Economics 711

1997 Midterm Answer Key

PART A

Q1 (i) In this question we have a Marshallian demand function with arguments $C(\mathbf{p}, m) = C(p, w, w\bar{L})$. We can determine this function from the solution to

$$\max_{\{C,L\}} \quad \min\{C, L\} \quad \text{s.t.} \quad pC + wL = w\bar{L}.$$

Clearly, in this problem we have $u = C = L$. Substituting this into the budget constraint gives, $pC + wC = w\bar{L}$, and rearranging yields

$$C(p, w, w\bar{L}) = \frac{w\bar{L}}{p + w}.$$

(ii) Substituting the Marshallian demand into the utility function yields indirect utility,

$$v(p, w, w\bar{L}) = \min\left\{\frac{w\bar{L}}{p + w}, \frac{w\bar{L}}{p + w}\right\} = \frac{w\bar{L}}{p + w}.$$

(iii) We know from the "basic four" that, in general, $v(\mathbf{p}, m) = v(\mathbf{p}, e(\mathbf{p}, u)) = u$. So in this case we have $v(p, w, e(p, w, u)) = u$, or

$$\frac{e(p, w, u)}{p + w} = u.$$

Rearranging, we derive the expenditure function,

$$e(p, w, u) = u(p + w).$$

(iv) The definition of the money metric indirect utility function is $\mu(\mathbf{p}; \mathbf{q}, m) = e(\mathbf{p}, v(\mathbf{q}, m))$. In this case we have

$$\mu((p_2, w_2); (p_1, w_1), w_1\bar{L}) = e((p_2, w_2), v(p_1, w_1, w_1\bar{L})) = v(p_1, w_1, w_1\bar{L})(p_2 + w_2) = \frac{w_1\bar{L}}{p_1 + w_2}(p_2 + w_2)$$

Q2 (i) A cost function, $C(y, \mathbf{w})$, derived from some convex monotonic technology has the following four properties:

(1) hd1 in $\mathbf{w}$.

(2) Nondecreasing in $\mathbf{w}$, nondecreasing in y.

(3) Concave in $\mathbf{w}$.

(4) Continuous in $\mathbf{w}$.

Now (1) says $C(y, t\mathbf{w}) = tC(y, \mathbf{w})$, so we have

$$C(y, t(w_1, w_2)) = y[atw_1 + btw_2 + c(tw_3)^{5/8}(tw_4)^{1/4}] = t[y(aw_1 + bw_2 + t^{-1/8}cw_3^{5/8}w_4^{1/4})].$$

We clearly require $c = 0$. So we now have $C(y, w_1, w_2) = y(aw_1 + bw_2)$. Checking (2) yields

$$\frac{\partial C(y,w)}{\partial w_1} = ya \geq 0, \qquad \frac{\partial C(y,w)}{\partial w_2} = yb \geq 0,$$

which implies $a \geq 0$, $b \geq 0$. Now note that we have a function that is linear in w_1 and w_2, so it is both concave and continuous, satisfying (3) and (4) above.

(ii) We have $C(y, w_1, w_2) = y(\alpha w_1 + \beta w_2)$, so from basic duality theory we know that the underlying technology is Leontief, of the form

$$f(x_1, x_2) = \min\left\{\frac{x_1}{\alpha}, \frac{x_2}{\beta}\right\}.$$

Arguing backwards, and with an example, let $\alpha = 1$ and $\beta = 2$. We know that $x_1 = x_2/2 = y$ at any cost minimising combination of factors. So to produce one unit of y we need one unit of x_1 and two units of x_2. The cost of one unit y is therefore $w_1 + 2w_2$. So we get a linear cost function, $C(y, w_1, w_2) = y(w_1 + 2w_2)$. It can be seen can see that this argument follows for any α and β.

Q3 (i) The consumer will only accept F if her expected utility is higher than if she is not insured. That is, F must satisfy the following inequality,

$$\log(W - F) \geq p\log(W - h) + (1 - p)\log(W + h).$$

Since the logarithmic function is monotonically increasingm, the largest such F will be the one satisfying the following equality,

$$\log(W-F) = p\log(W-h) + (1-p)\log(W+h).$$

(ii) From above we know F satisfies

$$\begin{aligned}
\log(W-F) &= p\log(W-h_0W) + (1-p)\log(W+h_0W) \\
&= p\log(W(1-h_0)) + (1-p)\log(W(1+h_0)) \\
&= p\log(W) + (1-p)\log(W) + p\log(1-h_0) + (1-p)\log(1+h_0) \\
&= \log(W) + p\log(1-h_0) + (1-p)\log(1+h_0)
\end{aligned}$$

Applying the exponential function to both sides and rearranging yields

$$\frac{F}{W} = 1 - (1-h_0)^p(1+h_0)^{(1-p)}.$$

Note that F is not exactly "independent" of W. What is true is that F/W is independent of W. The second part of the question asks for the effect of increasing h_0 on F/W. Taking the derivative of the above equation, we have

$$\frac{\partial\left(\frac{F}{W}\right)}{\partial h_0} = p(1-h_0)^{(p-1)}(1+h_0)^{(1-p)} - (1-p)(1-h_0)^p(1+h_0)^{-p}.$$

This derivative is positive if

$$p(1-h_0)^{(p-1)}(1+h_0)^{(1-p)} - (1-p)(1-h_0)^p(1+h_0)^{-p} \geq 0.$$

Rearranging, we have

$$\frac{p}{1-p}\left(\frac{(1-h_0)^{p-1}}{(1-h_0)^p}\right)\left(\frac{(1+h_0)^{1-p}}{(1+h_0)^{-p}}\right) \geq 0, \qquad \text{or} \qquad \frac{p}{1-p} \geq \frac{1-h_0}{1+h_0}.$$

So whether F/W varies positively with h_0 depends on the value of p. For example, if $p = 1/2$, then F/W increases with h_0. But for a small enough probability of loss we have F/W decreasing with h_0.

PART B

Q4 (a) The consumer's problem is

$$\max_q \quad pu(W - L - \pi q + q) + (1-p)u(W - \pi q).$$

The FOC is

$$p(1-\pi)u'(W - L - \pi q + q) - (1-p)\pi u'(W - \pi q) = 0.$$

Rearranging, we have

$$\left(\frac{1-p}{p}\right)\left(\frac{\pi}{1-\pi}\right) = \frac{u'(W - L - \pi q + q)}{u'(W - \pi q)}. \tag{1}$$

Since $\pi = p$, the LHS of (1) is equal to 1. Therefore $u'(W - L - \pi q + q) = u'(W - \pi q)$. We know $u''(\cdot) < 0$, so $u'(\cdot)$ is a strictly decreasing function, and therefore one-to-one. So $W - L - \pi q + q = W - \pi q$, implying $q^* = L$. This, of course, is just the Full Insurance Principle.

(b) We can rearrange the LHS of (1) as $\left(\frac{\pi}{p}\right)\left(\frac{1-p}{1-\pi}\right)$. Since $\pi > p$, we have $\frac{\pi}{p} > 1$ and $\frac{1-p}{1-\pi} > 1$. This implies the LHS of (1) is greater than 1, so $u'(W - L - \pi q + q) > u'(W - \pi q)$. Using the fact that $u'(\cdot)$ is a decreasing function, we have

$$W - L - \pi q + q < W - \pi q, \quad \text{implying } q < L.$$

So $q^{**} < L = q^*$. This makes sense – when insurance premiums are above actuarialy fair rates, risk averse consumers will less than fully insure against potential losses.

Q5 (a) (Note that the proof below that $a(\mathbf{p})$ must be constant is a little subtle. By assuming indirect utility satisfies hd0 in $(\mathbf{p}, m)$, it is shown that indirect utility cannot satisfy nonincreasing in $\mathbf{p}$ unless $a(\mathbf{p})$ is constant. It is *not sufficient* to simply state that hd0 fails – you must show that hd0 and nonincreasing in $\mathbf{p}$ cannot simultaneously hold.)

In order for $v(\mathbf{p}, m) = a(\mathbf{p})$ to satisfy hd0 in $(\mathbf{p}, m)$, we must have $a(\lambda\mathbf{p}) = a(\mathbf{p})$ for any $\lambda > 0$. Differentiating both sides of this equation wrt λ and evaluating at $\lambda = 1$ gives

$$\sum_{i=1}^{k} \frac{\partial a(\lambda\mathbf{p})}{\partial(\lambda p_i)} \frac{\partial(\lambda p_i)}{\partial\lambda} = \sum_{i=1}^{k} \frac{\partial a(\mathbf{p})}{\partial p_i} p_i \quad = \quad \sum_{i=1}^{k} \frac{\partial a(\mathbf{p})}{\partial\lambda} = 0.$$

Since all prices are positive, if $\partial a(\mathbf{p})/\partial p_i < 0$ for any good i, to maintain the above equality there must be at least one good j where $\partial a(\mathbf{p})/\partial p_j > 0$. But this would violate another of the indirect utility function's properties: nonincreasing in $\mathbf{p}$ implies $\partial a(\mathbf{p})/\partial p_i \leq 0$ for all i. So the only possibility is that $\partial a(\mathbf{p})/\partial p_i = 0$ for all i. That is, $a(\mathbf{p})$ is a constant function.

(b) We have $v(p_1, p_2, m) = p_1^\alpha p_2^\beta m$. Since $v(p_1, p_2, m)$ is nonincreasing in (p_1, p_2), we have

$$\frac{\partial(p_1^\alpha p_2^\beta m)}{\partial p_1} = \alpha p_1^{\alpha-1} p_2^\beta m \leq 0, \qquad \text{and} \qquad \frac{\partial(p_1^\alpha p_2^\beta m)}{\partial p_2} = \beta p_1^\alpha p_2^{\beta-1} m \leq 0.$$

This implies $\alpha \leq 0$, $\beta \leq 0$. Now, $v(p_1, p_2, m)$ is hd0 in (p_1, p_2, m). So

$$(\lambda p_1)^\alpha (\lambda p_2)^\beta (\lambda m) = \lambda^{\alpha+\beta+1} p_1^\alpha p_2^\beta m = p_1^\alpha p_2^\beta m,$$

implying that $\lambda^{\alpha+\beta+1} = 1$, or $\alpha + \beta = -1$.

(c) We can find the direct utility function by solving $u(x_1, x_2) = \min_{\{p_1,p_2\}} v(p_1, p_2, m)$ such that $p_1 x_1 + p_2 x_2 = m$. In Lagrangian form this problem is

$$\min_{\{p_1,p_2,\lambda\}} \mathcal{L} = m p_1^{-1/3} p_2^{-2/3} + \lambda(p_1 x_1 + p_2 x_2 - m).$$

The FOCs are

$$\frac{\partial\mathcal{L}}{\partial p_1}: \quad \frac{1}{3} m p_1^{-4/3} p_2^{-2/3} = \lambda x_1, \qquad \frac{\partial\mathcal{L}}{\partial p_2}: \quad \frac{2}{3} m p_1^{-1/3} p_2^{-5/3} = \lambda x_2.$$

Rearranging gives

$$\frac{1}{3} m p_1^{-1/3} p_2^{-2/3} = \lambda x_1 p_1, \qquad \frac{2}{3} m p_1^{-1/3} p_2^{-2/3} = \lambda x_2 p_2.$$

Adding together and using the budget constraint yields

$$m p_1^{-1/3} p_2^{-2/3} = \lambda(p_1 x_1 + p_2 x_2) = \lambda m,$$

so $\lambda = p_1^{-1/3} p_2^{-2/3}$. Substituting this back into the FOCs gives

$$\frac{1}{3} m p_1^{-1/3} p_2^{-2/3} = p_1^{-1/3} p_2^{-2/3} x_1 p_1, \qquad \frac{2}{3} m p_1^{-1/3} p_2^{-2/3} = p_1^{-1/3} p_2^{-2/3} x_2 p_2,$$

or

$$p_1 = \frac{1}{3}\frac{m}{x_1}, \qquad p_2 = \frac{2}{3}\frac{m}{x_2}.$$

Therefore

$$u(x_1, x_2) = m \left(\frac{1}{3}\frac{m}{x_1}\right)^{-1/3} \left(\frac{2}{3}\frac{m}{x_2}\right)^{-2/3} = \left(\frac{1}{3}\right)^{-1/3} \left(\frac{2}{3}\right)^{-2/3} x_1^{1/3} x_2^{2/3} = \left(\frac{3}{4^{1/3}}\right) x_1^{1/3} x_2^{2/3}.$$

From duality theory we know that Cobb-Douglas indirect utility, $v(p_1, p_2, m) = m p_1^{-\alpha} p_2^{-(1-\alpha)}$ corresponds to Cobb-Douglas direct utility of the form $u(x_1, x_2) = K x_1^{\alpha} x_2^{1-\alpha}$, so the parametric form of the direct utility function derived above is what we expect.

Q6 (a) Since $x(p, m) = 5 - 4p$ is not a function of m, we have $h(p, u) = 5 - 4p$. Hicksian demand is exactly Marshallian demand.

(b) The integrability equations are

$$\frac{\partial \mu(p; q, m)}{\partial p} = 5 - 4p, \qquad \text{with boundary condition} \;\; \mu(q; q, m) = m.$$

(c)

$$\mu(p; q, m) = \int_q^p (5 - 4t)dt + \mu(q; q, m) = 5t - 2t^2\Big|_{t=q}^{t=p} + m = 5p - 2p^2 + m - 5q + 2q^2.$$

(d) Since $\mu(p; q, m) = e(p, v(q, m))$, a sensible guess for $v(q, m)$ is

$$v(q, m) = m - 5q + 2q^2.$$

We can verify by checking Roy's identity:

$$-\frac{\partial v(q, m)/\partial q}{\partial v(q, m)/\partial m} = -\frac{-5 + 4q}{1} = 5 - 4q = x(q, m).$$