Average Switching Costs in Dynamic Logit Models

John Kennan

January 2008

There is an extensive literature on discrete choice models, in which agents choose one of a finite set of alternatives. The empirical relationship between the characteristics of these alternatives and the associated choice probabilities can be used to estimate the underlying preferences of the agent, and the estimated preferences can be used to predict choices in situations not observed in the data.

In dynamic models, there is typically one choice that represents the status quo, and there are switching costs associated with all of the other choices. When switching is rare in the data, estimated switching costs tend to be implausibly large. But switching does occur, and in many cases there is no observable reason for a switch, so that the observed choice must be attributed to unobserved payoff shocks, including random variations in switching costs. The question then arises as to how large the actual switching costs are, conditional on a switch being made. When the unobservables are drawn from the type I extreme value distribution, this question has a simple answer.

Suppose there are J alternatives, with payoffs $\tilde{v}_j = v_j + \zeta_j$, where $\{\zeta_j\}$ is a set of iid extreme value random variables. If u is uniformly distributed on [0,1], then y = - log(u) has the unit exponential distribution, and ζ = - log(y) has the type I extreme value distribution. Thus if u_j is defined by setting $\zeta_j = -\log(y_j)$ and $y_j = -\log(u_j)$, then $\{u_j\}$ is a set of iid random variables that are uniformly distributed on [0,1].

The following two results are well known (the proofs are given for completeness, since they are quite simple).

Lemma 1

$$\Pr\left(\tilde{v}_1 = \max_j \tilde{v}_j\right) = \frac{e^{v_1}}{\sum_{j=1}^{J} e^{v_j}}$$

Proof:

Let d_j be an indicator for the event that $\tilde{v}_j$ is maximal. Then

$$\begin{aligned}\Pr\left(\zeta_j + v_j \le \zeta_1 + v_1 \mid u_1\right) &= \Pr\left(y_j \ge \Delta_j y_1\right) \\ &= \Pr\left(u_j \le (u_1)^{\Delta_j}\right) \\ &= (u_1)^{\Delta_j}\end{aligned}$$

where $\Delta_j = \exp(v_j - v_1)$. Thus

$$\Pr\left(d_1 = 1\right) = \int \Pr\left(d_1 = 1 \mid u_1\right) du_1$$

$$\int \left(\prod_{j=2}^{J} \Pr\left(\zeta_j + v_j \le \zeta_1 + v_1 \mid u_1\right) \right) du_1$$

$$= \int_0^1 \left(u_1\right)^{A_1} du_1$$

where $A_1 = \Sigma_{j>1} \Delta_j$. This yields

$$\Pr\left(d_1 = 1\right) = \frac{1}{1 + A_1} = \frac{e^{v_1}}{\sum_{j=1}^{J} e^{v_j}}$$

Lemma 2

$$E\left(\max_j \tilde{v}_j\right) = \gamma + \log\left(\sum_{j=1}^{J} e^{v_j}\right)$$

where γ is the Euler constant.

Proof:

$$\frac{\partial}{\partial v_k} E\left(\max_j \tilde{v}_j\right) = \Pr\left(d_k = 1\right)$$

$$= \frac{e^{v_k}}{\sum_{j=1}^{J} e^{v_j}}$$

Integrating this with respect to v_k gives

$$E\left(\max_j \tilde{v}_j\right) = c + \log\left(\sum_{j=1}^{J} e^{v_j}\right)$$

where c is constant with respect to v_k. In fact since the choice of k is arbitrary, c is in fact

constant with respect to v_j for all j. Letting $v_j \to -\infty$ for $j > 1$ yields $c = E\zeta_1 = \gamma$.

The next result is not so well known, and it is somewhat surprising. In general, once it is known which alternative was chosen, the expected value of the agent's problem is not the same as the ex ante expectation of the maximal value. But when the shocks are drawn from the extreme value distribution, these two expected values are equal.

Lemma 3

$$E\left(v_1 + \zeta_1 \mid d_1 = 1\right) = \gamma + \log\left(\sum_{j=1}^{J} e^{v_j}\right)$$

Proof:

Define the random variable X as

$$X = \max_{j>1}\left(u_j^{\frac{1}{\Delta_j}}\right)$$

so that $d_1 = 1$ iff $X \le u_1$ (almost surely). The distribution function of X is

$$F(x) = \prod_{j=2}^{J} \Pr\left(u_j \le x^{\Delta_j}\right) = x^{A_1}$$

So

$$\begin{aligned}
\Pr\left(d_1 = 1\right) E\left(\zeta_1 \mid d_1 = 1\right) &= \int_0^1 \int_0^{u_1} -\log\left(-\log\left(u_1\right)\right) dF(x) du_1 \\
&= \int_0^1 -\log\left(-\log\left(u_1\right)\right)\left(u_1\right)^{A_1} du_1 \\
&= -\int_0^{\infty} \log\left(y_1\right)\exp\left(-A_1 y_1\right)\exp\left(-y_1\right) dy_1
\end{aligned}$$

Define $z = -\log((1+A_1)y_1)$. Then

$$\begin{aligned}
-\int_0^{\infty} \log\left(y_1\right)\exp\left(-A_1 y_1\right)\exp\left(-y_1\right) dy_1 &= \frac{1}{1+A_1}\int_{-\infty}^{\infty}\left(z + \log(1+A_1)\right) e^{-z} e^{-e^{-z}} dz \\
&= \frac{\gamma + \log(1+A_1)}{1+A_1}
\end{aligned}$$

because exp(-z)exp(-exp(-z)) is the extreme value density function, and γ is the mean of the extreme value distribution. Thus

$$\frac{1}{1+A_1}E\left(\zeta_1 \mid d_1 = 1\right) = \frac{\gamma + \log(1+A_1)}{1+A_1}$$

and

$$\begin{aligned} E\left(\zeta_1 \mid d_1 = 1\right) &= \gamma + \log\left(1+A_1\right) \\ E\left(\zeta_1 + v_1 \mid d_1 = 1\right) &= \gamma + \log\left(e^{v_1} + e^{v_1}A_1\right) \\ &= \gamma + \log\left(\sum_{j=1}^{J} e^{v_j}\right) \end{aligned}$$

The point of this result can be illustrated as follows. Suppose there are two alternatives, with $v_1 = 3$ and $v_2 = 0$, and suppose the payoff shocks are drawn from a uniform two-point distribution, with support {-6,6}. Then alternative 2 is chosen if and only if $(\zeta_1, \zeta_2) = (-6,6)$, so that the realized payoff is $\tilde{v}_2 = 6$. If alternative 1 is chosen, the realized payoff is $\tilde{v}_1 = 9$ if $(\zeta_1, \zeta_2) \in \{(6,-6),(6,6)\}$ and $\tilde{v}_1 = -3$ if $(\zeta_1, \zeta_2) = (-6,6)$. Thus the average payoff given that alternative 1 is chosen is (⅔)(9)+(⅓)(-3) = 5.

(-6,-6)

Average Switching Costs

The next result shows that the expected gain from the optimal choice, relative to an arbitrary alternative that is not chosen, is a simple function of the probability of choosing the alternative.

Define

$$\bar{v} = \log\left(\sum_{j=1}^{J} e^{v_j}\right)$$

$$\rho_j = e^{v_j - \bar{v}}$$

Lemma 4

$$E\left(\tilde{v}_j \mid d_j = 1\right) - E\left(\tilde{v}_k \mid d_k = 0\right) = -\frac{\log\left(\rho_k\right)}{1-\rho_k}$$

Proof:

The expected payoff given that j is chosen is

$$E\left(\tilde{v}_j \mid d_j = 1\right) = \gamma + \bar{v}$$

The unconditional expected payoff for alternative k is

$$E\tilde{v}_k = \gamma + v_k$$

Thus

$$\begin{aligned}\gamma + v_k &= \rho_k E\left(\tilde{v}_k \mid d_k = 1\right) + \left(1-\rho_k\right) E\left(\tilde{v}_k \mid d_k = 0\right) \\ &= \rho_k\left(\gamma + \bar{v}\right) + \left(1-\rho_k\right) E\left(\tilde{v}_k \mid d_k = 0\right)\end{aligned}$$

$$E\left(\tilde{v}_k \mid d_k = 0\right) = \gamma + \frac{v_k}{1-\rho_k} - \frac{\rho_k}{1-\rho_k}\bar{v}$$

and

$$\begin{aligned}E\left(\tilde{v}_j \mid d_j = 1\right) - E\left(\tilde{v}_k \mid d_k = 0\right) &= \frac{\bar{v} - v_k}{1-\rho_k} \\ &= -\frac{\log\left(\rho_k\right)}{1-\rho_k}\end{aligned}$$

which proves the result.

This result implies that the expected gain relative to alternative k is decreasing in ρ_k. But the expected gain is never less than 1, reaching this in the limit as the probability of choosing k approaches 1.

In a dynamic model with a state variable x, and switching costs Δ(x, j), the average switching cost net of the difference in payoff shocks can be written as

$$\Delta(x,j) - E\left(\zeta_j - \zeta_0 \mid d_j = 1\right) = \frac{\log\left(\rho_0\left(x\right)\right)}{1-\rho_0\left(x\right)} + \beta\sum_{x'}\left(p(x' \mid x, d_j = 1) - p(x' \mid x, d_0 = 1)\right)\bar{v}(x')$$

where the zero subscript denotes the status quo, and p is the transition probability function.