UNIQUENESS OF POSITIVE FIXED POINTS FOR INCREASING CONCAVE FUNCTIONS ON $\mathbb{R}^n$: AN ELEMENTARY RESULT

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ABSTRACT. The square root function has a unique positive fixed point. This function has the following properties: it is strictly increasing and strictly concave, with $f(0) = 0$, $f(a) > a$ for $a < 1$, and $f(b) < b$ for $b > 1$. It is shown that any function from $\mathbb{R}^n$ to $\mathbb{R}^n$ satisfying these properties has a unique positive fixed point.

1. INTRODUCTION

In dynamic models, stationary equilibrium is typically described as a solution of the equation $x = f(x)$, where f is a mapping that determines the current state as a function of the previous state, or as a function of the expected future state. In many cases x is a finite-dimensional vector, and one is interested only in positive solutions, i.e., positive fixed points of f . This note shows that if f is increasing and concave, it has a unique positive fixed point. The proof is very simple.

Although not well-known, the argument is not new: a version can be found in Krasnosel'skii (1964), and economic applications of this result can be found in Coleman (1991) and Dutta, Mirman and Reffett (1999). But Krasnosel'skii's result (for a concave operator on a cone in a Banach space) is more difficult than the result given here, and it also imposes restrictions that are not needed in the finite-dimensional case.

In static models of general equilibrium in a competitive economy, a well-known result gives uniqueness of the equilibrium price vector when the system of excess demand functions satisfies a "gross substitutes" condition, which is weaker than monotonicity. The argument relies on a special property of demand functions, namely zero-degree homogeneity. It will be shown that essentially the same argument works more generally for concave functions in situations where the homogeneity property does not apply.

2. DEFINITIONS

Let f be a function from $\mathbb{R}^n$ to $\mathbb{R}^n$, and let g be the function $g(x) \equiv f(x) - x$. Then a fixed point of f is a solution of the equation $g(x) = 0$, and vice versa. Concavity of f means concavity of each component, considered as a function from

Date: December, 2000.

Key words and phrases. Fixed points, concavity.

I thank Rabah Amir, Bart Lipman, Peter Norman, Kevin Reffett, Larry Samuelson and Bill Sandholm for useful comments.

$\mathbb{R}^n$ to $\mathbb{R}$. Under this definition, concavity of f and g are equivalent properties. But quasiconcavity of f and g are not equivalent, and quasiconcavity of f is not sufficient for the uniqueness result, while quasiconcavity of g is more than sufficient, provided that $g(0) \geq 0$. Since quasiconcavity means convexity of upper contour sets, $g(x) = 0$ and $g(0) \geq 0$ implies that g is nonnegative everywhere on the line from x to 0. The following definition formalizes this property.

Definition 1. *The function $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is radially quasiconcave (“R-concave”) if $g(x) = 0$ and $x > 0$ and $0 \leq \lambda \leq 1$ implies $g(\lambda x) \geq 0$. If (in addition) $0 < \lambda < 1$ implies $g(\lambda x) > 0$, then g is strictly R-concave.*

Note that if f has a positive fixed point, R-concavity implies $f(0) \geq 0$. This property is essential for the uniqueness result: for example, the (one-dimensional) function $f(x) = 4\sqrt{x} - 3$ has fixed points at $x = 1$ and at $x = 9$.

In addition to concavity, a monotonicity assumption is needed. For example, the gross-substitutes assumption says that the excess demand $g(p)$ for any good is an increasing function of the prices of other goods. An equilibrium is a solution of the equation $g(p) = 0$, or a fixed point of the function $f(p) \equiv g(p) + p$. In general, $f_i(p)$ is not increasing in p_i . This motivates

Definition 2. *The function $g = (g_1, g_2, \dots, g_n) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is quasi-increasing if $y_i = x_i$ and $y_j \geq x_j$ for all j implies $g_i(y) \geq g_i(x)$.*

Note that g is quasi-increasing if and only if f is quasi-increasing.

3. THE UNIQUENESS RESULT

Theorem 1. *If g is a quasi-increasing and strictly R-concave function from $\mathbb{R}^n$ to $\mathbb{R}^n$ there is at most one positive vector x such that $g(x) = 0$.*

Proof. Suppose $g(x) = g(y) = 0$ with $x > 0$ and $y > 0$. Let

$$(3.1) \quad \gamma = \min \left\{ \frac{x_j}{y_j}, 1 \leq j \leq n \right\} = \frac{x_r}{y_r}.$$

Then $\gamma > 0$ because $x > 0$ and $y > 0$. If $\gamma \geq 1$ then $y \leq x$. Otherwise let $w = \gamma y$, with $g(w) > 0$ by strict R-concavity. Then $w \leq x$ and $w_r = x_r$, so $g_r(x) \geq g_r(w)$ because g is quasi-increasing. But this implies $0 = g_r(x) \geq g_r(w) > 0$, a contradiction. Thus $y \leq x$. The same argument with the roles of x and y reversed gives $x \leq y$, so $y = x$. $\square$

Corollary 1. *If f is a quasi-increasing function from $\mathbb{R}^n$ to $\mathbb{R}^n$, and if $g(x) \equiv f(x) - x$ is strictly R-concave, there is at most one positive vector x such that $f(x) = x$.*

This result does not imply that a fixed point of f exists. But if f is increasing, existence is implied by Tarski’s Theorem, subject to some additional conditions ensuring that f maps some compact rectangle to itself.¹

¹Note that Tarski’s Theorem does not require any concavity assumption.

Theorem 2 (Tarski). *Suppose f is an increasing function from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that $f(a) > a$ for some positive vector a , and $f(b) < b$ for some vector $b > a$. Then f has a positive fixed point.*

Proof. Let L be the rectangle $\{x \in \mathbb{R}^n : a \leq x \leq b\}$. Since $f(a) > a$ and $f(b) < b$ and f is increasing, it follows that $f(x) \in L$ for all $x \in L$. Let $A = \{x \in L : f(x) \geq x\}$. Then A is not empty because $a \in A$, and since $A \subseteq L$ and L is bounded above, A has a least upper bound $u \in L$ (where $u^i = \sup\{x^i : x \in A\}$). If $x \in A$, then $x \leq f(x)$ and $f(x) \leq f(u)$ since f is increasing, so $x \leq f(u)$. But then $u = \sup A \leq f(u)$, so $u \in A$. Also, $f(u) \leq f(f(u))$ by monotonicity, so $f(u) \in A$, and $f(u) \leq u$. Thus $f(u) = u$. $\square$

The following result (which is an immediate implication of Theorem 1 and Tarski's Theorem) gives simple sufficient conditions for existence and uniqueness of a positive fixed point.

Theorem 3. *Suppose f is an increasing and strictly concave function from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that $f(0) \geq 0$, $f(a) > a$ for some positive vector a , and $f(b) < b$ for some vector $b > a$. Then f has a unique positive fixed point.*

4. EXAMPLES

- (1) Katzman, Kennan and Wallace (2000) derive an equation of the form $x = f(x)$ for the state-contingent value of money in a random matching model, where $x \in \mathbb{R}^n$ is a nonnegative vector representing the value of money in each of n states. The function f is increasing and strictly concave, with $f(0) = 0$, $f(a) > a$ for some positive vector a , and $f(b) < b$ for some vector $b > a$. Thus there is a unique positive solution for the value of money.
- (2) Let $f_1(x_1, x_2) = f_2(x_1, x_2) = \sqrt{x_1} + \sqrt{x_2}$. The positive fixed point is $(4, 4)$, and the points a and b can be taken as $(1, 1)$ and $(9, 9)$. Note that f is not a contraction (since the derivatives are unbounded as x approaches zero), so the Banach contraction mapping theorem cannot be used to show uniqueness. The example can easily be extended to n dimensions, and the square root function can be replaced by any other increasing concave function $v(x)$ such that $v(0) = 0$, $v'(0) = \infty$, and $v(b) < b/N$ for some positive number N .
- (3) The system of differential equations $\dot{x} = g(x)$ is called *cooperative* if g is quasi-increasing. Smith (1995) discusses a cooperative system of the form $\dot{x}_1 = \varepsilon(x_2 - x_1) + r_1 x_1 \left(1 - \frac{x_1}{K_1}\right)$, $\dot{x}_2 = \varepsilon(x_1 - x_2) + r_2 x_2 \left(1 - \frac{x_2}{K_2}\right)$, with the interpretation that x_1 and x_2 are sub-populations of a single species, living on two different "patches", with migration between the patches at the rate ε , and logistic population growth within each patch; $K_i > 0$ is the "carrying capacity" of patch i , and r_i is the reproduction rate when the population is small. Theorem 1 implies that if this system has a positive equilibrium, it is unique. For example, if $K_1 = K_2 = K$, then $x = (K, K)$ is the unique positive equilibrium.

- (4) The (one-dimensional) functions $f^A(x) = \log(1+x)$ and $f^B(x) = x + \sqrt{x}$ are strictly increasing and strictly concave, but they do not have positive fixed points. Tarski's Theorem does not apply because there is no point a such that $f^A(a) > a$, and there is no point b such that $f^B(b) < b$.
- (5) Let $f_1(x_1, x_2) = \sqrt{x_1} + \log(1 + x_2)$, $f_2(x_1, x_2) = \sqrt{x_2} + 2\log(1 + x_1)$. Then f is increasing and strictly concave, so it has a unique positive fixed point (at $x_1 = 3.826748532$, $x_2 = 5.491806484$).
- (6) Let $f_1(x_1, x_2) = x_1(\frac{1}{2} - x_1) + x_2$, $f_2(x_1, x_2) = x_2(\frac{1}{2} - x_2) + x_1$. Then although f is not increasing everywhere, it is increasing on the set $B = \{(z, z) : 0 \leq z \leq \frac{3}{4}\}$, with $f(a) > a$ for $a = (\frac{1}{4}, \frac{1}{4})$ and $f(b) < b$ for $b = (\frac{3}{4}, \frac{3}{4})$. Thus f has a positive fixed point in B , and this is the unique positive fixed point of f because $g(x) = f(x) - x$ is strictly R-concave and quasi-increasing.

5. REMARKS

- (1) It is clear from the proof of Theorem 1 that the strict version of R-concavity can be replaced by the weak version, if the function is strictly quasi-increasing. Note that if f is the identity function, then g is R-concave but not strictly R-concave, and quasi-increasing but not strictly quasi-increasing; of course the uniqueness result fails spectacularly for this function.
- (2) Suppose $g(p)$ is a system of excess demand functions that satisfies the gross substitutes property: the demand for good i is strictly increasing in p_j , for $j \neq i$. Then the function g is strictly quasi-increasing. Moreover, g is homogeneous of degree zero, so it is R-concave (although not strictly R-concave). Thus if there is an equilibrium price vector, it is unique.
- (3) Under the hypotheses of Theorems 1 and 2, with the additional assumption that f is continuous (which is implied by concavity), the positive fixed point can be computed by iterating f from any starting point x^1 in A (and in particular, from the point a). This gives a sequence x^n that is increasing, and bounded above by b , so x^n converges to $x^0 = \sup(x^n)$. Since both sides of the equation $x^{n+1} = f(x^n)$ converge to x^0 , it follows (using continuity) that x^0 is the positive fixed point of f . Similarly, the positive fixed point can be computed by iterating from any starting point y^1 in the rectangle L such that $f(y^1) \leq y^1$ (and in particular, from the point b): in this case the sequence is decreasing, and bounded below by a .
- (4) Clearly, if g is a quasi-decreasing and strictly R-convex function (meaning that $-g$ is quasi-increasing and strictly R-concave) Theorem 1 implies that f has at most one positive fixed point. But since there is no analog of Tarski's Theorem for decreasing functions, existence of a fixed point is a more complicated problem in this case. Moreover, although it might seem from the one-dimensional case (e.g. $f(x) = x^2$) that Theorem 1 could be modified to yield a uniqueness result for increasing and convex functions, the plot below illustrates that the logic of Theorem 1 does not work in the

convex case.² The plot shows the curves $g_1(x) = 0, g_2(x) = 0$ for a pair of quadratic functions. Quasiconvexity of g implies $g(x) < 0$ along a ray toward one of the fixed points, but since the function is quasi-increasing, this does not give rise to the contradiction in the proof of Theorem 1.

FIGURE 1

- (5) In order to extend Theorem 1 to the infinite-dimensional case, it is necessary to add a condition ensuring that γ is strictly positive, where $\gamma = \sup \{t : ty \leq x\}$. Krasnosel'skiĭ (1964) assumes there is some point u_0 such that for all nonzero x , there are positive numbers α and β (depending on x) such that $\alpha u_0 \leq f(x) \leq \beta u_0$. Then if x and y are both fixed points of f , it is easy to show that γ must be positive. Let X be the set of infinite sequences of nonnegative real numbers, and let $f : X \rightarrow X$ be the function defined by $f_i(x) = \sqrt{x_i}$. Then f does not satisfy Krasnosel'skiĭ's condition. To see this, let $y_i = \frac{1}{2^i}$, $z_i = \frac{1}{4^i}$, and $v_i = \frac{1}{16^i}$. Then $y = f(z)$ and $z = f(v)$, so if the condition holds, there must be a point u_0 with $y \leq \beta u_0$ and $\alpha u_0 \leq z$. Then $ty \leq z$, where $t = \frac{\alpha}{\beta} > 0$. This means $t \leq \frac{z_i}{y_i}$ for all i , which gives $t \leq \frac{1}{2^i}$, for all i , and this is impossible if t is positive. Thus Krasnosel'skiĭ's condition is very restrictive; indeed the square root function can be regarded as the prototypical example of Theorem 1, and it is easy to show directly that there is a unique positive fixed point at $x_i \equiv 1$ in this case.

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²See Krasnosel'skiĭ and Zabreĭko (1984) for some results on uniqueness of fixed points of convex operators.

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