Chapter 17: Consumption

1 Exercise: Two-period Fisher Model of Consumption

Consider a consumer that lives for two periods, $t = 0$ and $t = 1$. This consumer wants to maximize utility over his or her lifetime, which is given by the function $U(c_0, c_1)$, where c_0 is consumption at time $t = 0$ and c_1 is consumption at time $t = 1$. With this lifetime utility function, assume that the consumer wants to uniformly smooth consumption across time. The consumer receives income y_0 at $t = 0$ and y_1 at $t = 1$, which is known ahead of time with certainty. The gross rate of return is $(1 + R)$, so \$1 saved at $t = 0$ yields \$$(1 + R)$ at $t = 1$; R is the real interest rate.

There are two consumers, Albert and Beatrice, who receive the following fixed income independent of R:

	y_0	y_1
Albert	\$100	\$100
Beatrice	\$0	\$210

a) You observe consumption levels:

	c_0	c_1
Albert	\$100	\$100
Beatrice	\$100	\$100

Solve for R.

Beatrice's budget constraint:

$$(1 + R)c_0 + c_1 = (1 + R)y_0 + y_1$$

$$(1 + R)(100) + 100 = 210 \Rightarrow 100R = 10 \Rightarrow R = 0.10 = 10\%$$

b) Suppose that the interest rate increases. What will happen to c_0 and c_1 for Albert? Is he better or worse off as a result of the change in R?

For Albert, c_0 and c_1 will be unchanged because he is already uniformly smoothing his consumption with no borrowing. Albert's welfare is not affected by the increase in R.

c) Again, suppose that the interest rate increases. What will happen to c_0 and c_1 for Beatrice? Is she better or worse off as a result of the change in R?

An increase in the interest rate makes consumption relatively more expensive in the first period. For Beatrice, she has to borrow more against her future income to finance a given level of c_0. Her lifetime income does not change. Therefore, there is no income effect and the substitution effect says that Beatrice shifts consumption from $t = 0$ to $t = 1$; $c_0 \downarrow$, $c_1 \uparrow$, and she is worse off as a result of the increase in R (because her new consumption allocation is less smooth).

2 Exercise: Consumption Function

Let's go though a few alternatives to the Keynesian consumption function, $C = \bar{C} + MPC(Y - T)$.

a) Define W = current wealth; R = years to retirement; Y = yearly income; T = remaining years of life. Write Modigliani's life-cycle consumption function and average propensity to consume.

Assuming perfect consumption smoothing, we can write that:

$$C = \frac{\text{lifetime income}}{T}$$

$$C = \frac{W + RY}{T}$$

Consumption function (life-cycle):

$$C = (\frac{1}{T})W + (\frac{R}{T})Y$$

Average propensity to consume (life-cycle):

$$APC \equiv \frac{C}{Y} = (\frac{1}{T})\frac{W}{Y} + \frac{R}{T}$$

b) Define Y^P = permanent income; Y^T = transitory income; α = fraction of Y^P consumed annually. Write Friedman's permanent income consumption function and average propensity to consume.

Consumption function (permanent income):

$$C = \alpha Y^P$$

Average propensity to consume (permanent income):

$$APC = \frac{C}{Y} = \alpha\frac{Y^P}{Y}$$

c) Assume: $\bar{C} = 0$, $T = 0.2Y$; $W = 0$, $T = R + 10$; $Y^P = 0.75Y$. You observe that $\frac{C}{Y} = 0.35$ in aggregate data for households. Solve for MPC, R, and α.

Solving for MPC (Keynesian):

$$C = \bar{C} + MPC(Y - T) = 0 + MPC(Y - 0.2Y) = 0.8(MPC)(Y)$$

$$\frac{C}{Y} = 0.8(MPC) = 0.35 \Rightarrow MPC = \frac{0.35}{0.8} = 0.4375$$

Solving for R (life-cycle):

$$\frac{C}{Y} = (\frac{1}{T})\frac{W}{Y} + \frac{R}{T} = (\frac{1}{R+10})\frac{0}{Y} + \frac{R}{R+10} = \frac{R}{R+10} = 0.35$$

$$R = 0.35(R+10) \;\Rightarrow\; R(1-0.35) = 3.5 \;\Rightarrow\; R = \frac{3.5}{0.65} = 5.385$$

Solving for α (permanent income):

$$\frac{C}{Y} = \alpha\frac{Y^P}{Y} = \alpha\frac{0.75Y}{Y} = 0.75\alpha = 0.35$$

$$\alpha = \frac{0.35}{0.75} = 0.4\bar{6}$$

3 Exercise: Intertemporal Consumption

Consider a consumer that lives for two periods, $t = 0$ and $t = 1$. This consumer wants to maximize utility over his or her lifetime, which is given by the following function.
Lifetime utility:

$$U(c_0, c_1) = c_0^{\frac{1}{2}} + \beta c_1^{\frac{1}{2}} \tag{1}$$

where c_0 is consumption at time $t = 0$ and c_1 is consumption at time $t = 1$. $0 < \beta < 1$ is some constant less than one; the consumer is impatient, preferring to consume today.
The consumer receives income y_0 at $t = 0$ and y_1 at $t = 1$, which is known ahead of time with certainty. The gross rate of return is $(1 + R)$, so \$1 saved at $t = 0$ yields \$$(1 + R)$ at $t = 1$; R is the real interest rate. This means that we can write c_1 in terms of y_0, y_1, and c_0; all the income that is left over at time $t = 1$ is consumed.
Budget constraint:

$$(1 + R)(y_0 - c_0) + y_1 = c_1$$

Since $(y_0 - c_0)$ is saved in the first period, $(1 + R)(y_0 - c_0)$ plus new income y_1 can be used for consumption in the second period. Because you want to maximize utility, you'll consume all of your income in the second period.

a) Let $\beta = 1 + R = 1$. Write out the utility function and budget constraint under this assumption. Argue that $c_0 = c_1 = \frac{y_0 + y_1}{2}$ (complete consumption smoothing) is best in terms of maximizing utility. How did you arrive at your answer? *(hint: think about what happens if $c_0 \neq c_1$)*

Lifetime utility:

$$U(c_0, c_1) = c_0^{\frac{1}{2}} + c_1^{\frac{1}{2}}$$

$$\frac{\partial U(c_0, c_1)}{\partial c_0} = \frac{1}{2} c_0^{-\frac{1}{2}}$$

$$\frac{\partial U(c_0, c_1)}{\partial c_1} = \frac{1}{2} c_1^{-\frac{1}{2}}$$

Budget constraint:

$$(y_0 - c_0) + y_1 = c_1 \;\Rightarrow\; y_0 + y_1 = c_0 + c_1$$

With $\beta = 1 + R = 1$, we can transfer consumption directly from one period to the next. If lifetime utility is maximized, we shouldn't be able to come up with some transfer of consumption (or income) from one period to the next to make the consumer better off. Therefore, we want to equate marginal utility at $t = 0$ with marginal utility at $t = 1$ so the consumer is indifferent to any such transfers (at the margin).

$$\frac{\partial U(c_0, c_1)}{\partial c_0} = \frac{\partial U(c_0, c_1)}{\partial c_1}$$

$$\frac{1}{2}c_0^{-\frac{1}{2}} = \frac{1}{2}c_1^{-\frac{1}{2}} \Rightarrow c_0^{-\frac{1}{2}} = c_1^{-\frac{1}{2}} \Rightarrow c_0 = c_1$$

Use the budget constraint to solve for c_0 and c_1 in terms of income.

$$2c_0 = 2c_1 = y_0 + y_1 \Rightarrow c_0 = c_1 = \frac{y_0 + y_1}{2}$$

If $c_0 \neq c_1$, then $\frac{1}{2}c_0^{-\frac{1}{2}} \neq \frac{1}{2}c_1^{-\frac{1}{2}}$ and $\frac{\partial U(c_0,c_1)}{\partial c_0} \neq \frac{\partial U(c_0,c_1)}{\partial c_1}$; this can't maximize lifetime utility, and you should transfer consumption between periods until $c_0 = c_1$ and the marginal utilities are equated at $t = 0, 1$.

b) Consider the general case with no assumptions on β or $(1 + R)$. First, let's use the budget constraint to eliminate c_1 as something you have to choose. Write out $U(c_0)$, lifetime utility as a function of only c_0 and income. *(hint: substitute the budget constraint into the utility function for c_1)*

Lifetime utility:

$$U(c_0, c_1) = c_0^{\frac{1}{2}} + \beta c_1^{\frac{1}{2}}$$

Budget constraint:

$$(1 + R)(y_0 - c_0) + y_1 = c_1$$

Substitute the budget constraint into the utility function, replacing c_1 as a variable that you need to choose.

$$U(c_0) = c_0^{\frac{1}{2}} + \beta[(1 + R)(y_0 - c_0) + y_1]^{\frac{1}{2}}$$

c) Maximize $U(c_0)$ with respect to c_0 and solve for the utility-maximizing (c_0^*, c_1^*) as a function of income. *(hint: set $U'(c_0) = 0$ and solve for c_0, then solve for c_1 using the budget constraint; you don't need to simplify)*

$$U(c_0) = c_0^{\frac{1}{2}} + \beta[(1 + R)(y_0 - c_0) + y_1]^{\frac{1}{2}}$$

$$U'(c_0) = \frac{\partial U(c_0)}{\partial c_0} = \frac{1}{2}c_0^{-\frac{1}{2}} + \frac{1}{2}\beta[(1 + R)(y_0 - c_0) + y_1]^{-\frac{1}{2}}(-(1 + R)) = 0$$

$$\beta(1 + R)[(1 + R)(y_0 - c_0) + y_1]^{-\frac{1}{2}} = c_0^{-\frac{1}{2}}$$

$$\beta^2(1 + R)^2[(1 + R)(y_0 - c_0) + y_1]^{-1} = c_0^{-1}$$

$$\beta^2(1 + R)^2 c_0 = (1 + R)(y_0 - c_0) + y_1$$

$$c_0[(1 + R) + \beta^2(1 + R)^2] = (1 + R)y_0 + y_1$$

$$c_0^* = \frac{(1+R)y_0 + y_1}{(1+R)[1+\beta^2(1+R)]}$$

$$c_1^* = (1+R)(y_0 - c_0^*) + y_1 = (1+R)y_0 + y_1 - \frac{(1+R)y_0 + y_1}{1+\beta^2(1+R)}$$

$$c_1^* = ((1+R)y_0 + y_1)[1 - \frac{1}{1+\beta^2(1+R)}]$$

d) Compute partial derivatives $\frac{\partial c_0^*}{\partial \beta}$ and $\frac{\partial c_0^*}{\partial R}$. Can you sign them? Interpret. *(hint: again, either write c_0^* as a product and use the product rule or keep c_0^* as a fraction and use the quotient rule)*

$$c_0^* = [(1+R)y_0 + y_1][(1+R) + \beta^2(1+R)^2]^{-1}$$

Calculating $\frac{\partial c_0^*}{\partial \beta}$

$$\frac{\partial c_0^*}{\partial \beta} = [(1+R)y_0 + y_1](-1)[(1+R) + \beta^2(1+R)^2]^{-2}(2\beta(1+R)^2) + [(1+R) + \beta^2(1+R)^2]^{-1}(0)$$

$$\frac{\partial c_0^*}{\partial \beta} = -\frac{2\beta(1+R)^2[(1+R)y_0 + y_1]}{[(1+R) + \beta^2(1+R)^2]^2} < 0$$

As β increases, the consumer is more patient and more willing to delay consumption until the next period; c_1^* increases and c_0^* decreases.

Calculating $\frac{\partial c_0^*}{\partial R}$

$$\frac{\partial c_0^*}{\partial R} = [(1+R)y_0 + y_1](-1)[(1+R) + \beta^2(1+R)^2]^{-2}(1 + 2\beta^2(1+R)) + [(1+R) + \beta^2(1+R)^2]^{-1}(y_0)$$

$$\frac{\partial c_0^*}{\partial R} = \frac{y_0}{(1+R) + \beta^2(1+R)^2} - \frac{[(1+R)y_0 + y_1](1 + 2\beta^2(1+R))}{[(1+R) + \beta^2(1+R)^2]^2}$$

Cannot sign $\frac{\partial c_0^*}{\partial R}$ without specific parameter values. We usually claim that consumption is negatively related to the real interest rate, but here it is ambiguous due to competing income (higher return to saving) and substitution (consumption in the first period is relatively more expensive) effects.

e) A typical Keynesian consumption function of form $C_t = \bar{C} + MPC(Y_t - T_t)$ at time t has consumption today depending only on current disposable income. Using your previous results, discuss why this is incomplete if consumers are intertemporal utility-maximizers. Propose an alternative. *(hint: what did utility-maximizing consumption depend on in the previous parts?)*

In the two period intertemporal utility maximization problem, consumption at time $t = 0$ depends on y_0, y_1, β, and R. We'd expect that current consumption depends on lifetime expected future income, expectations *about future interest rates, and so on. The Keynesian consumption function is incomplete because it does not include a term for expected future income. A possible alternative would be* $C_t = \frac{1}{T}\sum_{t'=t}^{t+T} E_t(Y_{t'} - T_{t'})$, *where the consumer lives for* T *periods and tries to smooth consumption completely across all periods of his or her life* ($E_t(.)$ *is a prediction about some future variable, like income, made at time* t*; we aren't worrying about interest rates here).*