

Average Switching Costs in Dynamic Logit Models

John Kennan

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There is an extensive literature on discrete choice models, in which agents choose one of a finite set of alternatives. The empirical relationship between the characteristics of these alternatives and the associated choice probabilities can be used to estimate the underlying preferences of the agent, and the estimated preferences can be used to predict choices in situations not observed in the data.

In dynamic models, there is typically one choice that represents the status quo, and there are switching costs associated with all of the other choices. When switching is rare in the data, estimated switching costs tend to be implausibly large. But switching does occur, and in many cases there is no observable reason for a switch, so that the observed choice must be attributed to unobserved payoff shocks, including random variations in switching costs. The question then arises as to how large the actual switching costs are, conditional on a switch being made. When the unobservables are drawn from the type I extreme value distribution, this question has a simple answer.

Suppose there are J alternatives, with payoffs $\tilde{v}_j = v_j + \zeta_j$, where $\{\zeta_j\}$ is a set of iid extreme value random variables. If u is uniformly distributed on $[0,1]$, then $y = -\log(u)$ has the unit exponential distribution, and $\zeta = -\log(y)$ has the type I extreme value distribution. Thus if u_j is defined by setting $\zeta_j = -\log(y_j)$ and $y_j = -\log(u_j)$, then $\{u_j\}$ is a set of iid random variables that are uniformly distributed on $[0,1]$.

The following two results are well known (the proofs are given for completeness, since they are quite simple).

Lemma 1

$$\Pr\left(\tilde{v}_1 = \max_j \tilde{v}_j\right) = \frac{e^{v_1}}{\sum_{j=1}^J e^{v_j}}$$

Proof:

Let d_j be an indicator for the event that \tilde{v}_j is maximal. Then

$$\begin{aligned}\Pr\left(\zeta_j + v_j \leq \zeta_1 + v_1 \mid u_1\right) &= \Pr\left(y_j \geq \Delta_j y_1\right) \\ &= \Pr\left(u_j \leq (u_1)^{\Delta_j}\right) \\ &= (u_1)^{\Delta_j}\end{aligned}$$

where $\Delta_j = \exp(v_j - v_1)$. Thus

$$\begin{aligned} \Pr(d_1 = 1) &= \int \Pr(d_1 = 1 | u_1) du_1 \\ &= \int \left(\prod_{j=2}^J \Pr(\zeta_j + v_j \leq \zeta_1 + v_1 | u_1) \right) du_1 \\ &= \int_0^1 (u_1)^{A_1} du_1 \end{aligned}$$

where $A_1 = \sum_{j>1} \Delta_j$. This yields

$$\Pr(d_1 = 1) = \frac{1}{1 + A_1} = \frac{e^{v_1}}{\sum_{j=1}^J e^{v_j}}$$

Lemma 2

$$E\left(\max_j \tilde{v}_j\right) = \gamma + \log\left(\sum_{j=1}^J e^{v_j}\right)$$

where γ is the Euler constant.

Proof:

$$\begin{aligned} \frac{\partial}{\partial v_k} E\left(\max_j \tilde{v}_j\right) &= \Pr(d_k = 1) \\ &= \frac{e^{v_k}}{\sum_{j=1}^J e^{v_j}} \end{aligned}$$

Integrating this with respect to v_k gives

$$E\left(\max_j \tilde{v}_j\right) = c + \log\left(\sum_{j=1}^J e^{v_j}\right)$$

where c is constant with respect to v_k . In fact since the choice of k is arbitrary, c is in fact

constant with respect to v_j for all j . Letting $v_j \rightarrow -\infty$ for $j > 1$ yields $c = E\zeta_1 = \gamma$.

The next result is not so well known, and it is somewhat surprising. In general, once it is known which alternative was chosen, the expected value of the agent's problem is not the same as the ex ante expectation of the maximal value. But when the shocks are drawn from the extreme value distribution, these two expected values are equal.

Lemma 3

$$E(v_1 + \zeta_1 | d_1 = 1) = \gamma + \log\left(\sum_{j=1}^J e^{v_j}\right)$$

Proof:

Define the random variable X as

$$X = \max_{j>1} \left(u_j^{\frac{1}{A_j}}\right)$$

so that $d_1 = 1$ iff $X \leq u_1$ (almost surely). The distribution function of X is

$$F(x) = \prod_{j=2}^J \Pr(u_j \leq x^{A_j}) = x^{A_1}$$

So

$$\begin{aligned} \Pr(d_1 = 1) E(\zeta_1 | d_1 = 1) &= \int_0^1 \int_0^{u_1} -\log(-\log(u_1)) dF(x) du_1 \\ &= \int_0^1 -\log(-\log(u_1)) (u_1)^{A_1} du_1 \\ &= -\int_0^\infty \log(y_1) \exp(-A_1 y_1) \exp(-y_1) dy_1 \end{aligned}$$

Define $z = -\log((1+A_1)y_1)$. Then

$$\begin{aligned} -\int_0^\infty \log(y_1) \exp(-A_1 y_1) \exp(-y_1) dy_1 &= \frac{1}{1+A_1} \int_{-\infty}^\infty (z + \log(1+A_1)) e^{-z} e^{-e^{-z}} dz \\ &= \frac{\gamma + \log(1+A_1)}{1+A_1} \end{aligned}$$

because $\exp(-z)\exp(-\exp(-z))$ is the extreme value density function, and γ is the mean of the extreme value distribution. Thus

$$\frac{1}{1 + A_1} E(\zeta_1 | d_1 = 1) = \frac{\gamma + \log(1 + A_1)}{1 + A_1}$$

and

$$\begin{aligned} E(\zeta_1 | d_1 = 1) &= \gamma + \log(1 + A_1) \\ E(\zeta_1 + v_1 | d_1 = 1) &= \gamma + \log(e^{v_1} + e^{v_1} A_1) \\ &= \gamma + \log\left(\sum_{j=1}^J e^{v_j}\right) \end{aligned}$$

The point of this result can be illustrated as follows. Suppose there are two alternatives, with $v_1 = 3$ and $v_2 = 0$, and suppose the payoff shocks are drawn from a uniform two-point distribution, with support $\{-6, 6\}$. Then alternative 2 is chosen if and only if $(\zeta_1, \zeta_2) = (-6, 6)$, so that the realized payoff is $\tilde{v}_2 = 6$. If alternative 1 is chosen, the realized payoff is $\tilde{v}_1 = 9$ if $(\zeta_1, \zeta_2) \in \{(6, -6), (6, 6)\}$ and $\tilde{v}_1 = -3$ if $(\zeta_1, \zeta_2) = (-6, 6)$. Thus the average payoff given that alternative 1 is chosen is $(2/3)(9) + (1/3)(-3) = 5$.

Average Switching Costs

Define

$$\begin{aligned} \bar{v} &= \log\left(\sum_{j=1}^J e^{v_j}\right) \\ \rho_j &= e^{v_j - \bar{v}} \end{aligned}$$

Lemma 4

$$E(\tilde{v}_j | d_j = 1) - E(\tilde{v}_0 | d_0 = 0) = -\frac{\log(\rho_0)}{1 - \rho_0}$$

Proof:

The expected payoff given that j is chosen is

$$E(\tilde{v}_j | d_j = 1) = \gamma + \bar{v}$$

The unconditional expected payoff for the status quo is

$$E\tilde{v}_0 = \gamma + v_0$$

Thus

$$\begin{aligned} \gamma + v_0 &= \rho_0 E(\tilde{v}_0 | d_0 = 1) + (1 - \rho_0) E(\tilde{v}_0 | d_0 = 0) \\ &= \rho_0 (\gamma + \bar{v}) + (1 - \rho_0) E(\tilde{v}_0 | d_0 = 0) \\ E(\tilde{v}_0 | d_0 = 0) &= \gamma + \frac{v_0}{1 - \rho_0} - \frac{\rho_0}{1 - \rho_0} \bar{v} \end{aligned}$$

and

$$\begin{aligned} E(\tilde{v}_j | d_j = 1) - E(\tilde{v}_0 | d_0 = 0) &= \frac{\bar{v} - v_0}{1 - \rho_0} \\ &= -\frac{\log(\rho_0)}{1 - \rho_0} \end{aligned}$$

In a dynamic model with a state variable x , and switching costs $\Delta(x, j)$, the average switching cost net of the difference in payoff shocks can be written as

$$\Delta(x, j) - E(\zeta_j - \zeta_0 | d_j = 1) = \frac{\log(\rho_0(x))}{1 - \rho_0(x)} + \beta \sum_{x'} (p(x' | x, d_j = 1) - p(x' | x, d_0 = 1)) \bar{v}(x')$$

where p is the transition probability function.