An Elementary Fixed Point Theorem for Increasing Concave Functions on $\mathbb{R}^n$

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Given a nice function f mapping a nice domain to itself, the existence of a fixed point is given by Brouwer's theorem. Uniqueness of the fixed point is a desirable property in applications, but this requires much stronger conditions. The usual route to uniqueness is Banach's contraction mapping theorem, which requires that f shrinks distances by a factor strictly less than 1. To establish the contraction property, Blackwell's Monotonicity and Discounting conditions are frequently used. The purpose of this expository note is to show that if f is increasing, uniqueness can alternatively be established by showing that f is concave. Although these are strong conditions, they hold in some applications where f is not a contraction (see, for example, Katzman, Kennan and Wallace (1999)). Moreover, the proof of both existence and uniqueness is very simple. Although not well-known, the argument is not new: a more general version (for a concave operator on a cone in a Banach space) can be found in Krasnosel'skii (1964), and an economic application of this result can be found in Dutta, Mirman and Reffett (1999).

Theorem

Suppose $f = (f^1, f^2, \ldots, f^n)$ is a function from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that

(i) f is increasing

(ii) for each i, f^i is a strictly concave function from $\mathbb{R}^n$ to $\mathbb{R}$

(iii) $f(0) \geq 0$

(iv) there is a positive vector a such that $f(a) > a$

(v) there is a vector $b > a$ such that $f(b) < b$.

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Then there exists a unique positive vector x such that $f(x) = x$.

Proof: Let L be the rectangle $\{y \in \mathbb{R}^n : a \le y \le b\}$. Since $f(a) > a$ and $f(b) < b$ and f is increasing, it follows that $f(y) \in L$ for all $y \in L$. Define the function $g(y) \equiv f(y) - y$, and let $A = \{y \in L : g(y) \ge 0\}$. Then A is not empty because $a \in A$, and since $A \subseteq L$ and L is bounded above (by b), A has a least upper bound $u \in L$ (where $u^i = sup\{y^i : y \in A\}$). It will be shown that u is a fixed point of f.

If $y \in A$, then $y \le f(y)$ and $f(y) \le f(u)$ since f is increasing, so $y \le f(u)$. But then $f(u)$ is an upper bound for A, so $u \le f(u)$ because u is the least upper bound for A, and thus $u \in A$. Also, $f(u) \le f(f(u))$ by monotonicity, so $f(u) \in A$, and $f(u) \le u$. This proves that $f(u) = u$.

To show uniqueness, suppose $x > 0$ is any fixed point of f. Note that the strict concavity of f implies that g is strictly concave.

Suppose $y > 0$ and $g(y) \ge 0$. Let

$$\alpha = \min\left\{\frac{x^j}{y^j}, 1 \le j \le n\right\} = \frac{x^r}{y^r}. \tag{1}$$

Then $\alpha > 0$ because $x > 0$ and $y > 0$. If $\alpha \ge 1$ then $y \le x$. Otherwise let $w = \alpha y$, with $g(w) > 0$ because g is strictly concave, and $g(y) \ge 0$ and $g(0) \ge 0$. Then $w \le x$ and $w^r = x^r$, so $g^r(x) - g^r(w) = f^r(x) - f^r(w) \ge 0$ because f is increasing. But this implies $0 = g^r(x) \ge g^r(w) > 0$, a contradiction. Thus $y > 0$ and $g(y) \ge 0$ implies $y \le x$.

Now if $y > 0$ is a fixed point of f then, since $g(x) = 0$, the same argument with the roles of x and y reversed gives $x \le y$, so $y = x$. This completes the proof.

Example: Let $f^1(y_1, y_2) = f^2(y_1, y_2) = \sqrt{y_1} + \sqrt{y_2}$. This is evidently not a contraction, since the derivatives are unbounded as y approaches zero. The positive fixed point is $(4, 4)$, and the points a and b can be taken as $(1, 1)$ and $(9, 9)$. The example can easily be extended to n dimensions, and the square root function can be replaced by any other increasing concave function $v(y)$ such that $v(0) = 0, v'(0) = \infty$, and $v(b) < b/N$ for some positive number N.

Remarks:

1. The existence part of the above proof is lifted from Tarski (1955), and is included merely to make the proof self-contained.

2. The fixed point x can be computed by iterating the function f from any starting point x^1 in the rectangle L such that $f(x^1) \ge x^1$ (and

in particular, from the point a.) This gives a sequence x^n that is increasing, and bounded above by b, so x^n converges to $x^0 = sup(x^n)$. Since the concavity of f implies continuity, and since both sides of the equation $x^{n+1} = f(x^n)$ converge to x^0, it follows that x^0 is the fixed point of f. Similarly, the fixed point can be computed by iterating the function f from any starting point x^1 in the rectangle L such that $f(x^1) \leq x^1$ (and in particular, from the point b): in this case the sequence is decreasing, and bounded below by a.

3. Assumptions (iv) and (v) are used only to establish existence; given a positive fixed point, uniqueness requires only assumptions (i)-(iii). Moreover, it is clear from the proof that (i) can be weakened to require only that f^i is increasing in x_j, for $j \neq i$.

References

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