Assignment #4*

Econ 302: Intermediate Macroeconomics

November 1, 2009

1 Aggregate Demand / Aggregate Supply

Long-run aggregate supply curve (vertical):

$$Y_{LR} = 3000 \tag{1}$$

Short-run aggregate supply curve (horizontal):

$$P_{SR} = 1 \tag{2}$$

Aggregate demand curve:

$$Y = \frac{3M}{P} \tag{3}$$

Money supply:

$$M = 1000 \tag{4}$$

1.1 If the economy is initially in long-run equilibrium, what are the values of $P_{LR,1}$ and $Y_{LR,1}$?

$$LRAS = AD \Rightarrow Y_{LR,1} = 3000 = \frac{3M}{P_{LR,1}}$$

$$P_{LR,1} = \frac{3M}{3000} = \frac{3(1000)}{3000} = 1$$

1.2 What is the velocity of money in initial long-run equilibrium $(P_{LR,1}, Y_{LR,1})$?

$$MV = PY \Rightarrow V_{LR,1} = \frac{P_{LR,1}Y_{LR,1}}{M} = \frac{(1)(3000)}{(1000)} = 3$$

*Kelly ; UW-Madison. TAs Lihan Liu and Scott Swisher.

1.3 Now suppose a supply shock moves the short-run aggregate supply curve to $P = 1.5$ (still horizontal). What is the new short-run equilibrium $(P_{SR,2}, Y_{SR,2})$?

$$P_{SR,2} = 1.5$$

$$SRAS = AD \Rightarrow Y_{SR,2} = \frac{3M}{P_{SR,2}} = \frac{3(1000)}{1.5} = 2000$$

1.4 If the aggregate demand and long-run aggregate supply curves are unchanged, what is the long-run equilibrium $(P_{LR,2}, Y_{LR,2})$ after the supply shock?

The long-run aggregate supply and aggregate demand curves determine long-run equilibrium. Since these curves are not affected by the supply shock, our answer does not change from the previous part (1.1) without the supply shock.

$$P_{LR,2} = P_{LR,1} = 1$$

$$Y_{LR,2} = Y_{LR,1} = 3000$$

1.5 Suppose that after the supply shock, the Federal Reserve wants to hold output at its long-run level. What level of the money supply M' would be required to achieve this in the short-run?

$$Y_{SR} = \frac{3M'}{P_{SR}} = 3000$$

$$M' = \frac{(3000)(P_{SR})}{3} = \frac{(3000)(1.5)}{3} = 1500$$

2 Growth Accounting / Solow Residual

Assume the following functional form for the aggregate production function:

$$Y = AK_p^{\alpha_1} K_h^{\alpha_2} L^{\alpha_3} \tag{5}$$

Here, K_p denotes physical capital; K_h human capital; L labor. The constant returns to scale condition $\alpha_1 + \alpha_2 + \alpha_3 = 1$ holds.

2.1 Write $\%\Delta Y$ in terms of $\%\Delta A$, $\%\Delta K_p$, $\%\Delta K_h$, and $\%\Delta L$ (hint: $\Delta \log(x) \approx \%\Delta x$).

$$\log(Y) = \log(A) + \alpha_1 \log(K_p) + \alpha_2 \log(K_h) + \alpha_3 \log(L)$$

$$\Delta \log(Y) = \Delta \log(A) + \alpha_1 \Delta \log(K_p) + \alpha_2 \Delta \log(K_h) + \alpha_3 \Delta \log(L)$$

$$\%\Delta Y = \%\Delta A + \alpha_1 \%\Delta K_p + \alpha_2 \%\Delta K_h + \alpha_3 \%\Delta L$$

2.2 Based on your answer to the previous part, write $\%\Delta A$ in terms of the other variables. Interpret this as the growth rate of the "Solow residual". In what sense is A a residual here?

$$\%\Delta A = \%\Delta Y - (\alpha_1 \%\Delta K_p + \alpha_2 \%\Delta K_h + \alpha_3 \%\Delta L)$$

Productivity A is not observable, so it has to be estimated by using other variables that we can gather data for. In this case, productivity growth is the component of output growth that can't be explained by growth in the observable factors (assuming, for now, that human capital is observable).

2.3 <u>For this part only</u>, assume that $\%\Delta Y = 0.05$, $\%\Delta K_p = 0.02$, $\%\Delta K_h = 0.04$, $\%\Delta L = 0.01$, $\alpha_1 = 0.2$, $\alpha_2 = 0.6$, $\alpha_3 = 0.2$. Compute $\%\Delta A$. What is the most important source of output growth here?

$$\%\Delta A = \%\Delta Y - (\alpha_1 \%\Delta K_p + \alpha_2 \%\Delta K_h + \alpha_3 \%\Delta L)$$

$$\%\Delta A = (0.05) - [(0.2)(0.02) + (0.6)(0.04) + (0.2)(0.01)]$$

$$\%\Delta A = 0.02$$

Based solely on an interpretation of the numerical values, human capital (education, skills, training, etc.) is the most important (i.e. largest) source of output growth, assuming that the aggregate production function is correct (this includes our estimates of α_1, α_2, and α_3).

2.4 Now, let's write everything in per-worker terms. Define $y = \frac{Y}{L}$; $k_p = \frac{K_p}{L}$; $k_h = \frac{K_h}{L}$. Write y as a function of k_p and k_h only.

$$Y = AK_p^{\alpha_1} K_h^{\alpha_2} L^{\alpha_3}$$

$$\frac{Y}{L} = A(\frac{K_p}{L})^{\alpha_1} L^{\alpha_1} (\frac{K_h}{L})^{\alpha_2} L^{\alpha_2} L^{\alpha_3 - 1}$$

$$y = A(k_p)^{\alpha_1} (k_h)^{\alpha_2} L^{\alpha_1 + \alpha_2 + \alpha_3 - 1}$$

$$y = Ak_p^{\alpha_1} k_h^{\alpha_2} L^{1-1} = Ak_p^{\alpha_1} k_h^{\alpha_2} L^0 = Ak_p^{\alpha_1} k_h^{\alpha_2}$$

2.5 Write $\%\Delta y$ in terms of $\%\Delta A$, $\%\Delta k_p$, and $\%\Delta k_h$.

$$\log(y) = \log(A) + \alpha_1 \log(k_p) + \alpha_2 \log(k_h)$$

$$\Delta \log(y) = \Delta \log(A) + \alpha_1 \Delta \log(k_p) + \alpha_2 \Delta \log(k_h)$$

$$\%\Delta y = \%\Delta A + \alpha_1 \%\Delta k_p + \alpha_2 \%\Delta k_h$$

2.6 Based on your answer to (2.5), write $\%\Delta y$ in terms of $\%\Delta A$, $\%\Delta K_p$, $\%\Delta K_h$, and $\%\Delta L$. *(hint: $\%\Delta(\frac{A}{B}) \approx \%\Delta A - \%\Delta B$)*

$$\%\Delta y = \%\Delta A + \alpha_1 \%\Delta k_p + \alpha_2 \%\Delta k_h$$

$$\%\Delta y = \%\Delta A + \alpha_1 \%\Delta(\frac{K_p}{L}) + \alpha_2 \%\Delta(\frac{K_h}{L})$$

$$\%\Delta y = \%\Delta A + \alpha_1 \%\Delta(K_p) + \alpha_2 \%\Delta(K_h) - (\alpha_1 + \alpha_2)\%\Delta L$$

2.7 Based on your answer to (2.5), write $\%\Delta A$ in terms of $\%\Delta y$, $\%\Delta k_p$, and $\%\Delta k_h$ only. In what sense is A a residual here?

$$\%\Delta y = \%\Delta A + \alpha_1 \%\Delta k_p + \alpha_2 \%\Delta k_h$$

$$\%\Delta A = \%\Delta y - (\alpha_1 \%\Delta k_p + \alpha_2 \%\Delta k_h)$$

In this case, productivity growth is the component of output per worker growth that can't be explained by growth in the observable factors physical/human capital per worker (assuming, for now, that human capital is observable). $\%\Delta A$ based on the per-worker production function is the same as $\%\Delta A$ in (2.2)

2.8 If human capital K_h is unobservable (can't gather data for it), we set $\%\Delta k_h = 0$ and try to estimate productivity growth using the equation: $\%\Delta\hat{A} = \%\Delta y - \alpha_1\%\Delta k_p$. Assuming that our data on y, α_1, and k_p are accurate, will our estimated value $\%\Delta\hat{A}$ be too high or too low relative to the true $\%\Delta A$? *(hint: you should have three cases based on $\%\Delta k_h$)*

We have three cases based on the sign of $\%\Delta k_h$ (or $\%\Delta k_h = 0$):

$$\%\Delta\hat{A} - \%\Delta A = [\%\Delta y - \alpha_1\%\Delta k_p] - [\%\Delta y - (\alpha_1\%\Delta k_p + \alpha_2\%\Delta k_h)] = \alpha_2\%\Delta k_h$$

$$\%\Delta k_h > 0 \;\Rightarrow\; \%\Delta\hat{A} > \%\Delta A$$

$$\%\Delta k_h = 0 \;\Rightarrow\; \%\Delta\hat{A} = \%\Delta A$$

$$\%\Delta k_h < 0 \;\Rightarrow\; \%\Delta\hat{A} < \%\Delta A$$

2.9 Is it possible that $\%\Delta\hat{A} < 0$? Interpret. Is this consistent with the idea that A is the level of technology available in an economy?

Based on the previous part, it is possible that $\%\Delta\hat{A} < 0$ if $\%\Delta y < \alpha_1\%\Delta k_p$; this is especially likely to occur if $\%\Delta k_h < 0$. If A stands for the overall level of technology in an economy, then the level of technology is declining; A is decreasing in the sense of technological regress. Justifying $\%\Delta\hat{A} < 0$ is difficult because it implies that we are forgetting or losing production technologies. Since World War II, continual technological improvement has been the norm; with the advent of information technology (i.e. the Internet), the loss of technology seems unrealistic. Therefore, $\%\Delta\hat{A} < 0$ is not consistent with the interpretation of A as the overall level of technology, at least in the past century. If you don't want to attach any meaning to A, the Solow residual is just unexplained output growth based on the Cobb-Douglas assumption for the aggregate production function.

3 Solow Model with Population Growth

For this problem, try to keep time subscript t on all the relevant (i.e. changing over time) variables out of steady-state. This will help you for the next problem. Assume that the following variables are constant over time: depreciation rate δ; savings rate s; population growth rate n. Because population growth is non-zero, the labor force is now a function of time.
Definitions:

$$y_t = \frac{Y_t}{L_t} \tag{6}$$

$$k_t = \frac{K_t}{L_t} \tag{7}$$

$$c_t = \frac{C_t}{L_t} \tag{8}$$

$$i_t = \frac{I_t}{L_t} \tag{9}$$

Aggregate production function in per-worker terms:

$$y_t = f(k_t) \tag{10}$$

Labor force (population) growth equation:

$$L_t = (1+n)L_{t-1} \tag{11}$$

3.1 Write c_t and i_t as functions of k_t.

$$c_t = y_t - i_t = (1-s)y_t = (1-s)f(k_t)$$

$$i_t = y_t - c_t = sy_t = sf(k_t)$$

3.2 Write the law of motion for k_t, $\Delta k_t \equiv k_{t+1} - k_t$.

$$\Delta k_t = k_{t+1} - k_t = i_t - (n+\delta)k_t = sf(k_t) - (n+\delta)k_t$$

3.3 What is the law of motion for c_t, $\Delta c_t \equiv c_{t+1} - c_t$; the law of motion for i_t, $\Delta i_t \equiv i_{t+1} - i_t$? *(hint: should be a function of k_t and constants)*

$$\Delta c_t = c_{t+1} - c_t = (1-s)f(k_{t+1}) - (1-s)f(k_t) = (1-s)[f(k_t+\Delta k_t) - f(k_t)] = (1-s)[f(sf(k_t) - (n+\delta-1)k_t) - f(k_t)]$$

$$\Delta i_t = i_{t+1} - i_t = sf(k_{t+1}) - sf(k_t) = s[f(k_t + \Delta k_t) - f(k_t)] = s[f(sf(k_t) - (n+\delta-1)k_t) - f(k_t)]$$

3.4 Assume a steady-state in k_t; $k_t = \bar{k} = k_{ss}$. What are Δk_t, Δc_t, and Δi_t?

$$k_{t+1} = k_t = k_{ss} \;\Rightarrow\; \Delta k_t = \Delta c_t = \Delta i_t = 0$$

3.5 Continue to assume a steady-state in k_t. Write $\frac{\Delta K_t}{K_t}$, $\frac{\Delta L_t}{L_t}$, $\frac{\Delta Y_t}{Y_t}$, $\frac{\Delta C_t}{C_t}$, and $\frac{\Delta I_t}{I_t}$ in terms of constants. *(hint: these are not per-worker quantities, so they will have non-zero growth rates)*

The capital stock in per-worker terms is constant, so K_t must be growing at the rate of population growth n. From the population growth equation, L_t also grows at rate n. Output per worker is constant in steady-state, so Y_t grows at rate n as well. The same argument applies to consumption and investment per worker, so C_t and I_t grow at rate n.

$$\%\Delta K_t = \%\Delta L_t = \%\Delta Y_t = \%\Delta C_t = \%\Delta I_t = n$$

3.6 Continue to assume a steady-state in k_t. What are the steady-state levels k_{ss}, y_{ss}, c_{ss}, i_{ss}? *(hint: you can leave k_{ss} in your answers)*

$$\Delta k_t = 0 \;\Rightarrow\; sf(k_{ss}) = (n+\delta)k_{ss}$$

$$y_{ss} = f(k_{ss})$$

$$c_{ss} = (1-s)y_{ss} = (1-s)f(k_{ss}) = f(k_{ss}) - (n+\delta)k_{ss}$$

$$i_{ss} = sy_{ss} = sf(k_{ss}) = (n+\delta)k_{ss}$$

3.7 Solve for the "golden rule" level of the (per-worker) capital stock, k_{gr}, provided that $f(k_t) = k_t^{\alpha}$. *(hint: maximize c_{ss} as a function of k_{ss} from the previous part; your function for c_{ss} shouldn't contain savings rate s; your answer should depend only on constants)*

$$c_{ss} = f(k_{ss}) - (n+\delta)k_{ss}$$

$$\frac{\partial}{\partial k_{ss}}(c_{ss}) = f'(k_{ss}) - (n+\delta) = 0$$

$$f'(k_{gr}) = \alpha k_{gr}^{\alpha-1} = (n+\delta)$$

$$\alpha = (n+\delta)k_{gr}^{1-\alpha}$$

$$k_{gr} = \left(\frac{\alpha}{n+\delta}\right)^{\frac{1}{1-\alpha}}$$

3.8 Solve for the savings rate s_{gr} that supports k_{gr} from the previous part. ***(hint: use the law of motion for the capital stock; your answer should depend only on constants)***

$$\Delta k_t = sf(k_t) - (n+\delta)k_t = 0$$

$$s_{gr}f(k_{gr}) - (n+\delta)k_{gr} = 0$$

$$s_{gr} = \frac{(n+\delta)k_{gr}}{f(k_{gr})} = \frac{(n+\delta)(\frac{\alpha}{n+\delta})^{\frac{1}{1-\alpha}}}{(\frac{\alpha}{n+\delta})^{\frac{\alpha}{1-\alpha}}} = (n+\delta)(\frac{\alpha}{n+\delta})^{\frac{1-\alpha}{1-\alpha}} = \alpha$$

3.9 Let $\delta = 0.02$, $n = 0.01$, and $\alpha = 0.3$. Compute k_{gr} and s_{gr}.

$$k_{gr} = (\frac{\alpha}{n+\delta})^{\frac{1}{1-\alpha}} = (\frac{0.3}{0.01+0.02})^{\frac{1}{1-0.3}} = 26.827$$

$$s_{gr} = \alpha = 0.3$$

4 Computational Solow Model (do Q3 first)

From parts (3.1) and (3.2) of Q3, you should be able to implement the Solow model with population growth in Excel.
Parameter values and initial conditions:

$$f(k_t) = k_t^{\alpha} \tag{12}$$

$$\delta = 0.02 \tag{13}$$

$$n = 0.01 \tag{14}$$

$$\alpha = 0.3 \tag{15}$$

$$s = 0.15 \tag{16}$$

$$k_{t=0} = k_0 = 2 \tag{17}$$

General hints:

1. You can leave everything in per-worker terms. The $-nk_t$ term in the law of motion is all you need to include to account for L_t growing at rate n.
2. Make a table of t, k_t, y_t, c_t, and i_t in the usual way. You already know how y_0, c_0, and i_0 depend on k_0; write analogous formulas in Excel. Your formulas should refer to cells (relative references), so don't substitute in the numbers directly. Fill in k_0.
3. Use the law of motion to write k_1 in terms of k_0 and constants (again, refer to cells). Copy your row formulas for k_t, y_t, c_t, and i_t down to run the model. Now you can see convergence to the steady-state!

4.1 Graph k_t vs. time for $0 \leq t \leq 300$. What is the steady-state level of the capital stock?

See Excel file. The steady-state level of the capital stock is near 10; $k_{ss} = 9.966$.

4.2 Let $s = s_{gr}$, where s_{gr} is the golden rule savings rate that you found in part (3.9). Graph k_t vs. time for $0 \leq t \leq 300$. Is the steady-state level of the capital stock now the golden rule level found in (3.9)?

$$s_{gr} = \alpha = 0.3$$

See Excel file. The steady-state level of the capital stock is near 27; $k_{ss} = 26.827$. *This is equal to* k_{gr} *from part (3.9).*

4.3 Continue with $s = s_{gr}$. At $t = 300$, a new production technology is introduced such that $\alpha_{t \geq 300} = 0.4$. Graph k_t, c_t, and i_t vs. time for $300 \leq t \leq 600$. What is the new steady-state level of the capital stock?

See Excel file. The new steady-state level of the capital stock is near 46.5; $k_{ss} = 46.416$ *with* $\alpha = 0.4$.

4.4 Continue with $s = s_{gr}$ and $\alpha_{t \geq 300} = 0.4$. A natural disaster at time $t = 600$ occurs such that $\delta_{t \geq 600} = 0.10$ and $n_{t \geq 600} = -0.01$. Graph k_t, c_t, and i_t vs. time for $600 \leq t \leq 900$. What is the new steady-state level of the capital stock?

See Excel file. The new steady-state level of the capital stock is near 7.5; $k_{ss} = 7.438$ *with* $\alpha = 0.4$, $\delta = 0.10$, $n = -0.01$.

4.5 In words, describe how you would modify your Excel file to account for s, n, and δ varying over time. You do not have to implement this in Excel.

Make new columns for s_t, n_t, *and* δ_t *in the Excel file, and have the equations for* k_t, y_t, c_t, *and* i_t *refer to the contemporaneous values for* s_t, n_t,*and* δ_t.