Chapter 8: Solow Model II [1]

1 Solow Model (Population Growth, Technological Change)

1.1 Effective Units

Discrete time indexed by subscript t; $t = 0, 1, 2, ...$
Cobb-Douglas production function: $Y_t = F(K_t, L_tE_t) = K_t^\alpha(L_tE_t)^{1-\alpha}$
Output per effective worker: $y_t \equiv \frac{Y_t}{L_tE_t}$; capital per effective worker: $k_t \equiv \frac{K_t}{L_tE_t}$; consumption per effective worker: $c_t \equiv \frac{C_t}{L_tE_t}$; investment per effective worker: $i_t \equiv \frac{I_t}{L_tE_t}$.

$$Y_t = K_t^\alpha(L_tE_t)^{1-\alpha} \;\Rightarrow\; \frac{Y_t}{L_tE_t} = (\frac{K_t}{L_tE_t})^\alpha \;\Rightarrow\; y_t = k_t^\alpha$$

$$C_t = (1-s)Y_t \;\Rightarrow\; \frac{C_t}{L_tE_t} = (1-s)\frac{Y_t}{L_tE_t} \;\Rightarrow\; c_t = (1-s)y_t$$

$$I_t = sY_t \;\Rightarrow\; \frac{I_t}{L_tE_t} = s\frac{Y_t}{L_tE_t} \;\Rightarrow\; i_t = sy_t$$

1.2 Assumptions

1. Labor-augmenting technological progress with constant growth rate g: $E_{t+1} = (1+g)E_t$
2. Population grows at constant rate n: $L_{t+1} = (1+n)L_t$
3. Exogenous, constant savings rate: $s_t = \bar{s}$
4. Exogenous, constant depreciation rate: $\delta_t = \bar{\delta}$
5. No government/international sectors: $G_t = T_t = X_t = M_t = 0$

$$G_t = NX_t = 0 \;\Rightarrow\; Y_t = C_t + I_t$$

1.3 Law of Motion for the Capital Stock

How does the stock of physical capital k_t change over time? If we can solve for the dynamics of k_t (how k_t changes with time), then we know y_t, c_t, and i_t across all periods in terms of k_t.
"Capital gain": investment, $i_t = sf(k_t) = sk_t^\alpha$
"Capital loss": depreciated capital plus loss due to population growth and technological change, $(n+g+\delta)k_t$
Law of motion (discrete time): $k_{t+1} - k_t \equiv \Delta k_t = i_t - (n+g+\delta)k_t = sk_t^\alpha - (n+g+\delta)k_t$

$$i_t < (n+g+\delta)k_t \;\Rightarrow\; \Delta k_t < 0$$

$$i_t = (n+g+\delta)k_t \;\Rightarrow\; \Delta k_t = 0 \;\Rightarrow k_{ss}$$

$$i_t > (n+g+\delta)k_t \;\Rightarrow\; \Delta k_t > 0$$

[1] Econ 302, Week 10, 11/6/2009; UW-Madison. TAs Lihan Liu and Scott Swisher.

1.4 Steady-state (Equilibrium)

Steady-state in the Solow model with population growth and technological change: in long-run equilibrium, capital per effective worker is constant according to the new law of motion $\Delta k_t = i_t - (n+g+\delta)k_t$.
Steady-state condition: the following equation defines a steady-state in the Solow model with population growth and technological change.
General case:

$$sf(k_{ss}) = (n+g+\delta)k_{ss} \;\Rightarrow\; \frac{k_{ss}}{f(k_{ss})} = \frac{s}{(n+g+\delta)} \tag{1}$$

Cobb-Douglas case:

$$sk_{ss}^{\alpha} = (n+g+\delta)k_{ss} \;\Rightarrow\; k_{ss} = [\frac{s}{(n+g+\delta)}]^{\frac{1}{1-\alpha}} \tag{2}$$

If this steady-state condition holds, the flows in to (investment) and out of (depreciation, population growth, productivity growth) k_t are constant.

$$k_t < k_{ss} \;\Rightarrow\; sf(k_t) > (n+g+\delta)k_t \;\Rightarrow\; \Delta k_t > 0$$

$$k_t = k_{ss} \;\Rightarrow\; sf(k_t) = (n+g+\delta)k_t \;\Rightarrow\; \Delta k_t = 0$$

$$k_t > k_{ss} \;\Rightarrow\; sf(k_t) < (n+g+\delta)k_t \;\Rightarrow\; \Delta k_t < 0$$

Steady-state quantities associated with k_{ss}: y_{ss}, c_{ss}, i_{ss} ($k^* : y^*$, c^*, i^*).

1.5 Policy and the Golden Rule k_{gr}

The Solow model still predicts that countries with higher rates of savings and investment will have higher levels of capital and output/income per effective worker in the long-run, *ceteris paribus*.
How to increase k_{ss}, and therefore y_{ss}?

1. Increase s: $s \uparrow \Rightarrow\; k_{ss} \uparrow \Rightarrow\; y_{ss} \uparrow$
2. Decrease δ: $\delta \downarrow \Rightarrow\; k_{ss} \uparrow \Rightarrow\; y_{ss} \uparrow$
3. Decrease n: $n \downarrow \Rightarrow\; k_{ss} \uparrow \Rightarrow\; y_{ss} \uparrow$
4. Decrease g: $g \downarrow \Rightarrow\; k_{ss} \uparrow \Rightarrow\; y_{ss} \uparrow$

"Golden rule" capital per unit of effective labor: The level of capital per unit of effective labor k_{gr} that maximizes $c_{ss} = f(k) - (n+g+\delta)k$.
First-order condition with respect to k:

$$\frac{\partial y}{\partial k} = (n+g+\delta) \;\Rightarrow\; f'(k_{gr}) = \alpha k_{gr}^{\alpha-1} = (n+g+\delta)$$

$$k_{gr}^{1-\alpha} = \frac{\alpha}{(n+g+\delta)} \;\Rightarrow\; k_{gr} = [\frac{\alpha}{(n+g+\delta)}]^{\frac{1}{1-\alpha}}$$

1.6 Steady-state Growth Rates

Variable	Symbol	Steady-state growth rate
Labor Force	L_t	n
Labor Efficiency	E_t	g
Capital per effective worker	$k_t = \frac{K_t}{L_t E_t}$	0
Output per effective worker	$y_t = \frac{Y_t}{L_t E_t}$	0
Consumption per effective worker	$c_t = \frac{C_t}{L_t E_t}$	0
Investment per effective worker	$i_t = \frac{I_t}{L_t E_t}$	0
Output per worker	$\frac{Y_t}{L_t}$	g
Total Output	Y_t	$n + g$

2 Exercise: Solow Model (Population Growth, Technological Change)

Consider the Solow growth model with population growth and technological change. Time is discrete and is indexed by subscript t.

a) Rewrite aggregate production function $Y_t = 20K_t^{\frac{1}{2}}(E_t L_t)^{\frac{1}{2}}$ in effective units.

b) Let $K_0 = L_0 = 100$; $E_0 = 1$. What is k_0? *(note: $t = 0$ here)*

c) Using your answer from the previous part, solve for the initial level of GDP Y_0.

d) Provided that $\delta = 0.2$ and $n = g = 0.15$, what savings rate is necessary to sustain k_0 as a steady-state? *(hint: set $k_0 = k_{ss}$ in the steady-state condition and solve for s)*

e) Using your answer from the previous part, solve for c_{ss} and i_{ss}.

f) What is Y_1, the level of real GDP next year? *(note: $t = 1$ here)*

g) What is $\frac{K_1}{L_1}$, the capital-labor ratio next year?

h) What is the "golden rule" level of k for this economy? Recall that the golden rule level of the capital stock per effective worker k_{gr} maximizes consumption per effective worker in steady-state.