Prelim Question – Summer 2008 – Dan Quint
Suggested Solutions (And Common Mistakes)

Consider the classic Cournot model of quantity competition, with asymmetric marginal costs. Two firms $i \in \{1, 2\}$ simultaneously choose production levels $q_i \in \Re^+$; each firm i produces those units at a constant marginal cost of c_i per unit, and sells them at the market price, which is determined by the inverse demand function

$$P = \max\{0, \quad 100 - q_1 - q_2\}$$

1. *Calculate firm i's best-response given a production level q_j of his opponent. (Be sure to account for the cases where it is optimal to set $q_i = 0$.)*

What makes this a little tricky is that there are two "corners" to worry about: where price goes from being $100 - q_1 - q_2$ to 0, and where one player's best-response goes from being the maximizer of a concave function to being 0. As it happens, the second case contains the first, so there end up only being two cases to worry about. If $100 - q_j \geq c_i$, then any positive q_i will result in $P > c_i$ and negative profits, so player i's unique best-response is 0. If $100 - q_j < c_i$, on the other hand, player i can achieve strictly positive profits, and his best-response coincides with the maximizer of $q_i(100 - q_j - q_i - c_i)$. Taking the first-order condition gives the best-response mapping

$$BR_i(q_j) = \begin{cases} \frac{1}{2}(100 - q_j - c_i) & if \;\; q_j < 100 - c_i \\ 0 & otherwise \end{cases}$$

2. *Let $c_1 = 25$ and $c_2 = 55$. Find the Nash equilibrium in which both players produce, and calculate both firms' profits.*

If both firms produce in equilibrium, that is, $q_1 > 0$ and $q_2 > 0$, then $q_1 = \frac{1}{2}(100 - q_2 - c_1) = \frac{1}{2}(75 - q_2)$ and $q_2 = \frac{1}{2}(100 - q_1 - c_2) = \frac{1}{2}(45 - q_1)$. Solving these simultaneously gives $q_1 = 35$ and $q_2 = 5$, leading to price $P = 100 - 35 - 5 = 60$ and profits $\pi_1 = 35(60 - 25) = 35^2 = 1225$ and $\pi_2 = 5(60 - 55) = 25$.

(As it happens, this is the unique equilibrium: it's easy to verify that $BR_i(BR_j(0)) > 0$, so there cannot be an equilibrium in which only one firm produces.)

3. *If firm 1 produced 45 units, firm 2's best-response would be not to produce at all. Calculate firm 1's profits in this event. Is it higher or lower than your answer to part 2? Is $(q_1, q_2) = (45, 0)$ an equilibrium? Why or why not?*

If $q_1 = 45$ and $q_2 = BR_2(45) = 0$, then $P = 55$ and $\pi_1 = 45(55 - 25) = 45 * 30 = 1350$, which is greater than 1225 found above. However, this is not an equilibrium, since firm 1 is not playing a best-response: $BR_1(0) = \frac{75}{2} \neq 45$.

Now let G be any two-player simultaneous-move game, with strategy spaces $A_i = \Re^+$ and payoff functions $u_i : A_i \times A_j \rightarrow \Re$ which are continuous and differentiable. Let G_1 be a variation on the game G where player 1 moves first, then player 2 observes 1's action and moves second. (When G is the Cournot game, G_1 is known as the Stackelberg game.)

4. *Let $BR_i : A_j \rightrightarrows A_i$ be player i's best-response correspondence for the game G, that is, $BR_i(a_j) = \arg\max_{a_i \in A_i} u_i(a_i, a_j)$. Show that if BR_2 is single-valued ($BR_2(a_1)$ is a singleton for every a_1), then player 1's* **payoff** *in any subgame-perfect equilibrium of G_1 is at least as high as his payoff in any pure-strategy Nash equilibrium of G.*

Since BR_2 is single-valued, in any subgame-perfect equilibrium of G_1, if firm 1 plays a_1, firm 2 must play $BR_2(a_1)$. Thus,

$$u_1^{SPE} = \max_{a_1} u_1(a_1, BR_2(a_1))$$

Let $\left(a_1^{NE}, a_2^{NE}\right)$ be any Nash equilibrium of G; since $a_2^{NE} = BR_2(a_1^{NE})$,

$$u_1^{NE} = u_1\left(a_1^{NE}, BR_2(a_1^{NE})\right) \leq \max_{a_1} u_1(a_1, BR_2(a_1)) = u_1^{SPE}$$

(To put it another way, if $u_1^{SPE} < u_1^{NE}$, player 1 in G_1 would have a profitable deviation to a_1^{NE}, since this would have to be followed by a_2^{NE}. To put it yet another way, player 1's highest payoff in any pure-strategy Nash equilibrium of G can be written as

$$\max_{a_1, a_2} u_1(a_1, a_2) \quad \textit{subject to} \quad a_1 = BR_1(a_2) \;\; \textit{and} \;\; a_2 = BR_2(a_1)$$

while player 1's payoff in any SPE of G_1 can be written as

$$\max_{a_1, a_2} u_1(a_1, a_2) \quad \textit{subject to} \quad a_2 = BR_2(a_1)$$

which is the same objective function but fewer constraints and therefore has a higher value.)

5. *Now suppose BR_1 and BR_2 are both single-valued. Show that if BR_2 is weakly decreasing and u_1 is weakly decreasing in a_2, then player 1's* **strategy** *in any subgame-perfect equilibrium of G_1 is at least as high as his strategy in any pure-strategy Nash equilibrium of G.*

This is easiest to show by contradiction. Suppose that a_1^{SPE} was played by player 1 in some SPE of G_1, and $a_1^{SPE} < a_1^{NE}$. Since BR_1 is single-valued and $a_1^{NE} = BR_1(a_2^{NE})$, a_1^{SPE} is **not** a best-response to a_2^{NE}, so

$$u_1\left(a_1^{SPE}, a_2^{NE}\right) < u_1\left(a_1^{NE}, a_2^{NE}\right)$$

But if BR_2 is weakly decreasing, $BR_2(a_1^{SPE}) \geq BR_2(a_1^{NE}) = a_2^{NE}$; so if u_1 is decreasing in a_2,

$$u_1\left(a_1^{SPE}, BR_2(a_1^{SPE})\right) \leq u_1\left(a_1^{SPE}, a_2^{NE}\right)$$

By transitivity, then,

$$u_1^{SPE} = u_1\left(a_1^{SPE}, BR_2(a_1^{SPE})\right) < u_1\left(a_1^{NE}, a_2^{NE}\right) = u_1^{NE}$$

which contradicts our finding in part 4.

(By the way, a distinction that several people missed: BR_1 single-valued does **not** automatically imply that $\arg\max_{a_1} u_1\left(a_1, BR_1(a_1)\right)$ is single-valued.)

6. *Let G be the Cournot game described above, with marginal costs c_1 and c_2, and suppose G has a unique equilibrium in which both firms produce strictly positive quantities. Show that the statements in parts 4 and 5 hold strictly: firm 1 produces strictly more in the Stackelberg game than in the simultaneous-move Cournot game, and earns strictly higher profits.*

First, three facts about the Cournot game, which can be established easily from part 1:

- u_1 and BR_2 are continuous
- Wherever $q_1 > 0$ and $P > 0$, u_1 is differentiable with respect to q_2, with $\frac{\partial u_1}{\partial q_2} = -q_1$
- Wherever $BR_2(q_1) > 0$, BR_2 is differentiable, with slope $-\frac{1}{2}$

So wherever q_1 and $BR_2(q_1)$ are both strictly positive, $u_1(q_1, BR_2(q_1))$ (as a function of one variable) is differentiable, with derivative

$$\frac{d}{dq_1} u_1(q_1, BR_2(q_1)) = \frac{\partial u_1}{\partial q_1} + \frac{\partial u_1}{\partial q_2} BR_2'(q_1) = \frac{\partial u_1}{\partial q_1}(q_1, BR_2(q_1)) + \frac{1}{2} q_1$$

By assumption, both firms produce in the equilibrium of G, so q_1^{NE} and $BR_2(q_1^{NE})$ are strictly positive; so for q_1 close to q_1^{NE}, the derivative expression above holds.

But since q_1^{NE} is a static best-response to q_2^{NE}, the first derivative $\frac{\partial u_1}{\partial q_1}$ must be 0 at (q_1^{NE}, q_2^{NE}); so now we know that $u_1(q_1, BR_2(q_1))$ is continuous and differentiable with strictly positive derivative in a neighborhood of q_1^{NE}. But this means that for ϵ small enough,

$$u_1(q_1^{NE} + \epsilon, BR_2(q_1^{NE} + \epsilon)) > u_1(q_1^{NE}, BR_2(q_1^{NE}))$$

This means that $u_1^{SPE} > u_1^{NE}$. It also means that $q_1^{SPE} \neq q_1^{NE}$; since the Cournot game satisfies all the assumptions made in part 5,[1] we already know that $q_1^{SPE} \geq q_1^{NE}$, so now $q_1^{SPE} > q_1^{NE}$ and we're done.

(A number of people tried to explicitly solve for the Stackelberg equilibrium, either for $(c_1, c_2) = (25, 55)$ or in general, by substituting $q_2 = \frac{1}{2}(100 - q_1 - c_2)$ into firm 1's profit function and maximizing

$$q_1 \left(100 - q_1 - \frac{1}{2}(100 - q_1 - c_2) - c_1 \right)$$

This approach is spot on conceptually, but a problem in practice because it ignores the non-negativity constraint on BR_2. For example, with $c_1 = 25$ and $c_2 = 55$, maximizing the function above gives $q_1 = \frac{105}{2}$. This suggests that moving first, firm 1 should produce 52.5 units, because this will lead firm 2 to produce $\frac{1}{2}(100 - 52.5 - 55) = -3.75$ units. When the nonnegativity constraint is accounted for, firm 1 is better off producing 45 units – just enough to get firm 2 to stay out of the market, but no more.)

[1] Well, not exactly, I guess – due to the kink where $P = 0$, payoff functions in the Cournot game are not everywhere differentiable. They are, however, differentiable "everywhere that matters," and we didn't need differentiability to prove part 5 anyway.