**Econ 711 Part II - Midterm Exam - Answer Key (Unofficial)**

## PART 1

**Q1** See Varian pp323-4.

**Q2** As stated the proposition is false. The correct statement and proof of the Second Theorem of Welfare Economics is on p329 of Varian.

**Q3** Again the statement is false. The correct statement and proof goes under the title of "Pareto efficiency and welfare maximization", and is on pp334-5 of Varian, where the weights are $a$'s rather than $\theta$'s. The interpretation of the weight is that it is the inverse of the marginal utility of income.

**Q4** The reason for assuming that agents hold positive amounts of all goods is to rule out Arrow's exceptional case (see question 17.3, Varian p336). It is not necessary to assume that preferences are strictly convex. Non-convexities in the preferences allow for PE allocations that are impossible to get as Walrasian equilibria. See for example, Problem Set 1 (from 1995 Micro August Prelim).

**Q5** This question is a slight variation of the question asked on the 1994 711 Final. If we let the price of labour, $p_l = 1$, then we can easily find that $p_x = 1/4$ using Samuelson's Nonsubstitution Theorem. We can then solve the cost minimisation problem for the firm producing speeches and find the optimal ratio of inputs. Substituting in for $x$ and using the Nonsubstitution Theorem we have that $p_y = 1$. Solving the utility optimisation problem to find the demands for the two goods, $x_i = 1$ and $y_i = 3/4$.

## PART 2

**Q1 (a)** Since $u_i(\cdot)$ is strictly concave for all consumers $i$, all consumers are risk averse. And since $\sum_i \omega_{k1}^i = \sum_i \omega_{k2}^i$ for all $k$ goods, there is no aggregate risk. So intuitively we would expect any Arrow-Debreu equilibrium allocation $\hat{\mathbf{x}}$ to satisfy $\hat{x}_{k1}^i = \hat{x}_{k2}^i$ for all consumers $i$ and all goods $k$. That is, we would expect all consumers to fully insure. This can be shown rigorously.

Since the second welfare theorem applies, we know that at any interior equilibrium the Pareto efficiency condition, $\text{MRS}_{k1,k2}^i = \text{MRS}_{k1,k2}^j$, will hold for all consumers $i$ and $j$. Since

$$\text{MRS}_{k1,k2}^i = \frac{\partial U_i/\partial x_{k1}^i}{\partial U_i/\partial x_{k2}^i} = \frac{\pi_1}{\pi_2}\frac{\partial u_i/\partial x_{k1}^i}{\partial u_i/\partial x_{k2}^i},$$

we have

$$\frac{\pi_1}{\pi_2}\frac{\partial u_i/\partial x_{k1}^i}{\partial u_i/\partial x_{k2}^i} = \frac{\pi_1}{\pi_2}\frac{\partial u_j/\partial x_{k1}^j}{\partial u_j/\partial x_{k2}^j}$$

for all consumers $i$ and $j$. Since the probabilities are objective they cancel out of the equation:

$$\frac{\partial u_i/\partial x^i_{k1}}{\partial u_i/\partial x^i_{k2}} = \frac{\partial u_j/\partial x^j_{k1}}{\partial u_j/\partial x^j_{k2}}.$$

Now since $u_i(\cdot)$ is strictly concave for all consumers $i$, $\partial u_i/\partial x^i_{ks}$ is a one-to-one function. So $\partial u_i/\partial x^i_{k1} = \partial u_i/\partial x^i_{k2}$ if and only if $x^i_{k1} = x^i_{k2}$.

Now if $\partial u_i/\partial x^i_{k1} \neq \partial u_i/\partial x^i_{k2}$ for one consumer $i$, then this inequality must hold for all consumers. So if $x^i_{k1} \neq x^i_{k2}$ for one consumer $i$, then $x^j_{k1} \neq x^j_{k2}$ for all consumers $j$. This would imply that $\sum_i x^i_{k1} \neq \sum_i x^i_{k2}$. But this would violate feasibility, since $\sum_i \omega^i_{k1} = \sum_i \omega^i_{k2}$ for all goods $k$. So we must have $x^i_{k1} = x^i_{k2}$ for all consumers $i$ and all goods $k$.

**(b)** With one physical good, we know that in equilibrium the price ratio equals each consumer's MRS. That is,

$$\frac{p_1}{p_2} = \frac{\pi_1}{\pi_2}\frac{\partial u_i/\partial x^i_1}{\partial u_i/\partial x^i_2} \text{ for all } i.$$

From part (a) we know that

$$\frac{\partial u_i/\partial x^i_1}{\partial u_i/\partial x^i_2} = 1 \text{ for all consumers } i, \text{ so } \frac{p_1}{p_2} = \frac{\pi_1}{\pi_2}.$$

**(c)** The statement of this question is ambiguous. It is not clear whether, for example, $\omega^2_2 = (0, 3)$ represents the endowments of both goods in state 2 for consumer 2, or the endowments of good 2 in both states for consumer 2. Both interpretations are solved for below.

First, consider the endowments

$$(\omega^1_{11}, \omega^1_{21}, \omega^1_{12}, \omega^1_{22}) = (1, 3, 1, 0), \qquad (\omega^2_{11}, \omega^2_{21}, \omega^2_{12}, \omega^2_{22}) = (2, 0, 2, 3),$$

where $\omega^i_{ks}$ is consumer $i$'s endowment of good $k$ in state $s$. Both consumers have Cobb-Douglas utility functions and are therefore risk averse. Further, there is no aggregate risk, i.e. $\omega^1_{k1} + \omega^2_{k1} = \omega^1_{k2} + \omega^2_{k2}$ for $k = 1, 2$. So we know from part (a) that both consumers will fully insure.

Using this fact, we can break the problem down into one where the consumers first trade to ensure their "psuedo-endowments" are the same in both states, and second we can abstact away from the separate states to consider trades between the two physical goods. Since both consumers' endowments of good 1 are the same in both states, these endowment will be their respective psuedo-endowments of good 1. And since both states are equally likely, full insurance implies that each consumer's pseudo-endowment of good 2 will be 3/2. That is, they will write a contract to insure that they each receive half the total endowments of good 2, regardless of which state actually occurs. Note that this trade implies that the relative price between good 2 in state 1 and good 2 in state 2 is 1.

So with the consumers' psuedo-endowments of $(\tilde{\omega}_1^1, \tilde{\omega}_2^1) = (1, 3/2)$ and $(\tilde{\omega}_1^2, \tilde{\omega}_2^2) = (2, 3/2)$, we can now consider trades between good 1 and good 2. Since $\text{MRS}_{12}^1 = \text{MRS}_{12}^2$ in equilibrium, the fact that the Edgeworth box is square (i.e., $\tilde{\omega}_1^1 + \tilde{\omega}_1^2 = \tilde{\omega}_2^1 + \tilde{\omega}_2^2$) and both consumers' utility functions are identical implies that $\text{MRS}_{12}^i = 1$. So the relative price between good 1 and good 2 is 1.

Now since Cobb-Douglas utility implies equal expenditure shares, and all relative prices are 1, we must have the equilibrium allocations as

$$(x_{11}^1, x_{21}^1, x_{12}^1, x_{22}^1) = (5/4, 5/4, 5/4, 5/4), \qquad (x_{11}^2, x_{21}^2, x_{12}^2, x_{22}^2) = (7/4, 7/4, 7/4, 7/4).$$

Consumer 2's higher endowment of good 1 enables him to enjoy a higher consumption level in equilibrium.

Now let's consider the other pattern of endowments,

$$(\omega_{11}^1, \omega_{21}^1, \omega_{12}^1, \omega_{22}^1) = (1, 1, 3, 0), \qquad (\omega_{11}^2, \omega_{21}^2, \omega_{12}^2, \omega_{22}^2) = (2, 2, 0, 3),$$

where $\omega_{ks}^i$ is consumer $i$'s endowment of good $k$ in state $s$. Using the method employed above, we see that full insurance implies that the relative price of good $k$ in state 1 and good $k$ in state 2 (for $k = 1, 2$) is 1. So the psuedo-endowments will be $(\tilde{\omega}_1^1, \tilde{\omega}_2^1) = (2, 1/2)$ and $(\tilde{\omega}_1^2, \tilde{\omega}_2^2) = (1, 5/2)$. Again we have $\text{MRS}_{12}^i = 1$. So the relative price between good 1 and good 2 is 1. Equal expenditure shares (with all relative prices being 1) resulting from Cobb-Douglas utility implies the equilibrium allocations are as above:

$$(x_{11}^1, x_{21}^1, x_{12}^1, x_{22}^1) = (5/4, 5/4, 5/4, 5/4), \qquad (x_{11}^2, x_{21}^2, x_{12}^2, x_{22}^2) = (7/4, 7/4, 7/4, 7/4).$$

**Q2 (a)** Since agents' utilities are $U(x_i, G) = x_i + \log G$, the Samuelson condition for this economy is

$$\sum_{i=1}^{N} \text{MRS}_{Gx}^i = \sum_{i=1}^{N} \frac{\partial U_i / \partial G}{\partial U_i / \partial x_i} = \sum_{i=1}^{N} \frac{1}{G} = \frac{N}{G} \qquad = \qquad 1.$$

So the Pareto efficient level of the public good is $G^* = N$. Since preferences are quasi-linear **and** $\sum_i m_i > N$, the Pareto efficient level of $G$ is unique. However the resources in the economy devoted towards consumption of the private good, $\sum_i m_i - N$, can be distributed in any fashion to the $N$ agents and we still have a Pareto efficient allocation.

So while the Pareto efficient level of $G$ is uniquely determined, there are multiple Pareto efficient allocations in the economy.

**(b)** If consumers face a price $p_i$ for their right to the consumption of the public good, then they wish to consume the amount of $G$ that equates their marginal rate of substitution with this price:

$$\text{MRS}^i_{Gx} = \frac{1}{G} \qquad = \qquad p_i.$$

So clearly we want $p_i = 1/G$ for all $i$. And with the optimal level of the public good being $G^* = N$, we have $p_i = 1/N$ for all $i$. So each consumer purchases her Lindahl share of $N$ units of the public good by paying $p_i \times N = 1/N \times N = 1$. So the Lindahl solution is the following allocation

$$G = N, \qquad x_i = m_i - 1, \quad \text{for all } i.$$

Clearly it is unique. The whole point of the Lindahl solution is to choose one particular allocation from amongst the set of Pareto efficient allocations.

**(c)** A core allocation is one that cannot be improved upon by any blocking coalition.

It is very important to recognise that the definition of the core has nothing to do with the Pareto set. What is true is that in an economy *with no externalities* the core is a subset of the Pareto set. However, when we have an externality (such as that created by the free rider problem in part (d) below), it is possible for core allocations to exist that are **not** Pareto efficient.

**(d)** If the blocking coalition cannot be excluded from consuming $G$ then we have a free rider externality. As a result, the core will be all allocations such that the level of $G$ is exactly 1. For if we have $G > 1$, then we have

$$\text{MRS}^i_{Gx} = \frac{\partial U_i / \partial G}{\partial U_i / \partial x_i} = \frac{1}{G} < 1.$$

This means that an agent acting out of self interest alone would wish to raise her $\text{MRS}^i_{Gx}$ by substituting away from $G$ towards $x_i$. Any agent who is making a positive contribution towards $G$ can do this by forming a blocking coalition with herself as the sole member, and reducing her contribution. (There must be at least one agent who is making a positive contribution, since the total amount of $G$ is greater than 1.)

Note that this implies that the Pareto efficient level of the public good, $G^* = N$, can never be part of core allocation (unless $N = 1$), because of the externality created by the free rider problem.

We also cannot have $G < 1$ in any core allocation because an agent could break away in a one-member blocking coalition and increase her contribution to the public good. (This is not very intuitive to think about, since it involves the exact opposite of free riding, but this case still needs to be ruled out when identifying the core.)

So the core consists solely of allocations where $G = 1$.

**(e)** The situation now, where a blocking coalition can be excluded from consuming $G$, has no free rider externality. So the core will be a subset of the Pareto set. Clearly a single consumer, say consumer 1, will not form a one-member blocking coalition if

$$m_1 - 1 + \log 1 \leq x_1 + \log N,$$

since $m_1 - 1 + \log 1$ is the level of utility consumer 1 would enjoy if she were optimally providing the public good solely to herself. (The optimal amount of $G$ for a one-member economy is $G = 1$.) Likewise, two consumers, say consumers 1 and 2, will not form a blocking coalition if

$$m_1 + m_2 - 2 + 2\log 2 \leq x_1 + x_2 + 2\log N,$$

since this inequality implies there is no way for 1 and 2 to redistribute the private good (after providing 2 units of the public good) so that they are both better off in the coalition. We can characterise the general condition for the core in this economy as follows. Denote a potential blocking coalition as $S$, with the number of members denoted by $N_S$. The core is those allocations such that no blocking coalition $S$ will form. That is,

$$\sum_{i \in S} m_i - N_S + N_S \log N_S \leq \sum_{i \in S} x_i + N_S \log N,$$

for any coalition $S$. Rearranging this inequality gives

$$\frac{1}{N_S} \sum_{i \in S} (m_i - x_i) \leq 1 + \log\left(\frac{N}{N_S}\right),$$

which says that in the core there can be no coalition whose average "contribution" to the public good is much bigger than 1. From this inequality we can see that the Lindahl solution is always in the core, since the Lindahl solution has $m_i - x_i = 1$ for all $i$, implying that the LHS of the above inequality is 1 for any coalition. It will also be the case that allocations where the distribution of the private good is not too far away from the Lindahl solution turn out to be core allocations. Intuitively, when an allocation is far away from the Lindahl solution, so that some members of the economy are contributing much more than "their fair share" towards the public good, then these consumers will find it beneficial to form a blocking coalition.