**Economics 711**

**1997 Midterm Answer Key**

**PART A**

**Q1 (i)** In this question we have a Marshallian demand function with arguments \( C(p, m) = C(p, w, wL) \). We can determine this function from the solution to

\[
\max_{\{C,L\}} \min\{C, L\} \quad \text{s.t.} \quad pC + wL = w\bar{L}.
\]

Clearly, in this problem we have \( u = C = L \). Substituting this into the budget constraint gives, \( pC + wC = w\bar{L} \), and rearranging yields

\[
C(p, w, w\bar{L}) = \frac{w\bar{L}}{p + w}.
\]

(ii) Substituting the Marshallian demand into the utility function yields indirect utility,

\[
v(p, w, w\bar{L}) = \min \left\{ \frac{w\bar{L}}{p + w}, \frac{w\bar{L}}{p + w} \right\} = \frac{w\bar{L}}{p + w}.
\]

(iii) We know from the "basic four" that, in general, \( v(p, m) = v(p, e(p, u)) = u \). So in this case we have \( v(p, w, e(p, w, u)) = u \), or

\[
\frac{e(p, w, u)}{p + w} = u.
\]

Rearranging, we derive the expenditure function,

\[
e(p, w, u) = u(p + w).
\]

(iv) The definition of the money metric indirect utility function is \( \mu(p; q, m) = e(p, v(q, m)) \). In this case we have

\[
\mu((p_2, w_2); (p_1, w_1), w\bar{L}) = e((p_2, w_2), v(p_1, w_1, w_1\bar{L})) = v(p_1, w_1, w_1\bar{L})(p_2 + w_2) = \frac{w_1\bar{L}}{p_1 + w_2}(p_2 + w_2)
\]

**Q2 (i)** A cost function, \( C(y, w) \), derived from some convex monotonic technology has the following four properties:
(1) hdl in \( \mathbf{w} \).

(2) Nondecreasing in \( \mathbf{w} \), nondecreasing in \( y \).

(3) Concave in \( \mathbf{w} \).

(4) Continuous in \( \mathbf{w} \).

Now (1) says \( C(y, tw) = tC(y, \mathbf{w}) \), so we have

\[
C(y, t(w_1, w_2)) = y[aw_1 + bw_2 + c(tw_3)^{5/8}(tw_4)^{1/4}] = t[y(aw_1 + bw_2 + t^{-1/8}cw_3^{5/8}w_4^{1/4})].
\]

We clearly require \( c = 0 \). So we now have \( C(y, w_1, w_2) = y(aw_1 + bw_2) \). Checking (2) yields

\[
\frac{\partial C(y, w)}{\partial w_1} = ya \geq 0, \quad \frac{\partial C(y, w)}{\partial w_2} = yb \geq 0,
\]

which implies \( a \geq 0, \ b \geq 0 \). Now note that we have a function that is linear in \( w_1 \) and \( w_2 \), so it is both concave and continuous, satisfying (3) and (4) above.

(ii) We have \( C(y, w_1, w_2) = y(\alpha w_1 + \beta w_2) \), so from basic duality theory we know that the underlying technology is Leontief, of the form

\[
f(x_1, x_2) = \min \left\{ \frac{x_1}{\alpha}, \frac{x_2}{\beta} \right\}.
\]

Arguing backwards, and with an example, let \( \alpha = 1 \) and \( \beta = 2 \). We know that \( x_1 = x_2/2 = y \) at any cost minimizing combination of factors. So to produce one unit of \( y \) we need one unit of \( x_1 \) and two units of \( x_2 \). The cost of one unit \( y \) is therefore \( w_1 + 2w_2 \). So we get a linear cost function, \( C(y, w_1, w_2) = y(w_1 + 2w_2) \). It can be seen can see that this argument follows for any \( \alpha \) and \( \beta \).

Q3 (i) The consumer will only accept \( F \) if her expected utility is higher than if she is not insured. That is, \( F \) must satisfy the following inequality,

\[
\log(W - F) \geq p \log(W - h) + (1 - p) \log(W + h).
\]
Since the logarithmic function is monotonically increasing, the largest such $F$ will be the one satisfying the following equality,

$$\log(W - F) = p\log(W - h) + (1 - p)\log(W + h).$$

(ii) From above we know $F$ satisfies

$$\log(W - F) = p\log(W - h_0W) + (1 - p)\log(W + h_0W)$$

$$= p\log(W(1 - h_0)) + (1 - p)\log(W(1 + h_0))$$

$$= p\log(W) + (1 - p)\log(W) + p\log(1 - h_0) + (1 - p)\log(1 + h_0)$$

$$= \log(W) + p\log(1 - h_0) + (1 - p)\log(1 + h_0)$$

Applying the exponential function to both sides and rearranging yields

$$\frac{F}{W} = 1 - (1 - h_0)^p(1 + h_0)^{(1-p)}.$$

Note that $F$ is not exactly “independent” of $W$. What is true is that $F/W$ is independent of $W$. The second part of the question asks for the effect of increasing $h_0$ on $F/W$. Taking the derivative of the above equation, we have

$$\frac{\partial}{\partial h_0} \left(\frac{F}{W}\right) = p(1 - h_0)^{(p-1)}(1 + h_0)^{(1-p)} - (1 - p)(1 - h_0)^p(1 + h_0)^{-p}.$$

This derivative is positive if

$$p(1 - h_0)^{(p-1)}(1 + h_0)^{(1-p)} - (1 - p)(1 - h_0)^p(1 + h_0)^{-p} \geq 0.$$

Rearranging, we have

$$\frac{p}{1 - p} \left(\frac{(1 - h_0)^{p-1}}{(1 - h_0)^p}\right) \left(\frac{(1 + h_0)^{1-p}}{(1 + h_0)^{-p}}\right) \geq 0, \quad \text{or} \quad \frac{p}{1 - p} \geq \frac{1 - h_0}{1 + h_0}.$$

So whether $F/W$ varies positively with $h_0$ depends on the value of $p$. For example, if $p = 1/2$, then $F/W$ increases with $h_0$. But for a small enough probability of loss we have $F/W$ decreasing with $h_0$. 

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PART B

Q4 (a) The consumer’s problem is

$$\max_{q} pu(W - L - \pi q + q) + (1 - p) u(W - \pi q).$$

The FOC is

$$p(1 - \pi)u'(W - L - \pi q + q) - (1 - p)\pi u'(W - \pi q) = 0.$$ \hfill (1)

Rearranging, we have

$$\left(\frac{1 - p}{p}\right) \left(\frac{\pi}{1 - \pi}\right) = \frac{u'(W - L - \pi q + q)}{u'(W - \pi q)}.$$ \hfill (1)

Since $\pi = p$, the LHS of (1) is equal to 1. Therefore $u'(W - L - \pi q + q) = u'(W - \pi q)$. We know $u''(\cdot) < 0$, so $u'(\cdot)$ is a strictly decreasing function, and therefore one-to-one. So $W - L - \pi q + q = W - \pi q$, implying $q^* = L$. This, of course, is just the Full Insurance Principle.

(b) We can rearrange the LHS of (1) as $\left(\frac{\pi}{p}\right) \left(\frac{1 - p}{1 - \pi}\right)$. Since $\pi > p$, we have $\frac{\pi}{p} > 1$ and $\frac{1 - p}{1 - \pi} > 1$. This implies the LHS of (1) is greater than 1, so $u'(W - L - \pi q + q) > u'(W - \pi q)$.

Using the fact that $u'(\cdot)$ is a decreasing function, we have

$$W - L - \pi q + q < W - \pi q,$$

implying $q < L$.

So $q^{**} < L = q^*$. This makes sense – when insurance premiums are above actuarially fair rates, risk averse consumers will less than fully insure against potential losses.

Q5 (a) (Note that the proof below that $a(p)$ must be constant is a little subtle. By assuming indirect utility satisfies hdo in $(p, m)$, it is shown that indirect utility cannot satisfy nonincreasing in $p$ unless $a(p)$ is constant. It is not sufficient to simply state that hdo fails – you must show that hdo and nonincreasing in $p$ cannot simultaneously hold.)
In order for $v(p, m) = a(p)$ to satisfy $\text{hd}0$ in $(p, m)$, we must have $a(\lambda p) = a(p)$ for any $\lambda > 0$. Differentiating both sides of this equation wrt $\lambda$ and evaluating at $\lambda = 1$ gives

$$
\sum_{i=1}^{k} \frac{\partial a(\lambda p)}{\partial (\lambda p_i)} \frac{\partial (\lambda p_i)}{\partial \lambda} = \sum_{i=1}^{k} \frac{\partial a(p)}{\partial p_i} p_i = \sum_{i=1}^{k} \frac{\partial a(p)}{\partial \lambda} = 0.
$$

Since all prices are positive, if $\frac{\partial a(p)}{\partial p_i} < 0$ for any good $i$, to maintain the above equality there must be at least one good $j$ where $\frac{\partial a(p)}{\partial p_j} > 0$. But this would violate another of the indirect utility function’s properties: nonincreasing in $p$ implies $\frac{\partial a(p)}{\partial p_i} \leq 0$ for all $i$. So the only possibility is that $\frac{\partial a(p)}{\partial p_i} = 0$ for all $i$. That is, $a(p)$ is a constant function.

(b) We have $v(p_1, p_2, m) = p_1^\alpha p_2^\beta m$. Since $v(p_1, p_2, m)$ is nonincreasing in $(p_1, p_2)$, we have

$$
\frac{\partial (p_1^\alpha p_2^\beta m)}{\partial p_1} = \alpha p_1^{\alpha-1} p_2^\beta m \leq 0, \quad \text{and} \quad \frac{\partial (p_1^\alpha p_2^\beta m)}{\partial p_2} = \beta p_1^\alpha p_2^{\beta-1} m \leq 0.
$$

This implies $\alpha \leq 0$, $\beta \leq 0$. Now, $v(p_1, p_2, m)$ is $\text{hd}0$ in $(p_1, p_2, m)$. So

$$
(\lambda p_1)^\alpha (\lambda p_2)^\beta (\lambda m) = \lambda^{\alpha+\beta+1} p_1^\alpha p_2^\beta m = p_1^\alpha p_2^\beta m,
$$

implying that $\lambda^{\alpha+\beta+1} = 1$, or $\alpha + \beta = -1$.

(c) We can find the direct utility function by solving $u(x_1, x_2) = \min_{\{p_1, p_2\}} v(p_1, p_2, m)$ such that $p_1x_1 + p_2x_2 = m$. In Lagrangian form this problem is

$$
\min_{\{p_1, p_2, \lambda\}} \mathcal{L} = m p_1^{-1/3} p_2^{-2/3} + \lambda (p_1 x_1 + p_2 x_2 - m).
$$

The FOCs are

$$
\frac{\partial \mathcal{L}}{\partial p_1} : \frac{1}{3} mp_1^{-4/3} p_2^{-2/3} = \lambda x_1, \quad \frac{\partial \mathcal{L}}{\partial p_2} : \frac{2}{3} mp_1^{-1/3} p_2^{-5/3} = \lambda x_2.
$$

Rearranging gives

$$
\frac{1}{3} mp_1^{-1/3} p_2^{-2/3} = \lambda x_1 p_1, \quad \frac{2}{3} mp_1^{-1/3} p_2^{-2/3} = \lambda x_2 p_2.
$$

Adding together and using the budget constraint yields

$$
mp_1^{-1/3} p_2^{-2/3} = \lambda (p_1 x_1 + p_2 x_2) = \lambda m,
$$

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so \( \lambda = p_1^{-1/3}p_2^{-2/3} \). Substituting this back into the FOCs gives

\[
\frac{1}{3}m p_1^{-1/3} p_2^{-2/3} = p_1^{-1/3} p_2^{-2/3} x_1 p_1, \quad \frac{2}{3}m p_1^{-1/3} p_2^{-2/3} = p_1^{-1/3} p_2^{-2/3} x_2 p_2,
\]

or

\[
p_1 = \frac{1}{3} m, \quad p_2 = \frac{2}{3} m.
\]

Therefore

\[
u(x_1, x_2) = m \left( \frac{1}{3} x_1 \right)^{-1/3} \left( \frac{2}{3} x_2 \right)^{-2/3} = \left( \frac{1}{3} \right)^{-1/3} \left( \frac{2}{3} \right)^{-2/3} x_1^{1/3} x_2^{2/3} = \left( \frac{3}{4^{1/3}} \right) x_1^{1/3} x_2^{2/3}.
\]

From duality theory we know that Cobb-Douglas indirect utility, \( v(p_1, p_2, m) = mp_1^{-a}p_2^{-(1-a)} \) corresponds to Cobb-Douglas direct utility of the form \( u(x_1, x_2) = K x_1^\alpha x_2^{1-\alpha} \), so the parametric form of the direct utility function derived above is what we expect.

**Q6 (a)** Since \( x(p, m) = 5 - 4p \) is not a function of \( m \), we have \( h(p, u) = 5 - 4p \). Hicksian demand is exactly Marshallian demand.

(b) The integrability equations are

\[
\frac{\partial \mu(p, q, m)}{\partial p} = 5 - 4p, \quad \text{with boundary condition } \mu(q; q, m) = m.
\]

(c)

\[
\mu(p, q, m) = \int_q^p (5 - 4t) dt + \mu(q; q, m) = 5t - 2t^2 \bigg|_{t=q}^{t=p} + m = 5p - 2p^2 + m - 5q + 2q^2.
\]

(d) Since \( \mu(p; q, m) = e(p, v(q, m)) \), a sensible guess for \( v(q, m) \) is

\[
v(q, m) = m - 5q + 2q^2.
\]

We can verify by checking Roy’s identity:

\[
\frac{\partial v(q, m)}{\partial q} = \frac{-5 + 4q}{1} = 5 - 4q = x(q, m).
\]