} \sum_{j \in S^c(i)} \pi_{ij}(\lambda - \theta_{ji})\Delta_j \\ &= \frac{\beta(\sigma - \sigma')}{\sigma\sigma'} \sum_{j \in S^c(i)} \pi_{ij}(\lambda - \theta_{ji})\Delta_j + \frac{\beta}{\sigma'} \sum_{j \in \mathbf{S}_+ - S^c(i)} \pi_{ij}(\lambda - \theta_{ji})\Delta_j, \end{aligned} \tag{8.5}$$

where $\sigma \equiv \gamma_i^1(S^c(i)) \leq \sigma' \equiv \gamma_i^1(\mathbb{S}_+)$. Now, let $x = \min_{j \in \mathbb{S}_+ - S^c(i)} \Delta_j$. Then, by (4.9),

$$D \geq \frac{\beta(\sigma - \sigma')}{\sigma'} x + \frac{\beta}{\sigma'}(\sigma' - \sigma)x = 0, \tag{8.6}$$

as required. ■

The next lemma describes some properties of a mapping whose fixed point is $\Delta(y,a)$. Let

$$h(x;y,a) \equiv b(y) + \beta C(a)x \text{ for } x \in R^S. \tag{8.7}$$

where the $S \times S$ matrix $C(a) \equiv [c_{ij}(a)]$ and the $S \times 1$ vector $b(y) \equiv [b_i(y)]$ are defined as

$$\frac{c_{ij}(a)}{\pi_{ij}} = (1-\lambda)[1 - \frac{1-\lambda}{N}a(0,0,i) - \frac{\theta_{ji}}{N}a(0,j,i) - \frac{\lambda-\theta_{ji}}{N}a(j,0,i)] +$$

$$(\lambda - \eta_{ij})\{1 - \frac{\lambda}{N}a(j,j,i) - \frac{(1-\lambda)}{N}[(1-m_i)a(j,0,i) + m_i a(0,j,i)]\} \tag{8.8}$$

and

$$\begin{aligned} Nb_i(y) &= (1-\lambda)^2\left[(1-m_i)u(y(0,0,i)) + m_i y(0,0,i)\right] + \\ &\quad (1-\lambda)\sum_{j=1}^{S}\pi_{ij}\left[\theta_{ji}u(y(0,j,i)) + (\lambda-\theta_{ji})y(j,0,i)\right] + \\ &\quad \sum_{j=1}^{S}\pi_{ij}(\lambda-\eta_{ij})\left[\theta_{ji}u(y(j,j,i)) + (\lambda-\theta_{ji})y(j,j,i)\right] + \\ &\quad (1-\lambda)\sum_{j=1}^{S}\pi_{ij}(\lambda-\eta_{ij})[(1-m_i)u(y(j,0,i)) + m_i y(0,j,i)]. \end{aligned} \tag{8.9}$$

Lemma 8. *Let $h(x;y,a)$ be given by (8.7)-(8.9). Then (i) $\Delta(y,a)$ is a fixed point of $h(\cdot;y,a)$; (ii) $h(\cdot;y,a)$ satisfies Blackwell's sufficient conditions for contraction; (iii) if $h(\Delta(y,a);y',a') \geq \Delta(y,a)$, then $\Delta(y',a') \geq \Delta(y,a)$; and (iv) if $y' \geq y$, then $\Delta(y',a) \geq \Delta(y,a)$.*

Proof. From (8.2), we have

$$\Delta_i = \sum_{j=1}^{S}\pi_{ij}[(1-\lambda)\delta_i(0,j) + (\lambda-\eta_{ij})\delta_i(1,j)], \tag{8.10}$$

where $\delta_i(\iota,j) \equiv v_1(\iota,j,i) - v_0(\iota,j,i)$. Also, from (8.1), we have

$$\begin{aligned} N\delta_i(\iota,j) &= N\beta\Delta_j + \theta_{ji}G_1(\iota,1,j,i) + (1-\lambda)(1-m_i)G_1(\iota,0,j,i) \\ &\quad - (\lambda-\theta_{ji})G_0(1,\iota,j,i) - (1-\lambda)m_i G_0(0,\iota,j,i). \end{aligned} \tag{8.11}$$

Then, if we substitute from (8.11) into (8.10) and use the definition of $G_k(\iota^c,\iota^p,j,i)$, it follows that $\Delta(y,a)$ satisfies the matrix equation,

$$\Delta(y,a) = b(y) + \beta C(a)\Delta(y,a). \tag{8.12}$$

By the definition of $h(\cdot;y,a)$, it follows that $\Delta(y,a)$ is a fixed point of h. Therefore, (i) is true.

As regards (ii), it follows from (8.8) that $c_{ij}(a)/\pi_{ij} \in (1-\eta_{ij})[1-N^{-1},1]$, where the lower endpoint is attained when money is always transferred and the upper endpoint when money is never transferred. Because $\eta_{ij} < 1$, it follows that $c_{ij}/\pi_{ij} \in (0,1]$. That implies that $h(\cdot;y,a)$ is increasing and satisfies discounting, Blackwell's sufficient conditions for contraction.

Claim (iii) is an obvious consequence of contraction. That is, if $h^{(n)}(\Delta(y,a);y',a') \geq \Delta(y,a)$, then $h^{(n+1)}(\Delta(y,a);y',a') = h[h^{(n)}(\Delta(y,a);y',a');y',a'] \geq h(\Delta(y,a);y',a') \geq \Delta(y,a)$, where $h^{(n)}$ is the n-th iterate of h, where the first inequality follows from monotonicity of h and the second inequality is the hypothesis, which serves as the initial condition for the induction step. But by the contraction property, $\lim_{n\to\infty} h^{(n)}(x;y',a') = \Delta(y',a')$, which implies $\Delta(y',a') \geq \Delta(y,a)$.

From (8.9), $b(y)$ is increasing in y. That and claim (iii) imply claim (iv). ■

Lemma 2. If (y,a) satisfies producer *ir* constraints, then $y(s,s',i) \leq \beta \max_i \Delta_i(y,a) \leq y^1_{\max}$.

Proof. From equations (8.1) we have

$$\begin{aligned}
N\delta_i(0,j) &= N\beta\,\Delta_j + \theta_{ji} z(y(0,j,i)) + (1-\lambda)\,(1-m_i)\,z(y(0,0,i)) \\
&\quad + (1-\lambda)\,[y(0,0,i) - a(0,0,i)\beta\Delta_j] + \theta_{ji}\,[y(0,j,i) - a(0,j,i)\beta\Delta_j] \\
&\quad + (\lambda - \theta_{ji})\,[y(j,0,i) - a(j,0,i)\beta\Delta_j] \\
N\delta_i(1,j) &= N\beta\,\Delta_j + \theta_{ji} z(y(j,j,i)) + (1-\lambda)\,(1-m_i)\,z(y(j,0,i)) \\
&\quad + \lambda\,[y(j,j,i) - a(j,j,i)\beta\Delta_j] + (1-\lambda)m_i\,[y(0,j,i) - a(0,j,i)\beta\Delta_j] \\
&\quad + (1-\lambda)\,(1-m_i)\,[y(j,0,i) - a(j,0,i)\beta\Delta_j]
\end{aligned} \tag{8.13}$$

where $\Delta_j = \Delta_j(y,a)$. In each of these equations, the sum of the coefficients of $\beta\,\Delta_j$ is non-negative. Also, each coefficient of the function z is non-negative. Therefore, if we let $K = \max_i \Delta_i(y,a)$ and let $\zeta = \max_{s,s',i} z(y(s,s',i))$, we get the following inequalities:

$$\begin{aligned}
N\delta_i(0,j) &\leq N\beta\,K + (1-m_j)\,\zeta + \,(1-\lambda)\,[y(0,0,i) - a(0,0,i)\beta K] \\
&\quad \theta_{ji}\,[y(0,j,i) - a(0,j,i)\beta K] + (\lambda - \theta_{ji})\,[y(j,0,i) - a(j,0,i)\beta K] \\
N\delta_i(1,j) &\leq N\beta\,K + (1-m_j)\,\zeta + \,\lambda\,[y(j,j,i) - a(j,j,i)\beta K] + \\
&\quad (1-\lambda)m_i\,[y(0,j,i) - a(0,j,i)\beta K] + (1-\lambda)\,(1-m_i)\,[y(j,0,i) - a(j,0,i)\beta K]\,,
\end{aligned}$$

where we used the definition of θ_{ji} to replace $\theta_{ji} + (1-\lambda)\,(1-m_i)$ by $1-m_j$. The producer *ir* constraints imply $y(s,s',i) \leq a(s,s',i)\beta K$, for all s,s',i. Therefore, $N\delta_i(\iota,j) \leq N\beta\,K + (1-m_j)\,\zeta$ for $\iota = 0,1$. Now from (8.10), we have

$$\Delta_i \leq \sum_{j=1}^{S} \pi_{ij}(1-\eta_{ij})\left[\beta\,K + \frac{1}{N}\,(1-m_j)\,\zeta\right] \tag{8.14}$$

Then because this holds for all i, and $\sum_j \pi_{ij}(1-\eta_{ij}) \leq 1$, and $1-m_j \leq 1-m_1$,

$$K \leq \frac{\zeta(1-m_1)}{N(1-\beta)}. \tag{8.15}$$

By the definition of $y^1_{\max}$ and the assumption $y^* \geq y^1_{\max}$, we have

$$\frac{\beta z(y^*)(1-m_1)}{N(1-\beta)} \leq y^*. \tag{8.16}$$

Then since $\zeta \leq z(y^*)$ it follows that

$$\beta K \leq \frac{\beta z(y^*)(1-m_1)}{N(1-\beta)} \leq y^*. \tag{8.17}$$

Now because the function z is increasing on the interval $[0, y^*]$ and because $y(s, s', i) \leq \beta K$, it follows that $\zeta \leq z(\beta K)$. Thus (8.15) implies

$$\beta K \leq \frac{\beta z(\beta K)(1 - m_1)}{N(1 - \beta)}. \tag{8.18}$$

This, in turn, implies $\beta K \leq y^1_{\max}$. ■

Proposition 1. If (y, a) is optimal and satisfies $u(\beta \min_s \Delta_s(y, a)) \geq y^1_{\max}$, then (i) $S^c(i) \equiv \mathbb{S}_+$ and (y, a) satisfies (5.1)-(5.4); (ii) for each i, $S^p(i)$ is not empty; and (iii) if $\Delta_j(y, a) = \Delta_k(y, a)$, then for each i either $j, k \in S^p(i)$ or $j, k \in \mathbb{S}_+ - S^p(i)$.

Proof. Suppose that (y, a) is optimal, but does not satisfy (i). Then consider (y', a') given by (5.1)-(5.4), but with $\Delta(y, a)$ inserted on the right sides of (5.1)-(5.4) and with $S^c(i)$ replaced by $\mathbb{S}_+$.

Our first task is to show that $y' \geq y$. For all but $Y^c(i)'$, this follows immediately from the fact that each component of y satisfies the relevant producer *ir* constraint. For $Y^c(i)'$, we have,

$$Y^c(i) \leq \frac{1}{\gamma^1_i(S^c(i))} \sum_{j \in S^c(i)} \pi_{ij}(\lambda - \theta_{ji})\beta\Delta_j \leq \frac{1}{\gamma^1_i(\mathbb{S}_+)} \sum_{j \in \mathbb{S}_+} \pi_{ij}(\lambda - \theta_{ji})\beta\Delta_j = Y^c(i)', \tag{8.19}$$

where the first inequality is by incentive feasibility of y and where the second is lemma 7.

We next show that $\Delta(y', a') \geq \Delta(y, a)$. By the third part of lemma 8, it is enough to show that $h_i(\Delta(y, a); y', a') \geq \Delta_i(y, a) = h_i(\Delta(y, a); y, a)$, where h_i is the i-th component of the mapping defined in (8.7). Because h_i is a sum of terms, we can deal one-at-a-time with the replacements given by (5.1)-(5.4). Because $y' \geq y$ and $h(x; y, a)$ is increasing in y, we need only examine situations in which the replacement of (y, a) by (y', a') involves replacing a no-trade outcome by a trade outcome.

(5.1)An informed consumer meets an uninformed producer:

Considering only the (5.1) substitution, (8.12) and the definition of $Y^c(i)'$. imply that

$$\frac{N}{1-\lambda}[h_i(\Delta(y, a); y', a') - h_i(\Delta(y, a); y, a)] = (1 - m_i) \sum_{j \in S^c(i)} \pi_{ij}(\lambda - \eta_{ij})[u(Y^c(i)') - u(Y^c(i))]$$

$$+(1 - m_i) \sum_{j \in \mathbb{S}_+ - S^c(i)} \pi_{ij}(\lambda - \eta_{ij})[u(Y^c(i)') - \beta\Delta_j] - \sum_{j \in S^c(i)} \pi_{ij}(\lambda - \theta_{ij})[Y^c(i) - \beta\Delta_j].$$

Because (y, a) satisfies the uninformed producer's *ir* constraint, $\sum_{j \in S^c(i)} \pi_{ij}(\lambda - \theta_{ij})[Y^c(i) - \beta\Delta_j] \leq 0$. Therefore,

$$\frac{N}{(1-\lambda)(1-m_i)}[h_i(\Delta(y, a); y', a') - h_i(\Delta(y, a); y, a)] \geq$$

$$\sum_{j \in S^c(i)} \pi_{ij}(\lambda - \eta_{ij})[u(Y^c(i)') - u(Y^c(i))] + \sum_{j \in \mathbb{S}_+ - S^c(i)} \pi_{ij}(\lambda - \eta_{ij})[u(Y^c(i)') - \beta\Delta_j] \tag{8.20}$$

The first term is nonnegative by (8.19). The second term is nonnegative because $Y^c(i)' \geq \beta \min_s \Delta_s$ and $u(\beta \min_s \Delta_s) \geq y^1_{\max} \geq \beta \max_s \Delta_s$ by hypothesis and lemma 2.

(5.2) An uninformed consumer meets an uninformed producer:

Here, by (8.12), the replacement of no trade by trade given by (*5.2*) implies

$$N[h_i(\Delta(y, a); y', a') - h_i(\Delta(y, a); y, a)] =$$

$$(1 - \lambda)^2(1 - m_i) \sum_{j=1}^{S} \pi_{ij}[u(y(0, 0, i)') - \beta\Delta_j] \geq 0. \tag{8.21}$$

The equality follows from the definition of $y(0,0,i)'$ and the inequality is implied by $y(0,0,i)' \geq \beta \min_s \Delta_s$ and $u(\beta \min_s \Delta_s) \geq y^1_{\max} \geq \beta \max_s \Delta_s$.

(5.3) An informed consumer meets an informed producer

Here, by (8.12), the replacement of no trade by trade given by (*5.3*) implies

$$N[h_i(\Delta(y,a);y',a') - h_i(\Delta(y,a);y,a)] = \sum_j \pi_{ij}(\lambda - \eta_{ij})\theta_{ji}[u(\beta\Delta_j) - \beta\Delta_j] \geq 0, \tag{8.22}$$

where the summation is over some set of current states.

We now have $y' \geq y$ and $\Delta(y',a') \geq \Delta(y,a)$. It follows that (y',a') satisfies all the producer *ir* constraints. Moreover, by lemma 2, $\beta\Delta(y',a') \leq y^1_{\max}$. Therefore, $u(\beta \min_s \Delta_s(y',a')) \geq u(\beta \min_s \Delta_s(y,a)) \geq y^1_{\max} \geq \beta \max_s \Delta_s(y',a')$. This implies that (y',a') satisfies all the consumer *ir* constraints. It also satisfies truth-telling for the consumer because it has trade in every state when the consumer is informed. But (y',a') may or may not satisfy truth-telling for producers. If it does, then we have a contradiction because $y^1_{\max} \geq y' \geq y$ and we have either increased output or replaced no-trade by trade, both of which increase Z.

If not, because $\Delta(y',a') \geq \Delta(y,a)$, then there is some previous state i and some current state r such that $a(0,r,i) = 0$ and

$$\beta\Delta_r(y',a') > Y^p(i)' \geq \beta\Delta_r(y,a) \tag{8.23}$$

where the second inequality holds because (y,a) is incentive-feasible. That is, the informed producer would like to get money in additional states when responding to the incentives implied by $\Delta(y',a')$ as opposed to those implied by $\Delta(y,a)$. If so, then define a new mechanism (y'',a'') that agrees with (y',a') except that

$$a(0,r,i)'' = 1 \text{ and } y(0,r,i)'' = Y^p(i)'. \tag{8.24}$$

for all pairs r, i satisfying (8.23). This increases the number of states with trade when the producer has private information. By (8.12), the replacement of no-trade by trade in state r implies

$$\frac{N}{1-\lambda}[h_i(\Delta(y,a);y'',a'') - h_i(\Delta(y,a);y,a)] =$$

$$\pi_{ir}\theta_{ri}[u(Y^p(i)') - \beta\Delta_r(y,a)] + \pi_{ir}(\lambda - \eta_{ir})m_i[Y^p(i)' - \beta\Delta_r(y,a)] \geq 0. \tag{8.25}$$

Therefore, by the third part of lemma 8, $\Delta(y'',a'') \geq \Delta(y,a)$. Thus (by repeating the arguments given above for (y',a')), the mechanism (y'',a'') satisfies all the constraints except possibly the producer's truth-telling constraints; if these are also satisfied then (y'',a'') is incentive-feasible and has higher output and, therefore, higher welfare than (y,a), contradicting the assumption that (y,a) is optimal. Otherwise, we amend (y'',a'') by adding states in which trade occurs. Since the producer's truth-telling constraints are satisfied if $S^p(i) = \mathbf{S}_+$, and since the number of states is finite, repetition of these steps must lead to a mechanism that satisfies the producer's truth-telling constraints, with higher output and therefore higher welfare than (y,a), a contradiction. Thus, condition (i) is necessary for an optimum.

Now suppose (y,a) is optimal, but $S^p(i)$ is empty. Let k be such that $\Delta_k(y,a) = \max_j \Delta_j(y,a)$. Then consider (y',a') that agrees with (y,a) except that $a(0,k,i)' = 1$ and $y(0,k,i)' = \beta\Delta_k(y,a)$. Then, as above, it can be shown that $\Delta(y',a') \geq \Delta(y,a)$, and it follows that (y',a') gives higher welfare and satisfies all the constraints except perhaps the truth-telling constraint for producers. However, if it fails to satisfy that constraint, then there are two possibilities. One is that (8.23) might hold with respect to a previous state $i_0 \neq i$ and, therefore, such that $S^p(i_0)' = S^p(i_0)$. In that case we can apply the argument above which leads to a contradiction. The other is that

$$\beta\Delta_r(y', a') > y(0, k, i)' = \beta\Delta_k(y, a) \geq \beta\Delta_r(y, a) \tag{8.26}$$

where the last inequality holds because $\Delta_k(y, a) = \max_j \Delta_j(y, a)$. But this, too, is a version of (8.23). Hence, again, we can define a new mechanism (y'', a'') that agrees with (y', a') except that $a(0, r, i)'' = 1$ and $y(0, r, i)'' = \beta\Delta_k(y, a)$ for all states r satisfying (8.26). Thus, in this case also, we can argue to a contradiction.

Finally, suppose (y, a) is optimal but does not satisfy (iii). Then the optimum is such that $S^p(i)$ for some i partitions states in such a way that there are two current states, say j and k with $\Delta_j = \Delta_k$ and $j \in S^p(i)$ and $k \notin S^p(i)$. By truth-telling, it follows that $\Delta_j = \min_{l \in S^p(i)} \Delta_l$. Therefore, by necessary condition (i), (y, a) has $Y^p(i) = \beta\Delta_j$. Then amend (y, a) by making the omitted state a trade state without changing output. The rest of the argument is exactly like the argument for necessary condition (i), and, therefore, produces a contradiction. ■

Lemma 3. Given $S^p(i)$ for each $i \in \mathbf{S}_+$, there exists a unique monetary mechanism that satisfies (5.1)-(5.4).

Proof. We will use the following result (see [4]): Suppose $f = (f^1, f^2, \ldots, f^n)$ is a function from R^n to R^n such that (i) f is increasing; (ii) for each i, f^i is a strictly concave function from R^n to R; (iii) $f(0) \geq 0$; (iv) there is a positive vector x^a such that $f(x^a) > x^a$; (v) there is a vector $x^b > x^a$ such that $f(x^b) < x^b$. Then there exists a unique positive vector x such that $f(x) = x$. Moreover, $x \in (x^a, x^b)$.

The proof proceeds by defining a mapping that satisfies conditions (i)-(v) and each of whose fixed points is a monetary mechanism that satisfies (5.1)-(5.4). Let f from R^S to R^S be defined as follows. For $\Delta \in R^S$ let $y = g(\Delta)$ be the mapping defined by (5.1)-(5.4). Then define $f(\Delta) \equiv b(g(\Delta)) + \beta C(a)\Delta$, where the vector b and the matrix C are as defined in (8.12). It follows that if Δ is a fixed point of f, then $g(\Delta)$ and the associated money transfers is a mechanism that satisfies (5.1)-(5.4) and vice versa. Therefore, for existence and uniqueness, the claims in the lemma, it is enough to show that $f(\Delta)$ satisfies conditions (i)-(v).

The mapping g is increasing and b is also increasing, so f is increasing. Because the *min* function is concave, it follows that g is concave. And because b is concave in y, $b(g(\Delta))$, the composition of two increasing concave functions, is concave. Also, $f^i(\Delta)$ can be written in the form

$$f^i(\Delta) = \frac{1}{N}\sum_{j=1}^{S} \pi_{ij}(\lambda - \eta_{ij})\theta_{ji} u(y(j, j, i)) + \Psi(\Delta),$$

where Ψ is a nonnegative concave function, $\pi_{ij}(\lambda - \eta_{ij})\theta_{ji}$ is positive, and $y(j, j, i) = \frac{\Delta_j}{\beta}$. Because u is strictly concave, this implies that f^i is strictly concave.

Let Δ^a be the constant vector with $\Delta^a_j = \frac{\varepsilon}{\beta}, j \in \mathbf{S}_+$. Then

$$f^i(\Delta^a) \geq \frac{u(\varepsilon)}{N}\sum_{j=1}^{S} \pi_{ij}(\lambda - \eta_{ij})\theta_{ji} \geq \frac{u(\varepsilon)}{N}\pi_{ii}\lambda\theta_{ii} = \frac{u(\varepsilon)}{N}\pi_{ii}\lambda^2(1 - m_i). \tag{8.27}$$

Because $u'(0) = \infty$, the ratio $\frac{u(\varepsilon)}{\varepsilon}$ becomes arbitrarily large as $\varepsilon \to 0$. Choose ε so that

$$\frac{u(\varepsilon)}{\varepsilon} > \frac{N}{\beta\pi_{ii}\lambda^2(1 - m_i)}. \tag{8.28}$$

Then

$$f^i(\Delta^a) > \frac{\varepsilon}{\beta} = \Delta^a. \tag{8.29}$$

Thus Δ^a satisfies assumption (iv) of the fixed point theorem.

Let Δ^b be the constant vector with $\Delta^b_j = \frac{y^*}{\beta}, j \in \mathbf{S}_+$. (Recall that $u'(y^*) = 1$.) Then $f^i(\Delta^b) < \frac{y^*}{\beta}$ by essentially the argument in the proof of lemma 5. Therefore, Δ^b satisfies assumption (v) of the fixed point theorem. Finally, because the mapping f satisfies condition (iii) with equality, f has a unique positive fixed point $\hat{\Delta}$. ■

Proposition 3. Let $\{\Pi_k\} \to I$ (the identity matrix) and be such that the corresponding sequence of invariant probability vectors $\{p_k\}$ converges to a strictly positive vector. Assume that the optimum for any Π in the neighborhood of $\Pi = I$ satisfies $u(\beta \min_s \Delta_s) \geq y^1_{\max}$. There exists K such that if $k > K$, then the optimum for Π_k has no trade between informed producers and uninformed consumers whenever the current state, j, exceeds the previous state, i.

Proof. Let $\{\Pi_k\} \to I$ be given and let $\{p_k\} \to p'$. For $\Pi = I$, define the function, $Z(I)$, to be Z with $p = p'$. The maximum of $Z(I)$ over incentive-feasible (y, a) is $y(\iota^c i, \iota^p i, , i) = y^i_{\max}$. (When $\Pi = I$, the maximum is obtained state by state and for each state, the result is as described in Section 3.) Denote that maximum $Z^*(I)$.

Now let $S^p(i)' = \{1, 2, ..., i\}$ for each i, and let y^k and Δ^k denote the unique monetary mechanism that satisfies (5.1)-(5.4) of proposition 1 when $S^p(i) = S^p(i)'$ and $\Pi = \Pi_k$; also, let y^∞ and Δ^∞ denote y^k and Δ^k for $\Pi = I$. Then, $y^\infty(\iota^c i, \iota^p i, , i) = y^i_{\max}$ and $\Delta^\infty_i = y^i_{\max}/\beta$. (This is verified by inserting these expressions into (5.1)-(5.4) and (8.12) and noticing that they hold.) Then, by lemma 4, $\lim_{k\to\infty} y^k(\iota^c i, \iota^p i, , i) = y^i_{\max}$ and $\lim_{k\to\infty} \Delta^k_i = y^i_{\max}/\beta$. That, in turn, implies that if k is sufficiently large, then y'_k is incentive feasible. (This follows because the limit result implies that if k is large, then $\Delta^k_i > \Delta^k_{i+1}$.) Now let $Z^*(\Pi)$ denote the maximized objective. It follows that for any $\varepsilon > 0$, there exists K such that $k > K$ implies $Z^*(\Pi_k) > Z^*(I) - \varepsilon$. We now show that if Π_k is close enough to I, then a mechanism which has trade when the current state j and previous state i satisfy $j > i$ gives a value of the objective that is less than $Z^*(I) - \varepsilon$, and, therefore, cannot be optimal.

If (y, a) satisfies producer *ir* constraints, then

$$\lim_{\Pi \to I} \Delta_i(y, a; \Pi) \leq y^i_{\max}/\beta. \tag{8.30}$$

This follows by taking the limit of $\Delta_i(y, a)$ as given by (8.12) and noting two things: only terms of the form $\lim y(\iota^c i, \iota^p i, i)$ and $\lim[a(\iota^c i, \iota^p i, i)\beta\Delta_i]$ appear and the *ir* constraints imply $\lim y(\iota^c i, \iota^p i, i) \leq \lim[a(\iota^c i, \iota^p i, i)\beta\Delta_i]$. Now suppose by contradiction that the no-trade claim is false. Then there exists a subsequence of the given sequence $\{\Pi_k\}$, say $\{\Pi_n\}$, such that $\{\Pi_n\} \to I$ and for each n the optimum has trade in some current state j, where $j > i$, the previous state. Let $Z''(\Pi_n)$ denote the implied value of Z. The producer truth-telling constraint and (8.30) imply that for sufficiently large n, either $a(0, i, i)^n = 0$ or $y(0, i, i)^n \leq y^j_{\max} + \varepsilon_0$, where $j > i$ and where $\varepsilon_0 > 0$ but is otherwise arbitrary. (Recall that $y^j_{\max} < y^i_{\max}$ if $j > i$ and that $y^i_{\max}$ depends on m_i, but not on Π.) But (8.30) also implies that for sufficiently large n, $y(\iota^c i, \iota^p i, i)^n \leq y^i_{\max} + \varepsilon_i$, where, again, $\varepsilon_i > 0$ but is otherwise arbitrary. Because a component of y appears with a non-vanishing coefficient in $\lim_{n\to\infty} Z(\Pi_n)$ if and only if it has the form $y(\iota^c i, \iota^p i, i)$, it follows that $\limsup\{Z''(\Pi_k)\} < Z^*(I)$, a contradiction. ■

9. Appendix B: Numerical Results

Tables 3 and 4 contain detailed results for examples 1 and 2. In particular, they contain the Markov processes for total output and the price level. The rows labeled $\frac{X_{ii'} - X_{ii}}{X_{ii}}$ contain percentage changes of switching states relative to not switching states, except when they pertain to Δ. For Δ, the entry is $\frac{\Delta_2 - \Delta_1}{\Delta_1}$. Also, all the meeting-specific outputs can be computed from the reported Δ using the applicable conditional expectations as given in proposition 1.

Table 3. Example 1: $\varepsilon = \frac{1}{10}$

λ	Z	j	Y_{1j}	Y_{2j}	P_{1j}	P_{2j}	Δ
1	.132033	1	.186857	.186857	4.405	4.405	.2691
		2	.180001	.180001	4.630	4.630	.2560
		$\frac{X_{ii'}-X_{ii}}{X_{ii}}$	-3.67%	3.81%	5.11%	-4.86%	-4.87%
$\frac{11}{20}$	.131924	1	.182068	.1792335	4.521	4.592	.2654
		2	.182349	.1792350	4.570	4.649	.2545
		$\frac{X_{ii'}-X_{ii}}{X_{ii}}$	0.15%	-0.0008%	1.09%	-1.23%	-4.13%
$\frac{1}{9}$	.131988	1	.184025	.177959	4.4725	4.6249	.2674
		2	.186891	.180175	4.4589	4.6251	.2553
		$\frac{X_{ii'}-X_{ii}}{X_{ii}}$	1.56%	-1.23%	-0.30%	-0.004%	-4.52%

Table 4. Example 2: $\varepsilon = \frac{1}{400}$

λ	Z	j	Y_{1j}	Y_{2j}	P_{1j}	P_{2j}	Δ
1	.132325	1	.205136	.205136	4.012	4.012	.29540
		2	.186943	.186943	4.458	4.458	.26587
		$\frac{X_{ii'}-X_{ii}}{X_{ii}}$	-8.87%	9.73%	11.10%	-9.99%	-9.99%
$\frac{11}{20}$	.132291	1	.205003	.190627	4.015	4.318	.29524
		2	.159842	.186940	4.171	4.458	.26584
		$\frac{X_{ii'}-X_{ii}}{X_{ii}}$	-22.03%	1.97%	3.89%	-3.14%	-9.96%
$\frac{1}{9}$	.132324	1	.205071	.184667	4.013	4.45691	.29538
		2	.205543	.186975	4.015	4.45693	.26586
		$\frac{X_{ii'}-X_{ii}}{X_{ii}}$	0.23%	-1.23%	0.03%	-0.0003%	-9.99%

The rows labeled $\frac{X_{ii'}-X_{ii}}{X_{ii}}$ contain percentage changes of switching states relative to not switching states, except when they pertain to Δ. For Δ, the entry is $\frac{\Delta_2-\Delta_1}{\Delta_1}$. In addition to impact effects, the outcomes associated with any pattern of realizations for the money supply can be identified. For example, if $\varepsilon = 0.1$ and $\lambda = \frac{11}{20}$ and if the sequence of money realizations starting at date $t-1$ is $(m_1, m_1, m_2, m_2, m_1)$, then the sequence for the price level starting at date t is $(4.521, 4.570, 4.649, 4.592)$. Also, all the meeting-specific outputs can be computed from the reported Δ using the applicable conditional expectations as given in proposition 1.

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