

## Firms and Production

We are going to delve into some economics today. Specifically we are going to talk about production and returns to scale.

firm - an organization that converts inputs such as labor, materials, and capital into outputs, the goods and services that it sells.

efficient production - a firm's production is efficient if it cannot produce its current level of output with fewer inputs given the existing state of knowledge about technology and the organization of production.

production function - the production function is a function that returns the maximum level of outputs for a given level of inputs.

**Example:**  $Q = \min(K, L)$  - capital and labor are perfect compliments. They must be used in equal quantities for efficient production.

**Example:**  $Q = K + L$  - capital and labor are perfect substitutes. You can replace one unit of capital with a unit of labor and not lose any output.

**Example:**  $Q = K^a L^b$  - Cobb-Douglas production function.

Returns to scale - refers to how much additional output can be obtained when we change all inputs proportionately.

- Decreasing returns to scale - when we double all inputs, output is less than doubled.

$$2 \cdot Q < F(2 \cdot k, 2 \cdot L)$$

A concrete example is the Cobb-Douglas production function ( $Q = K^a L^b$ ) with  $a + b < 1$ .

To see this multiple both inputs by  $t$

$$(t \cdot K)^a (t \cdot L)^b = t^{a+b} (K)^a (L)^b < t \cdot Q$$

- Constant returns to scale – when we double all inputs, output is exactly doubled.

$$2 \cdot Q = F(2 \cdot k, 2 \cdot L)$$

A concrete example is the Cobb-Douglas production function  $(Q = K^\alpha L^\beta)$  with  $\alpha + \beta < 1$ .

What about perfect substitutes?

What about perfect compliments?

- Increasing returns to scale – when we double all inputs, output is more than doubled

What production function that we have already talked about exhibits increasing returns to scale?

## Profit Maximization and Duality

What is the objective of most firms?

How do firms go about meeting this objective?

### Steps to profit maximization

- 1) The firm must determine how to produce output in an efficient manner in the sense that they are producing the maximum level of outputs for a given level of inputs – production engineers help with this. Basically they have to find their production function.
- 2) Figure out how to produce in a cost efficient way – Given input prices what combinations of different inputs should we employ to produce a given level of output.
- 3) Figure out how much output to produce.

### Lets consider step 2)

Step 2) says the firm has to find the cost function

Cost Function  $C(q)$  – a function that returns the least cost way of producing  $q$  units of output. You plug in a quantity and get back the least cost way of producing that quantity.

The cost function can be found by solving the following program

$$\min_{K,L} [w \cdot L + r \cdot K] \text{ subject to the constraint } [q = F(K, L)]$$

In English - minimize the cost of producing  $q$  units of output.

The values of  $K$  and  $L$  that solve the cost minimization problem are called conditional factor demands. They are conditional because they will generally depend on the level of outputs that need to be produced. Note that these conditional factor demands will generally be a function of the input prices ( $r$  and  $w$ ) and the quantity to be produced ( $q$ ).

Thus,

$$K = K(q : w, r)$$

$$L = L(q : w, r)$$

If we want the cost function we just multiply these conditional factor demands by their prices and sum. That is

$$C(q) = w \cdot L(q : w, r) + r \cdot K(q : w, r)$$

**Example:** For the Cobb-Douglas production function with two inputs  $q = K^a L^b$

$$L^* = \left( \frac{b \cdot r}{a \cdot w} \right)^{\frac{a}{a+b}} q^{\frac{1}{a+b}}$$

$$K^* = \left( \frac{a \cdot w}{b \cdot r} \right)^{\frac{b}{a+b}} q^{\frac{1}{a+b}}$$

$$C(q) = w \cdot L^* + r \cdot K^* = w \cdot \left( \frac{b \cdot r}{a \cdot w} \right)^{\frac{a}{a+b}} q^{\frac{1}{a+b}} + r \cdot \left( \frac{a \cdot w}{b \cdot r} \right)^{\frac{b}{a+b}} q^{\frac{1}{a+b}} \Rightarrow$$

$$C(q) = (a + b) \left( a^a \cdot b^b \right)^{\frac{-1}{a+b}} \cdot r^{\frac{a}{a+b}} \cdot w^{\frac{b}{a+b}} \cdot q^{\frac{1}{a+b}}$$

### Properties of the Costs Functions

- 1) The cost function is increasing in factor prices. If I increase w or r costs have to increase.
- 2) If I increase both w and r proportionately then cost increase by that factor of proportionality. For example – If I double input prices cost double. We will use this eventually.

### Links: Cost and Returns to Scale

It should be obvious that there are links between (technology) and production functions. Why?

In particular, there are links between returns to scale and cost functions. What are these links?

- If a technology exhibits *decreasing returns to scale* then average cost will be decreasing in output.
- If a technology exhibits *constant returns to scale* then average cost will be constant in output.
- If a technology exhibits *increasing return to scale* then average cost will be decreasing in output

### **Example:** Cobb-Douglas

$$C(q) = (a+b) \left( a^a \cdot b^b \right)^{\frac{-1}{a+b}} \cdot r^{\frac{a}{a+b}} \cdot w^{\frac{b}{a+b}} \cdot q^{\frac{1}{a+b}} \Rightarrow$$
$$AC(q) = \frac{C(q)}{q} = (a+b) \left( a^a \cdot b^b \right)^{\frac{-1}{a+b}} \cdot r^{\frac{a}{a+b}} \cdot w^{\frac{b}{a+b}} \cdot q^{\frac{1-(a+b)}{a+b}}$$

Is there decreasing, constant, or increasing returns to scale? It depends

- If  $a+b < 1$  there are decreasing returns to scale because average cost increase when output increases
- If  $a+b = 1$  there are constant returns to scale because average cost stay the same when output increases
- If  $a+b > 1$  there are increasing returns to scale because average cost decrease when output increases.

Because of these relationships we can rewrite the Cobb-Douglas cost function as

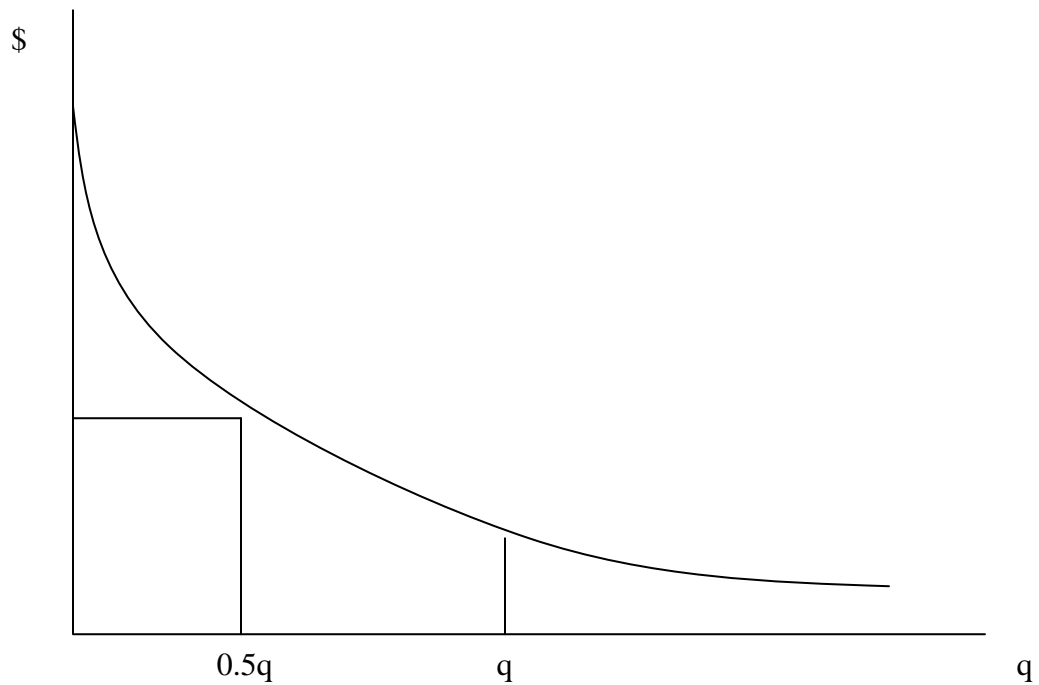
$$C(q) = \phi \cdot \left( a^a \cdot b^b \right)^{\frac{-1}{\phi}} \cdot r^{\frac{a}{\phi}} \cdot w^{\frac{b}{\phi}} \cdot q^{\frac{1}{\phi}}$$

where  $\phi$  is known as the scale parameter.

## Why Care About Returns to Scale?

The biggest thing is that when there are increasing returns to scale a firm's average cost of production is decreasing. There are important implications of this.

- If one firm with increasing returns to scale is capable of producing enough output for the entire market then there is a barrier to entry because this one firm could produce a level of output that would satisfy the market at less cost than any other firm.
- If one firm with increasing returns to scale is capable of producing enough output for the entire market then it may be better to have just one firm in the market and to regulate it in some way. Such a firm is called a natural monopoly and can be regulated in a number of ways.



## Nerlove's Model

Nerlove's basic setup is just Cobb-Douglas Technology with 3 inputs (capital, labor, and fuel).

Consider the Cobb-Douglas production with three inputs:

$$(A1) \quad q = A \cdot K^a L^b F^c$$

where  $A$  is a constant,  $L$  is labor,  $K$  is capital and  $F$  is fuel. In class it was shown that the cost function dual to this Cobb-Douglas production function is

$$(A2) \quad C(q, r, w, p_f) = h \cdot q^{\frac{1}{\phi}} \cdot r^{\frac{a}{\phi}} \cdot w^{\frac{b}{\phi}} \cdot p_f^{\frac{c}{\phi}}$$

where  $\phi = a + b + c$  is the returns to scale parameter and  $h = \phi \cdot [A \cdot a^a \cdot b^b \cdot c^c]^{-\frac{1}{\phi}}$ . Taking the log of this equation and adding an error term ( $e_i$ ) yields the equation estimated by Mark Nerlove (1963)

$$(A3) \quad \ln(C_i^*) = \beta_0 + \beta_q \cdot \ln(q_i) + \beta_k \cdot \ln(p_{ki}) + \beta_l \cdot \ln(p_{li}) + e_i$$

where

$$\begin{array}{ll} C_i^* = C_i / p_{fi} & \beta_0 = \ln(h) \\ p_{ki} = r_i / p_{fi} & \beta_q = 1/\phi \\ p_{li} = w_i / p_{fi} & \beta_k = a/\phi \\ & \beta_l = b/\phi \end{array}$$

In this model we know that if returns to scale is going to depend on the value of  $\phi = a + b + c$ .

- If  $\phi < 1 \Rightarrow$  decreasing returns to scale
- If  $\phi = 1 \Rightarrow$  constant returns to scale
- If  $\phi > 1 \Rightarrow$  increasing returns to scale

The nice feature of this model is that the coefficient on  $\ln(q_i)$  in the above regression is the inverse of the returns to scale parameter. Thus, when we estimate the model we get an estimate of returns to scale.

**Elasticity** – elasticity is a measure of the responsiveness of one variable with respect to another variable. The elasticity of  $y$  with respect to  $x$  is measured as the percentage change in  $y$  induced by a 1-percent change in  $x$ .

§ Anytime dependent variable and independent variable are in natural logs the coefficient on the logged independent variable is the elasticity of the un-logged dependent variable with respect to the un-logged independent variable.

**Example:** In Nerlov's base specification (A3) the  $\beta_q$  is the elasticity of costs with respect to output. If output changes by 1-percent, then costs changes by  $\beta_q$  percent.

Below are estimates of (A3)

```
reg lnc lnq lnpl lnpk
```

Source	SS	df	MS			
Model	294.667577	3	98.2225256	Number of obs =	145	
Residual	21.640321	141	.153477454	F( 3, 141) =	639.98	
Total	316.307898	144	2.19658262	Prob > F =	0.0000	
				R-squared =	0.9316	
				Adj R-squared =	0.9301	
				Root MSE =	.39176	

	lnc	lnq	lnpl	lnpk	_cons		
	Coef.	Std. Err.	t	P> t	[95% Conf. Interval]		
	.7206875	.0174357	41.33	0.000	.6862183	.7551567	
	.5929097	.2045722	2.90	0.004	.1884845	.9973349	
	-.0073811	.1907356	-0.04	0.969	-.3844523	.3696901	
	-4.690789	.8848715	-5.30	0.000	-6.440119	-2.941459	

What can we say from the estimates?

- Our estimate of the elasticity of cost with respect to output is 0.72. If output increases by 1-percent, cost will increase by an estimated 0.72 percent.
- Our estimate of the return to scale parameter is  $1/0.72 = 1.39 \Rightarrow$  increasing returns to scale.
- Our estimate of the elasticity of cost with respect to the price of labor is 0.59. If the price of labor increases by 1-percent, costs will increase by an estimated 0.59 percent.
- Our estimate of the elasticity of cost with respect to the price of capital is -0.007. If the price of capital increases by 1-percent, cost will decrease by an estimated -0.007 percent.
- We can also recover an estimate of the elasticity of cost with respect to the price of fuel. If the price of fuel increases by 1-percent, cost will increase by  $1 - \hat{\beta}_{lnpk} - \hat{\beta}_{lnpl} \approx 1 - 0.59 = 0.41$ . If the price of fuel increases by 1-percent, costs will increase by an estimated 0.41.