Consider the classic Cournot model of quantity competition, with asymmetric marginal costs. Two firms \( i \in \{1, 2\} \) simultaneously choose production levels \( q_i \in \mathbb{R}^+ \); each firm \( i \) produces those units at a constant marginal cost of \( c_i \) per unit, and sells them at the market price, which is determined by the inverse demand function

\[
P = \max \{0, 100 - q_1 - q_2\}
\]

1. Calculate firm \( i \)'s best-response given a production level \( q_j \) of his opponent. (Be sure to account for the cases where it is optimal to set \( q_i = 0 \).)

2. Let \( c_1 = 25 \) and \( c_2 = 55 \). Find the Nash equilibrium in which both players produce, and calculate both firms’ profits.

3. If firm 1 produced 45 units, firm 2’s best-response would be not to produce at all. Calculate firm 1’s profits in this event. Is it higher or lower than your answer to part 2? Is \((q_1, q_2) = (45, 0)\) an equilibrium? Why or why not?

Now let \( G \) be any two-player simultaneous-move game, with strategy spaces \( A_i = \mathbb{R}^+ \) and payoff functions \( u_i : A_i \times A_j \rightarrow \mathbb{R} \) which are continuous and differentiable. Let \( G_1 \) be a variation on the game \( G \) where player 1 moves first, then player 2 observes 1’s action and moves second. (When \( G \) is the Cournot game, \( G_1 \) is known as the Stackelberg game.)

4. Let \( BR_i : A_j \Rightarrow A_i \) be player \( i \)'s best-response correspondence for the game \( G \), that is, \( BR_i(a_j) = \arg \max_{a_i \in A_i} u_i(a_i, a_j) \). Show that if \( BR_2 \) is single-valued (\( BR_2(a_1) \) is a singleton for every \( a_1 \)), then player 1’s payoff in any subgame-perfect equilibrium of \( G_1 \) is at least as high as his payoff in any pure-strategy Nash equilibrium of \( G \).

5. Now suppose \( BR_1 \) and \( BR_2 \) are both single-valued. Show that if \( BR_2 \) is weakly decreasing and \( u_1 \) is weakly decreasing in \( a_2 \), then player 1’s strategy in any subgame-perfect equilibrium of \( G_1 \) is at least as high as his strategy in any pure-strategy Nash equilibrium of \( G \).

6. Let \( G \) be the Cournot game described above, with marginal costs \( c_1 \) and \( c_2 \), and suppose \( G \) has a unique equilibrium in which both firms produce strictly positive quantities. Show that the statements in parts 4 and 5 hold strictly: firm 1 produces strictly more in the Stackelberg game than in the simultaneous-move Cournot game, and earns strictly higher profits.