Testing Auction Models: Are Private Values Really Independent?

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Abstract

In this paper, we introduce and apply an approach to testing independence of values in standard auction settings. We show that the widely-used assumption that private values are independent (IPV) has testable implications in both first-price and English auction settings, which will be violated under the natural alternative of positve dependence. We provide a computationally straightforward and asymptotically valid closed-form approach to testing these implications on auction data. We apply our tests to data from much-studied United States Forest Service timber auctions, and find clear evidence to reject the IPV model in favor of a model of correlated private values.

1 Introduction

The independent private values (IPV) model is a workhorse in auction theory, and has been used as the basis for a vast array of empirical studies. If private values are independently distributed, a classic result in auction theory is that the optimal selling mechanism takes the form of a standard auction in which the only relevant design parameter is the reserve price. Independent private values also implies revenue-equivalence of the two most prevalent auction formats, first-price and English auctions; optimality of a reserve price strictly higher than the seller's valuation; and invariance of that optimal reserve price to the number of bidders present. However, the policy prescriptions of auction theory can differ substantially when there is intra-auction correlation among bidders' values. When values are correlated, the optimal auction no longer takes the form of a standard auction; and even in a standard auction format, the benefits of using a strategic reserve price can be substantially lowered or eliminated altogether.

The literature on non-parametric identification in first-price and English auctions (see, e.g., Athey and Haile (2007)), which is the basis for many empirical applications of auction theory, is also closely tied to the assumption that private values are independent. When private values may instead be correlated, new problems arise for identification which have only recently begun to be explored. An important message emerging from this literature is that policy inferences drawn from auction data under the assumption of IPV can be substantially misleading in environments where valuations are actually correlated.[1]

The assumption that bidders have private (as opposed to common) values can often be institutionally motivated; but there is typically little economic basis for assuming that these values are independent. While there is a large empirical literature that *assumes* independence, we are not aware of any papers that *test* this assumption despite its economic significance. Our paper aims to fill this void. Methodologically, we make two contributions:

- First, we establish testable implications of the IPV model for both first-price and English auctions.

In first-price auctions, the implications of IPV are straightforward: conditional on observed covariates, bids must be independent. In open-outcry English auctions, however, this transparency breaks down: actual bidding behavior can depend on the bids made by others, so correlation of observed bids need not imply correlation in values. However, we show it is still possible to derive meaningful and testable implications of independence of values in English auctions. The key, as in Athey and Haile (2002) and Haile and Tamer (2003), it to consider the distributions of *order statistics* of bids, rather than the individual bids themselves. We show that correlation among values will bring order statistics closer together relative to the IPV benchmark; based on this insight, we derive testable implica-

[1]Li, Perrigne, and Vuong (2002), working with simulated data which is affiliated, find that estimation using IPV would overstate bidder rents by 17% to 24%. Krasnokutskaya (2009), working with procurement data, finds that the reserve prices suggested by an IPV model would result in actual procurement costs 10-35% higher on average than the reserve prices calculated in a model accounting for unobserved heterogeneity. In Aradillas-López, Gandhi, and Quint (2011), we find that IPV analysis would cause the optimal reserve price to be overestimated on average by between 15% and 47%, and the increase in expected profit from setting the reserve price optimally (relative to simply setting reserve equal to the seller's valuation) to be overestimated by between 86% and 202%!

tions of the IPV model, and show that they will be violated when valuations are instead positively dependent (in a general sense which we define below).

- Second, we develop a unified nonparametric approach to testing these restrictions.

While the testable implication of independence in first-price auctions is a conditional independence restriction (a problem previously studied in the literature), our key testable implications for ascending auctions take the form of inequalities comparing non-linear transformations of conditional moments of the data, a problem not yet addressed in the conditional moment inequalities literature (see, e.g, Andrews and Shi (2011)). We provide a computationally simple and asymptotically closed-form appraoch to testing such inequalities given a sample of auction observations. We transform each conditional moment inequality into an equality restriction on an unconditional expectation, in a way that preserves all the information in the original inequalities. Test statistics are then constructed as sample analogs of the corresponding expectations; under standard regularity conditions, they are therefore asymptotically normal, allowing for the use of normal critical values. The same testing strategy would also be applicable to testing conditional moment inequalities that arise in other economic applications.[2]

We apply our tests to data from USFS timber auctions. The USFS timber data has several features that argue in favor of the plausibility of an IPV model. First, as we discuss later, there are specific institutional features that make private values (as opposed to common values) a natural assumption. Second, there is a rich vector of covariate information about each auction that precisely captures the public information available to the bidders; conditioning on such detailed covariates is often used to justify the assumption that any remaining private-value differences are independent. As a result, the USFS data has been extensively studied under the assumption of IPV. We nonetheless find clear evidence to reject independence of values in favor of correlation, in both the first-price and English auction data.

It is worthwhile to consider the consequences of testing for identification and estimation of auction models. As noted above, when values are correlated, policy implications drawn

[2] Such moment inequalities arise, for example, in testing heterogeneous treatment effects (Imbens and Wooldridge (2009), Lee and Whang (2009)), strategic complementarities in games (de Paula and Tang (2011), Aradillas-López and Gandhi (2011)), and asymmetric information in insurance settings (Chiappori, Jullien, Salanié, and Salanié (2006)).

from an IPV-based empirical approach can be misleading; thus, if the tests presented in this paper lead to rejection of the IPV model, an empirical strategy that allows for correlation is necessary. In first-price auctions, there are multiple existing approaches to identification with correlated values if more than one bid from each auction is observed.[3] In English auctions, however, observation of multiple bids is not enough, as only one bid per auction is tightly linked to a bidder's valuation. In Aradillas-López, Gandhi, and Quint (2011), we show instead how to exploit variation in the number of bidders N to partially identify a general model of correlated values. If there is positive dependence between bidder values and N (consistent with equilibrium play in standard models of endogenous entry), an upper bound on the seller's expected profit function is identified; if values satisfy the stronger assumption of being independent of N, both upper and lower bounds are identified, leading to bounds on the optimal reserve price as well. In the present paper, as part of our strategy for testing IPV, we provide a testable implication of independence between values and N that has power against endogenous entry. Applying the resulting test to the timber data, we fail to reject independence between values and N; thus, despite our rejection of IPV, there is evidence to support an empirical strategy that can still draw strong inferences from the English auction data.

The rest of the paper proceeds as follows. In Section 2, we derive testable implications of each of the standard auction models. In Section 3, we translate these implications into test statistics, based on sample analogs of related moment conditions; we then discuss the asymptotic properties of the tests, and conduct a Monte Carlo simulation to show their performance on finite samples. In Section 4, we apply our tests to the USFS timber data and present our results. Section 5 concludes. Appendix A contains proofs of the theoretical results (the testable implications of each model); the details of the formal tests and their implementation are given in Appendix B.

[3]Li, Perrigne, and Vuong (2002) show how to estimate a model of affiliated private values. Krasnokutskaya (2009) shows how to estimate a model of IPV with scalar, additively-separable unobserved heterogeneity; Hu, McAdams, and Shum (2009) extend this technique to IPV with non-separable unobserved heterogeneity, as well as to conditionally independent private values. This covers the three "standard" models of correlated values used empirically; as we point out below, our formulation of correlated values nests all three of these.

2 Testable Implications of Auction Models

Each auction in the data is characterized by a set of covariates describing that particular auction, X; a number of bidders, N; and a vector of bids, $\boldsymbol{B} = (B_1, \ldots, B_N)$. The joint distribution of the observables $(X, N, \boldsymbol{B})$ is thus nonparametrically identified by the data. In this section, we will seek restrictions on the conditional distribution of $\boldsymbol{B}$ given (X, N) which are implied by equilibrium play in different auction models. The subsequent section will then address the question of developing econometric tests of whether these restrictions are satisfied given a sample of auctions from the population distribution of $(X, N, \boldsymbol{B})$.

We will maintain the following assumption throughout:

Assumption 1. *Bidders have private values, and the joint distribution of these private values is symmetric.*

Thus, we will assume that bidders have private values $\boldsymbol{V} = (V_1, \ldots, V_N)$; let $F(\cdot|x, n)$ denote the joint distribution of these valuations, conditional on $X = x$ and $N = n$.[4] Symmetry imposes the additional restriction that $F(v_1, v_2, \ldots, v_n|x, n) = F(v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(n)}|x, n)$ for $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ any permutation. An independent private values (IPV) model would impose the additional restriction that $F(v_1, \ldots, v_n|x, n) = \prod_{i=1}^{n} F_V(v_i|x, n)$ for some univariate distribution F_V (which may or may not depend on n). Our primary focus is to understand and test the implications of the IPV assumption in the two most common and most-studied auction environments, first-price and ascending auctions for a single good.

2.1 First-Price Auctions

In a symmetric, IPV setting, first-price auctions have a unique Bayesian Nash equilibrium, which is symmetric and has strictly increasing strategies.[5] For a given realization (x, n) of (X, N), then, we can write $B(V_i|x, n)$ as bidder i's equilibrium bid. As recognized by Athey and Haile (2007) (section 5.5), conditional on a realization (x, n) of (X, N), independence of valuations $(V_1, \ldots, V_n)$ would imply independence of bids $(B(V_1|x, n), \ldots, B(V_n|x, n))$ as

[4]This is a slight abuse of notation, as the domain of $F : \Re_+^n \to [0, 1]$ depends on n, but the meaning should be clear.

[5]See Athey and Haile (2007), footnote 11. Such an equilibrium also exists, and is also unique, if private values are symmetric and affiliated.

well. Despite the transparency of this restriction on the joint distribution of bids, we are unaware of any implementations of this idea as a formal test. We capture this testable restriction in a way we will later show can be naturally implemented as an econometric test:

Proposition 1. *Let* (B^I, B^{II}) *denote two randomly selected bids from a given auction. Under IPV,*

$$\operatorname{cov}\left(B^I, B^{II} \mid (X,N) = (x,n)\right) = 0 \tag{1}$$

for all (x, n).

Any violation of the IPV model in the form of positive dependence among values would also lead to positive dependence among bids; under very general conditions, then, the above test will have power to reject independence when values are instead positively dependent.

2.2 English Auctions

English auctions create unique empirical challenges relative to first-price auctions. Since the winner will stop raising his bid as soon as the other bidders stop bidding, information about the highest bidder's willingness to pay is effectively censored: the top two bids will mechanically be close together, and the winner's valuation cannot be easily inferred. And similarly, losing bidders may not bid up to their valuations. A bidder with an intermediate valuation might make a low bid early (or not bid at all), then sit back and watch while multiple opponents bid against each other, knowing he could bid again later if the price was still favorable; once the price passed his valuation, he would not bid again, never having bid even close to his actual willingness-to-pay. For both these reasons, a bidder's actual bid may be a function of both his own and his opponents' valuations, and so bids will be correlated even if valuations are not. Thus, we cannot test the IPV model in ascending auctions simply by testing for conditional covariance among bids. However, in this section we show it is still possible to derive non-parametric testable implications of ascending auctions using the properties of order statistics.

Fixing $N = n$, let $V_{1:n} \leq V_{2:n} \leq \ldots \leq V_{n:n}$ denote the order statistics of the random vector of valuations $\boldsymbol{V}$, and $F_{k:n}(\cdot \mid x)$ the distribution of $V_{k:n}$ conditional on the realization $(X,N) = (x,n)$. Similarly, let $B_{1:n} \leq \ldots \leq B_{n:n}$ denote the order statistics of the random vector of bids $\boldsymbol{B}$, and $G_{k:n}(\cdot \mid x)$ the distribution of $B_{k:n}$ given $(X,N) = (x,n)$. As noted above, $G_{k:n}(\cdot \mid x)$ is identified from the data.

For $k \leq n$, define a function $\psi_{k:n} : [0,1] \to [0,1]$ by

$$\psi_{k:n}(s) = \frac{n!}{(n-k)!(k-1)!} \int_0^s t^{k-1}(1-t)^{n-k} dt$$

For $t \in (0,1)$, the integrand is positive, and so $\psi_{k:n}$ is strictly increasing everywhere and therefore invertible. Athey and Haile (2002) observe that if n independent random variables are drawn from a distribution $H(\cdot)$, the distribution of the k^{th}-lowest is $\psi_{k:n}(H(\cdot))$. Under an IPV model, then, for any k and n, $F_V(v|x,n) = \psi_{k:n}^{-1}(F_{k:n}(v|x))$.

2.2.1 Testing IPV with Fixed N

As noted above, a bidder's valuation does not uniquely determine his bid in an English auction, so inferring valuations from bids is not a straightforward exercise. To address this, Haile and Tamer (2003) introduced an "incomplete model" of bidding in ascending auctions. Rather than impose a unique mapping from primitives to equilibrium outcomes, Haile and Tamer (2003) impose only weak assumptions about bidder behavior, and aim to partially identify an IPV model. They assume bidders never bid higher than their valuations, which implies $B_{k:n} \leq V_{k:n}$; and they assume that bidders never allow the auction to end at a price they could profitably beat, which implies that for $k < n$, $V_{k:n} \leq B_{n:n} + \Delta$, where Δ is the minimum bid increment at the end of the auction. While Haile and Tamer (2003) assume IPV, these bidding assumptions are equivalent to bidders playing weakly undominated strategies; they therefore do not depend on bidders' beliefs or the joint distribution of valuations, and are therefore equally valid in any private-values setting. We will use "Haile-and-Tamer bidding" to describe bidding strategies which satisfy these two assumptions, but are otherwise unrestricted.

In order to create a test that will still have power even in such an unstructured model of bidding, we must place some structure on how we might expect independence to be violated, that is, what we see as the alternative hypothesis to IPV. We do this in a theoretically general and non-parametric way:

Assumption 2. *If valuations are not independent, then the joint distribution $F(\cdot \mid x, n)$ is such that for any realization $(X,N) = (x,n)$ and any v and i, $\Pr(V_i < v)$ is weakly increasing in the number of bidders $j \neq i$ such that $V_j < v$.*

This is the same "general correlated" model that we use for identification and estimation in Aradillas-López, Gandhi, and Quint (2011). In that paper, we show that Assumption

2 holds under all the standard models of symmetric, correlated private values: specifically, affiliated private values, conditionally-independent private values, and IPV with unobserved heterogeneity.

Haile-and-Tamer bidding implies that $G_{k:n}(v|x) \geq F_{k:n}(v|x)$ and $F_{n-1:n}(v|x) \geq G^{\Delta}_{n:n}(v|x)$, where $G^{\Delta}_{n:n}(\cdot|x)$ is the distribution of $B_{n:n} + \Delta$ (given $X = x$). Even though valuations are not uniquely pinned down, the model is still testable:

Proposition 2. *Under IPV and Haile-and-Tamer bidding, for any (x, n, v) and any $k \leq n-2$,*

$$\psi^{-1}_{k:n}\left(G_{k:n}(v|x)\right) \quad \geq \quad \psi^{-1}_{n-1:n}\left(G^{\Delta}_{n:n}(v|x)\right) \tag{2}$$

On the other hand, if values are not independent, then at any (x, n, v) where Assumption 2 holds strictly – that is, where $\Pr\left(V_i < v \mid X = x, N = n, \|\{j \neq i : V_j < v\}\| = m\right)$ is not the same for all m – then for $k \leq n-2$,

$$\psi^{-1}_{k:n}(F_{k:n}(v|x)) \quad < \quad \psi^{-1}_{n-1:n}(F_{n-1:n}(v|x))$$

and equation 2 will therefore be violated if there is sufficiently little slack in the "Haile and Tamer bounds".

The first part of Proposition 2 was noted by Haile and Tamer (2003) (Remark 2), who point out that it could be used as a test of the IPV model. The second part of the proposition, however, is new, and shows that this test can have power against positively-correlated values. As the proposition indicates, the power of the test depends on how close $G^{\Delta}_{n:n}$ is to $F_{n-1:n}$ and $G_{k:n}$ to $F_{k:n}$. If the top two bids are close together in most auctions (implying also that Δ is small), then the first inequality will not have much slack: since $G^{\Delta}_{n:n} \leq F_{n-1:n} \leq G_{n-1:n}$, if $G_{n-1:n}$ and $G^{\Delta}_{n:n}$ are close together, $F_{n-1:n}$ must be close to $G^{\Delta}_{n:n}$. Thus, the real concern is whether $G_{k:n}$ is close to $F_{k:n}$ for $k \leq n-2$ – that is, whether losing bidders other than the second-highest bid close to their valuations. Song (2004) considers the possibility that the "top two losers" bid close to their values, even if the others do not; this would be enough for our test to have power. Unfortunately, there is no easy way to check this in the data. And if only the highest losing bidder approaches his value, this test may have little power.

As a result, we consider another approach to testing the IPV model, which relies only on transaction prices (or the winning and highest losing bids) but requires variation in the number of bidders.

2.2.2 Testing IPV using Variation in N

Since this test relies on variation in the number of bidders, it requires an assumption about the nature of this variation. In particular, we will assume that variation in the number of bidders is independent of the realization of their valuations. To formalize this condition, let $F_m^n(\cdot \mid x)$ denote the joint distribution of m bidders drawn at random from an auction with n bidders, conditional on $X = x$.[6]

Definition. Values are **independent of** N if $F_m^n(\,\cdot \mid x) = F_m^{n'}(\,\cdot \mid x)$ for all (x, n, n', m).

Under the IPV model, this simply means that the marginal distribution $F_V\,(\,\cdot \mid x, n)$ does not depend on n. This assumption has also been used in Haile, Hong, and Shum (2003), Guerre, Perrigne, and Vuong (2009), and Gillen (2009), and has been termed an "exclusion restriction" since N is excluded from the distribution $F_V\,(\,\cdot \mid x)$.

This test will rely only on the distribution of the second-highest valuation, $F_{n-1:n}$, for various n. It is common practice to assume that this one distribution is learned exactly from transaction prices, rather than bounded by the observed bid distributions according to the Haile-and-Tamer assumptions. For ease of exposition, we will present the tests under the assumption that $B_{n:n} = V_{n-1:n}$; in Appendix A.2, we show how the same tests would be applied under the Haile and Tamer assumptions.

Proposition 3. *Assume $B_{n:n} = V_{n-1:n}$ and values are independent of N.*

(a) Under IPV, for any (x, n, n', v),

$$\psi_{n-1:n}^{-1}\left(G_{n:n}(v|x)\right) \quad = \quad \psi_{n'-1:n'}^{-1}\left(G_{n':n'}(v|x)\right) \tag{3}$$

(b) Under Assumption 2 (nonnegatively correlated private values), for any (x, n, n', v),

$$n > n' \quad \longrightarrow \quad \psi_{n-1:n}^{-1}\left(G_{n:n}(v|x)\right) \quad \geq \quad \psi_{n'-1:n'}^{-1}\left(G_{n':n'}(v|x)\right) \tag{4}$$

Further, equation 4 holds strictly wherever Assumption 2 holds strictly, that is, at all (x, n, n', v) where $n > n'$ and $\Pr(V_i < v \mid X = x, N = n, \|\{j \neq i : V_j < v\}\| = m)$ is not the same for all m.

Remark 1. Equation 3 was proposed by Athey and Haile (2002) as a possible basis for a test of the IPV model. (See also the discussion in Athey and Haile (2007).) The drawback

[6] Since $F(\,\cdot \mid x, n)$ is symmetric, $F_m^n(v_1, \ldots, v_m \mid x) = F\,(v_1, \ldots, v_m, \infty, \infty, \ldots, \infty \mid x, n)$.

of equation 3 as a standalone test of IPV, however, is that it is really a joint test of two assumptions: IPV *and* the exclusion restriction. That is, a rejection of equation 3 could follow from a violation either of IPV or of the exclusion restriction; next, we discuss how to disentangle the two.

Remark 2. Equation 4, on the other hand, is a new result, and contributes to our testing strategy in two ways. First, it ensures that equation 3 has power against all the standard models of positively-correlated values when the exclusion restriction holds. More importantly, it provides a testable implication of the exclusion restriction itself which does not depend on independence of values. In Appendix A.5, we show that equation 4 has power as a test of the exclusion restriction. Specifically, we show fairly general conditions under which a correlated private values model, combined with either of the two standard models of endogenous entry in auctions (those of Levin and Smith (1994) and Samuelson (1985)), would lead to a violation of equation 4.

Remark 3. Thus, if data violates equation 3 but satisfies equation 4, this supports the hypothesis that the failure of equation 3 is caused by a violation of IPV rather than a violation of the exclusion restriction. If this is indeed the case – values are correlated, but independent of the number of bidders – then both upper and lower bounds are identified for the seller's expected profit and optimal reserve price, using the approach laid out in Aradillas-López, Gandhi, and Quint (2011).[7]

To gain intuition for Proposition 3, consider what happens to the distribution of transaction prices as N increases. As N increases, transaction prices get stochastically higher (the distribution shifts to the right), since the price is set by the second-highest of a bigger group. (Pinkse and Tan (2005) refer to this as the *sampling effect.*) If values are IPV and F_V does not vary with n, Proposition 3 says that this must happen at a particular "speed" – that is, for each v, $F_{n-1:n}(v)$ must fall exactly fast enough so that $\psi^{-1}_{n-1:n}(F_{n-1:n}(v))$ remains constant.

Relative to that benchmark, correlation of values *slows down* the sampling effect – if values are correlated, then each incremental bidder has less impact on transaction price, as bidder values are more prone to be close together. So if values are correlated but independent

[7] In that paper, we also show that the same upper bound on profit, and a weaker upper bound on the optimal reserve price, still hold if the exclusion restriction is violated.

of N, $F_{n-1:n}(v)$ falls more slowly than under IPV, and $\psi^{-1}_{n-1:n}(F_{n-1:n}(v))$ therefore increases with n.

On the other hand, violations of the exclusion restriction would likely be due to a positive relationship between valuations and N – that is, endogenous participation favoring auctions for more-valuable prizes. This would augment the sampling effect, causing $F_{n-1:n}(v)$ to fall more quickly than under IPV; provided this effect was stronger than the slowing-down due to correlation, it would result in $\psi^{-1}_{n-1:n}(F_{n-1:n}(v))$ decreasing with n. As noted in Remark 2, we have shown that even in the presence of correlation, the test of equation 4 has power against a fairly wide class of "typical" violations of the exclusion restriction.

2.2.3 Testing IPV when the Exclusion Restriction Fails

When the exclusion restriction is rejected, of course, equation 3 no longer offers a test of IPV. Without any restriction on how $F_V(\,\cdot\,|x,n)$ can vary with n, the IPV model is just-identified from transaction price data, and therefore not testable. However, a natural restriction would be a general positive reslationship between the number of bidders and their valuations. This can be formalized as the following condition:[8]

Assumption 3. *If valuations are IPV but the distribution $F_V(\,\cdot\mid x,n)$ depends on n, then it does so in such a way that (for any x) $n > n'$ implies $F_V(\,\cdot\mid x,n) \succsim_{FOSD} F_V(\,\cdot\mid x,n')$.*

Proposition 4. *Assume $B_{n:n} = V_{n-1:n}$. Under IPV and Assumption 3,*

$$n > n' \quad \longrightarrow \quad \psi^{-1}_{n-1:n}\left(G_{n:n}(v|x)\right) \quad \leq \quad \psi^{-1}_{n'-1:n'}\left(G_{n':n'}(v|x)\right) \tag{5}$$

for all (x,n,n',v).

Observe that Proposition 4 gives the opposite conclusion as part (b) of Proposition 3; that is, relative to the benchmark of equation 3, violations of the exclusion restriction work in the opposite direction as correlation among values. When both the exclusion restriction and independence fail, equation 5 need not always have power as a test of IPV. Nevertheless, a rejection would serve as evidence against IPV.

[8] In Aradillas-López, Gandhi, and Quint (2011), we generalize this notion of valuations being "stochastically increasing in N" to settings with correlated values, and show conditions under which it follows from three different models of endogenous entry.

2.2.4 Example of Propositions 3 and 4

Equations 3, 4, and 5 are claims that $\psi_{n-1:n}^{-1}(G_{n:n}(v|x))$ is constant, increasing, or decreasing in n, respectively. To illustrate how $\psi_{n-1:n}^{-1}(G_{n:n}(v|x))$ behaves for various types of data-generating processes, we graph its value (as a function of v) for different values of N under four versions of a parametric example. For the example, there are no observable covariates X; bidder values are *i.i.d.* draws from a log-normal distribution, $\log(V_i) \sim N(\mu, \sigma^2)$, with $\sigma^2 = 0.5$ throughout but μ potentially variable. The four cases, shown in Figure 1, are:

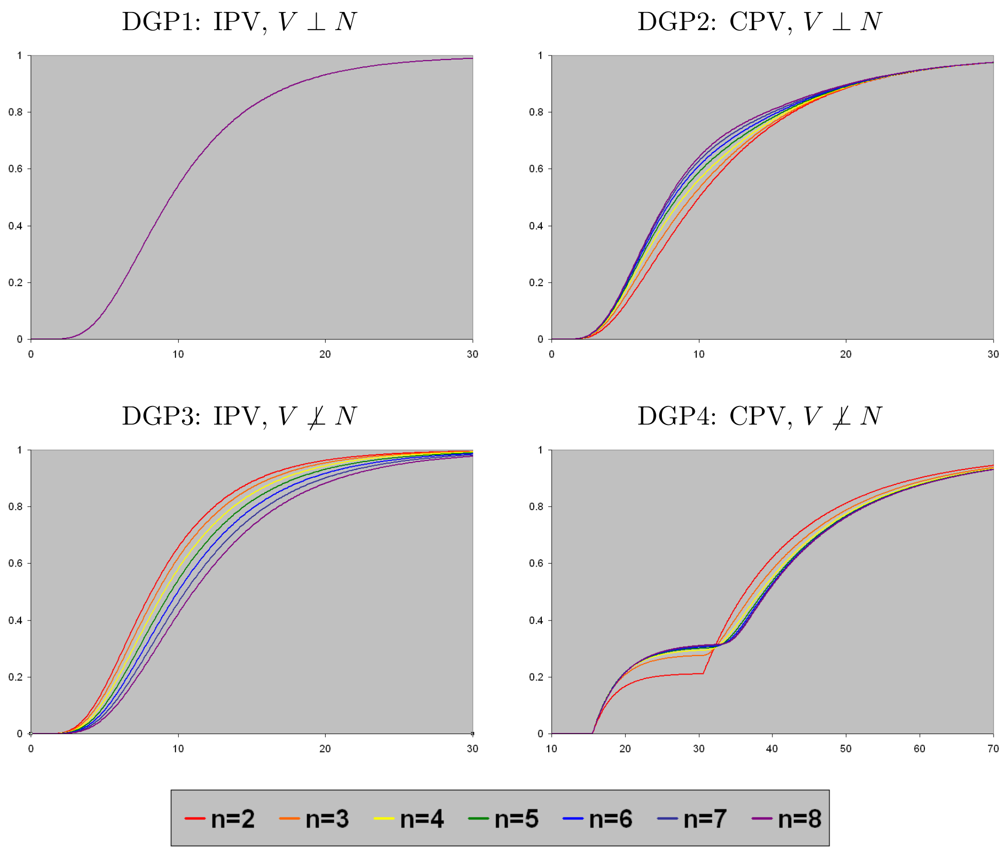
1. Values are IPV and F_V does not depend on N: specifically, $\mu = 2.25$ for every N
2. Values are independent of N, but correlated with each other via conditional independence: regardless of N, with probability $\frac{1}{2}$, $\mu = 2.0$, and with probability $\frac{1}{2}$, $\mu = 2.5$. (Variation in μ induces correlation among values.)
3. Values are IPV, but the distribution varies with N: specifically, $\mu = 2 + 0.05N$
4. Values are correlated with each other, and with N. $\mu = 2.5$ or 1.5 with probabilities $\frac{1}{3}$ and $\frac{2}{3}$ respectively, and N is determined endogenously via equilibrium play of the entry game described in Samuelson (1985). There are 12 potential bidders, each of whom learns μ and his own valuation before deciding whether to pay a cost of \$10 to participate in the auction. Bidders play a symmetric, cutoff-strategy equilibrium, with the cutoff value varying with μ;[9] this induces a positive relationship between N and μ, and therefore between N and valuations.

Consider what would happen if we ran our tests, in order, on each of these four data-generating processes. For DGP1, we would fail to reject equation 3, and conclude (correctly) that the data was consistent with both IPV and the exclusion restriction. For DGP2, we would reject equation 3 but fail to reject equation 4, (correctly in this case) rejecting IPV but not the exclusion restriction. For DGP3, we would reject equations 3 and 4, but fail to reject equation 5, concluding (correctly in this case) that the data was consistent with an IPV model violating the exclusion restriction. Finally, for DGP4, we would reject all three tests, concluding (correctly) that both IPV and the exclusion restriction failed.[10]

[9] When $\mu = 2.5$, the entry cutoff is 30.57, which is exceeded by 9.7% of bidders; when $\mu = 1.5$, the cutoff is 15.54, which is exceeded by 3.9% of bidders. By Bayes' Law, then, $\Pr(\mu = 2.5|N = n)$ is increasing in n.

[10] When neither IPV nor the exclusion restriction hold, it is not necessarily the case that equations 4 and 5 will both be violated: a similar example based on a different entry model (that of Levin and Smith (1994))

Figure 1: $\psi_{n-1:n}^{-1}\left(G_{n:n}(v)\right)$ against v under four scenarios



2.3 Summarizing the Tests

Table 1 below summarizes the five tests we are proposing: which environment each one applies to, what model it tests, and what assumptions it relies on about equilibrium bidding. For English auction data where the top two bids were not consistently close together, we would replace equations 3, 4, and 5 with their analogs given in Appendix A.2 – relaxing the assumption that $V_{n-1:n}$ was revealed by transaction price, and instead assuming it was bounded by the Haile-and-Tamer restrictions.

Table 1: Overview of Our Proposed Tests

Eq.	**Test of**	**Auctions**	N	**Bidding assumptions**
1	IPV	First-Price	Fixed	Equilibrium
2	IPV	English	Fixed	Haile-and-Tamer bidding
3	IPV $\wedge$ $V \perp N$	English	Variable	Transaction price $= V_{n-1:n}$
4	$V \perp N$	English	Variable	Transaction price $= V_{n-1:n}$
5	IPV	English	Variable	Transaction price $= V_{n-1:n}$

We assume symmetric private values throughout. Tests 4 and 5 also rely on Assumptions 2 and 3.

3 Testing These Restrictions on Auction Data

We assume the researcher observes a random sample $(X_i, N_i, \boldsymbol{B}_i)_{i=1}^{L}$ of auction outcomes from either first-price or English auctions. We are interested in testing whether the data is consistent with a data-generating process satisfying the implications of each model derived above (equations 1-5). A quick inspection shows that the testable implications of the various models are of two types:

(A) *equality* restrictions (equations 1 and 3), and

(B) *inequality* restrictions (equations 2, 4 and 5)

leads to distributions satisfying equation 5 everywhere.

The equality restrictions are similar to restrictions seen in nonparametric specification tests (see, e.g., Fan and Li (1996) and Lavergne (2001)), with the exception (in equation 3) that the equality in question involves nonlinear transformations of nonparametric functionals. This problem is generally not considered in specification tests, which typically involve *direct* comparisons between functionals (nonparametric regressions), not between nonlinear transformations of them. However, the fact that the implications to be tested involve equalities will allow us to design econometric tests in a relatively straightforward way.

The inequality restrictions are more challenging, as they involve inequalities comparing *nonlinear transformations* of nonparametric functionals, a problem that has not been considered in existing work on nonparametric tests of monotonicity and dominance. So far, the literature has focused on *direct* inequality comparisons between functionals (distributions or expectations)[11], but not between nonlinear transformations of them.

Our approach to covariates also differs somewhat from the existing literature. To illustrate this, consider the restriction that $m(x) \leq 0$ for some nonparametrically identified function m. To address this question, the existing econometric literature typically takes a Cramér-von Mises (CVM) type of approach,[12] focusing on L_p-based test-statistics of the form $\int_{\mathcal{X}} \max\{m(x), 0\}^p \cdot w(x)dx$, where the set $\mathcal{X}$ and the weighting function $w(\cdot)$ are *prespecified by the researcher.* In contrast, our test statistics will effectively use the data itself as the weighting function, focusing instead on $\int \max\{m(x), 0\}dF(x) = E\left[\max\{m(X), 0\}\right]$, which can be estimated using sample averages.[13] This eliminates the need for numerical

[11]See Ghosal, Sen, and Vaart (2000), Barrett and Donald (2003), Hall and Yatchew (2005), Lee, Linton, and Whang (2009), Lee and Whang (2009), Delgado and Escanciano (2010), Bennett (2011), and Lee, Song, and Whang (2011).

[12]Lee, Linton, and Whang (2009) use a Kolmogorov-Smirnov (KS) approach to test for first-order stochastic dominance between distribution functions. To our knowledge a KS-based procedure has not been used to test monotonicity restrictions nonparametrically when the functional of interest $m(x)$ is not a distribution function, due to the computational and theoretical complexities of this type of approach.

[13]While we believe that using $Supp(X)$ as the target testing range is a sensible approach in our setting, there are other scenarios where an L_p-based criterion is uniquely appropriate. For instance, consider an *i.i.d.* sample $(Y_i)_{i=1}^{L} \sim F$, where we want to test if $F = F^*$, where F^* is a given (parametric) distribution. In this setting, the CVM criterion $\int |F(x) - F^*(x)|dF^*(x)$ is more appropriate than $\int |F(x) - F^*(x)|dF(x) = E\left[|F(x) - F^*(x)|\right]$, since the relevant resting range should be the support of the candidate distribution F^*, not that of F. This is not the nature of our problem, since our only goal is to find out if the distribution that produced our data (whatever it may be) satisfies the testable implications of auction models described previously.

integration, which can be computationally costly, especially when x includes multiple covariates (as in many auction data sets). In addition, basing our testing procedures on expected values will produce pivotal statistics and allow us to use critical values from the standard normal distribution, avoiding the need to obtain them via resampling.[14] Of course, this means that we will actually be testing whether $m(x) \leq 0$ *almost* everywhere, i.e., with probability 1 with respect to the distribution of x induced by the data-generating process, as violations of the inequality outside the support of the data or on a measure-zero subset would not be detected.

Within the context of auctions, the closest related work is Jun, Pinkse, and Wan (2010), who develop a nonparametric test for affiliation.[15] To our knowledge, their procedure is the first such nonparametric test capable of handling continuously-distributed bids without the need for discretization. The test-statistic they propose is also of the CVM type, but it is not designed to condition on observable continuous covariates. Our tests require a procedure capable of conditioning on a potentially rich collection of covariates X with some continuously-distributed elements. For all the reasons described above, we will develop econometric tests specifically tailored to our needs.

3.1 Reframing Testable Implications as Mean-Zero Restrictions

Our goal is to present a unified approach to the five tests proposed above. We begin by showing that we can re-frame all of them as mean-zero restrictions on appropriately-defined functions. By construction, if a particular restriction is violated, the expectation in question will be strictly positive. In particular, there is no loss of information (power) by re-expressing the observable implications in terms of these mean-zero restrictions. As we mentioned above, basing our econometric tests on expected values will make them computationally easy to implement relative to alternative (e.g., L_p-based) procedures, particularly if the set of covariates X is rich.

Below, "almost all" and "almost everywhere" will consistently mean with probability 1 with respect to the probability distribution of (X, N) or $(X, N, B_{N:N})$ induced by the

[14] The asymptotically pivotal nature of our test-statistics (under the null hypothesis) would potentially allow for asymptotic refinements using bootstrap instead of normal critical values; but these improvements might come at a substantial computational cost.

[15] The applicability of the affiliation test in Jun, Pinkse, and Wan (2010) goes beyond auctions, but the latter is perhaps the most natural economic application.

data-generating process.

3.1.1 Equation 1

We are testing whether $\text{cov}(B^I, B^{II}|X,N) = 0$ holds for all realizations (x,n) of (X,N). Let B_i^I and B_i^{II} be two bids chosen at random from the vector $\boldsymbol{B}_i$ of bids in auction i, and let $\nu^I(x,n) = E(B^I|(X,N)=(x,n))$ and $\nu^{II}(x,n) = E(B^{II}|(X,N)=(x,n))$.[16] Consider the expected value

$$E\left[\left(B^I B^{II} - \nu^I(X,N)\nu^{II}(X,N)\right)\left(E(B^I B^{II}|X,N) - \nu^I(X,N)\nu^{II}(X,N)\right)\right] \qquad (1')$$

By iterated expectations, this is equal to

$$E_{X,N}\left[\left(E(B^I B^{II}|X,N) - \nu^I(X,N)\nu^{II}(X,N)\right)^2\right] = E_{X,N}\left[\text{cov}^2(B^I, B^{II}|X,N)\right]$$

Thus, the expected value $1'$ is nonnegative, and is zero if and only if equation 1 holds at almost all $(x,n) \in Supp(X,N)$. Our test statistic for equation 1 will be based on a sample analog estimator for $1'$: complete details are in the appendix, but roughly, our test statistic will be calculated as

$$\sum_{i=1}^{L} \frac{1}{L}\left(B_i^I B_i^{II} - \widehat{\nu}^I(X_i,N_i)\widehat{\nu}^{II}(X_i,N_i)\right)\left(\widehat{E}(B^I B^{II}|X_i,N_i) - \widehat{\nu}^I(X_i,N_i)\widehat{\nu}^{II}(X_i,N_i)\right)$$

where $\widehat{\nu}$ and $\widehat{E}(B^I B^{II}|X,N)$ are nonparametric kernel-based estimates. In the appendix, we will show conditions under which this test statistic will be asymptotically normal, and so, after a normalization, we can compare its value to critical values from the standard normal distribution.

3.1.2 Equation 3

The other equality test works similarly. We are interested in whether, for every (x,n,n',v), $\psi_{n-1:n}^{-1}(G_{n:n}(v|x)) = \psi_{n'-1:n'}^{-1}(G_{n':n'}(v|x))$.

We abuse notation slightly by letting $B_{k:N_i}$ denote the k^{th}-lowest bid from auction i. Let Z_i denote the transaction price of auction i, $B_{N_i:N_i}$.[17] Let

$$\lambda_{n,n'}(z|x) \quad \equiv \quad \psi_{n-1:n}\left(\psi_{n':n'-1}^{-1}\left(G_{n':n'}(z|x)\right)\right)$$

[16]Since B^I and B^{II} were chosen at random, $\nu^I(x,n) = \nu^{II}(x,n)$; they are defined separately because they will be estimated on different samples.

[17]Ideally, we would test each restriction over all v in the support of all bidders' valuations. Under the models being tested here, this support matches the support of transaction prices $B_{N:N}$.

so that equation 3 is equivalent to $G_{n,n}(z|x) = \lambda_{n,n'}(z|x)$. Take two observations $i \neq j$, and consider the expected value

$$E\left[\sum_{n' \neq N_i} \left(\mathbb{1}\{B_{N_i:N_i} \leq Z_j\} - \lambda_{N_i,n'}(Z_j|X_i)\right) \cdot \left(G_{N_i:N_i}(Z_j|X_i) - \lambda_{N_i,n'}(Z_j|X_i)\right)\right] \tag{3'}$$

By iterated expectations, this is equal to $E\left[\sum_{n' \neq N_i} \left(G_{N_i:N_i}(Z_j|X_i) - \lambda_{N_i,n'}(Z_j|X_i)\right)^2\right]$; thus, $3'$ is nonnegative, and is 0 if and only if equation 3 holds almost everywhere. We will therefore test equation 3 using the sample analog estimator of $3'$.

3.1.3 Equations 2, 4, and 5

The three inequality tests will be handled slightly differently: in particular, we do not replace distribution functions with indicator functions, but directly estimate the degree to which each inequality appears to be violated at each point. This is done purely for computational convenience, as it will enable us to use simple averages instead of higher order sums (U-statistics) in their implementation.[18] Define the three expected values

$$E\left[\sum_{k \leq N_i - 2} \max\left\{\psi^{-1}_{N_i-1:N_i}\left(G^{\Delta}_{N_i:N_i}(Z_i|X_i)\right) - \psi^{-1}_{k:N_i}\left(G_{k:N_i}(Z_i|X_i)\right)\ , 0\right\}\right] \tag{2'}$$

$$E\left[\sum_{n' < N_i} \max\left\{\psi^{-1}_{n'-1:n'}\left(G_{n':n'}(Z_i|X_i)\right) - \psi^{-1}_{N_i-1:N_i}\left(G_{N_i:N_i}(Z_i|X_i)\right)\ , 0\right\}\right] \tag{4'}$$

$$E\left[\sum_{n' < N_i} \max\left\{\psi^{-1}_{N_i-1:N_i}\left(G_{N_i:N_i}(Z_i|X_i)\right) - \psi^{-1}_{n'-1:n'}\left(G_{n':n'}(Z_i|X_i)\right)\ , 0\right\}\right] \tag{5'}$$

By construction, all three are nonnegative, and each is equal to 0 if and only if the corresponding restriction holds almost everywhere.

3.2 Asymptotic Properties and Inference

We implement our tests as sample analogs of the expected values described above, *conditional on* the covariates (X, N) (in the case of equation 1) or (X, N, Z) (equations 2, 3, 4, and 5) being within a prespecified subset of their support. (This is necessary to get the desired asymptotic properties, as the transformations $\psi^{-1}_{k:n}$ are not differentiable at 0 and 1,

[18] As Appendix B illustrates, the equality test described above does indeed require the type of expression shown in $(3')$ if the aim is to have a pivotal, asymptotically normal test-statistic.

and we therefore limit the testing range to the space where the estimated CDFs are bounded away from 0 and 1.) The details of the testing range $\mathcal{T}$ we selected, and how we actually implement the tests, are given in the appendix. For each equation $s \in \{1, 2, 3, 4, 5\}$, we will calculate a test statistic $\widehat{t}_s$ which will have, under conditions given in the appendix, the following asymptotic properties:

- For the *equality tests* ($s \in \{1, 3\}$), as $L \to \infty$,
  - (i) $\widehat{t}_s \xrightarrow{d} \mathcal{N}(0, 1)$ if equation s holds almost everywhere in $\mathcal{T}$
  - (ii) $\widehat{t}_s \xrightarrow{p} +\infty$ otherwise
- For the *inequality tests* ($s \in \{2, 4, 5\}$), as $L \to \infty$,
  - (i) $\widehat{t}_s \xrightarrow{p} 0$ if equation s holds *strictly* almost everywhere in $\mathcal{T}$
  - (ii) $\widehat{t}_s \xrightarrow{d} \mathcal{N}(0, 1)$ if equation s holds almost everywhere in $\mathcal{T}$, and holds *with equality* on a positive measure of $\mathcal{T}$
  - (iii) $\widehat{t}_s \xrightarrow{p} +\infty$ otherwise

Based on these asymptotic features, for any target significance level α, we reject the auction model in question if $\widehat{t}_s > z_\alpha = \Phi^{-1}(1 - \alpha)$, where $\Phi(\cdot)$ is the standard normal CDF. Under this rejection rule, the asymptotic probability of rejecting a valid model will be exactly α for the equality tests, and at most α for the inequality tests; and the asymptotic probability of rejecting a model when the corresponding equation does not hold almost everywhere will be 1.

3.3 Monte Carlo Experiments

In order to investigate the finite-sample power and size properties of the test-statistics proposed, we applied the tests of equations 4 and 5 on simulated data generated according to the four data-generating processes described in Section 2 and illustrated in Figure 1. The results are summarized in Tables 2 and 3 below. (The last two columns give the true (theoretical) asymptotic rejection rates for each test.)

Examining Table 2, both tests perform well on the second and third data-generating processes, where one of the two inequalities holds strictly everywhere: even on sample sizes as low as 200, the tests have reasonable size and power. On DGP 1, where both inequalities being tested hold everywhere with equality, both tests give a large number of false rejections

Table 2: Empirical Rejection Rates for Monte Carlo Exercise, DGP1-3

	$L = 200$		$L = 600$		$L = 1000$		$L \to \infty$	
Data	Eq. 4	Eq. 5	Eq. 4	Eq. 5	Eq. 4	Eq. 5	Eq. 4	Eq. 5
DGP1	38.1%	54.1%	9.9%	15.4%	4.7%	5.5%	5%	5%
DGP2	9.3%	83.9%	0.5%	78.0%	0.0%	81.2%	0%	100%
DGP3	91.4%	7.6%	99.7%	0.1%	100.0%	0.0%	100%	0%

Based on 5% target significance level, 1000 simulated samples of size L for each L and each DGP, and tuning parameter value $b_L = 0.005$.

when $L = 200$, but behave much better at $L = 600$, and are close to their asymptotic target rejection levels at $L = 1000$.

Table 3: Empirical Rejection Rates for Monte Carlo Exercise, DGP4

$L = 200$		$L = 600$		$L = 1000$		$L = 2000$		$L \to \infty$	
Eq. 4	Eq. 5	Eq. 4	Eq. 5	Eq. 4	Eq. 5	Eq. 4	Eq. 5	Eq. 4	Eq. 5
10.7%	19.8%	30.2%	16.7%	42.7%	21.2%	70.3%	23.2%	100%	100%

Based on 5% target significance level, 1000 simulated samples of size L for each L and each DGP, and tuning parameter value $b_L = 0.005$.

On DGP 4, each inequality being tested is violated on a range of v, which means that each inequality also holds on a portion of the range. As Table 3 shows, the tests therefore have little power on small samples. At $L = 2000$, however, the test has reasonable power to reject equation 4, though less to reject equation 5. (This is in line with Figure 1, which shows that equation 4 is violated for a wider range of v.)[19]

Along with kernel and bandwidth choices, the inequality tests all depend on the value of a "tuning parameter" b_L defined and discussed in Appendix B. While the effect of b_L vanishes asymptotically, on small samples, the choice represents a tradeoff between the power and

[19]That the test of equation 4 has more power than the test of equation 5 on DGP 4 – that is, that in the event both assumptions are violated, we seem better able to detect a violation of independence between valuations and N than a violation of IPV – is not a general result. For different values of the parameters in the example, the test of equation 5 might have more power. For a different example based on the entry game of Levin and Smith (1994), the test has more power against equation 4 but none against equation 5.

size of the test. In Tables B.1-B.4 in Appendix B.7, we show Monte Carlo results for various choices of b_L; here, we show results for one such choice.

4 Application to USFS Timber Data

4.1 Timber Auctions

We apply our tests to data from timber auctions run by the Unites States Forest Service. These are auctions for the right to harvest timber on a tract of public land. Auctions are heterogeneous due to both differences in the tracts themselves (for example, in the type and density of timber present) and differences in the lease contracts (such as the length of the lease).

Prior to each auction, the Forest Service conducts a "cruise" of the tract and publishes detailed information on the tract for potential bidders. The timber data therefore includes a rich set of auction covariates, corresponding to the information the bidders had about the tract. It has often been argued that these covariates therefore capture all *systematic* demand shifters for a tract of timber, and that any remaining variation in valuations is likely bidder-specific (such as differences in costs and capacity) and hence independent. We can now explicitly test this claim and evaluate whether independence or correlation better models valuations in the timber data after we control for these covariates. Another appealing feature of the data is that since the USFS conducts both ascending and first-price auctions, we are able to conduct our analysis across both auction formats, which allows us to study whether conclusions concerning the statistical nature of valuations is consistent across formats.

A number of other papers have studied Forest Service auctions empirically. Nearly all have done so within the framework of independent private values.[20] Two recent papers, however, have found indirect evidence of correlation among valuations. Athey, Levin, and Seira (2011) estimate a model allowing for unobserved heterogeneity on data from first-price auctions, noting that "an extension along these lines appears crucial as... we estimate implausibly high bid margins when we fail to account for within-auction bid correlation."[21]

[20] See, e.g., Baldwin, Marshall, and Richard (1997); Haile (2001); Haile, Hong, and Shum (2003); Lu and Perrigne (2008); Athey and Levin (2001); and Haile and Tamer (2003).

[21] Another key insight of Athey, Levin, and Seira (2011) is that there are ex-ante asymmetries between two

In Aradillas-López, Gandhi, and Quint (2011), we estimate a model allowing for correlated values on English auction data; we find the estimates (of expected profit as a function of reserve price) differ significantly from estimates made under the assumption of independence.

Thus, while independence is a standard assumption in empirical work, both in general and applied to these particular auctions, there does exist some recent evidence to suggest this assumption might be worrisome. Here, we directly examine the testable implications of independence on this data.

4.2 Data

Data on all USFS timber auctions held between 1978 and 1996 was made available to us by Phil Haile. We focus on the auctions held between 1982 and 1990, as the reserve price policy in place was stable during that period, and the reserve prices used were generally recognized not to be binding, allowing us to infer the number of potential bidders from the number who submitted bids.[22] We use auctions from Region 6 (mostly Oregon), which relative to other regions provides a large sample of English auctions.

We use the same conventions as Haile and Tamer (2003) (which were motivated by the previous literature) to select auctions most likely to satisfy the assumption of private values. In particular, we focus on sales whose contracts expire within a year, to minimize the effect of resale possibilities on valuations. And we focus on scaled sales, where bids are in dollars per unit of timber actually harvested, and therefore common-value uncertainty about the total amount of timber should not affect valuations. Within this selection of auctions, there are both first-price and ascending auctions, allowing us to apply both tests to the same environment.

For each auction in the sample, in addition to the number of bidders and their bids,

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types of bidders: mills and loggers. As we discuss in Aradillas-López, Gandhi, and Quint (2011), our model can accommodate this type of asymmetry, if we imagine each bidder being independently and randomly either a miller or a logger, so that the marginal distribution of each bidder's valuations is the appropriate mixture between a "mill distribution" and a "logger distribution". In first-price auctions, bids depend on beliefs about one's opponents, so the identities of the other bidders would affect bidding; but in ascending auctions, bidding is in dominant strategies and this information has no effect.

[22] Campo, Guerre, Perrigne, and Vuong (2002) write, "It is well known that this reserve price does not act as a screening device to participating," and perform analysis that confirms that "the possible screening effect of the reserve price is negligible" (p. 33). See also Haile (2001), Froeb and McAfee (1988), and Haile and Tamer (2003).

our data contains detailed covariate information about the auction from the government's cruise report. We control for four auction covariates which have been emphasized in the previous literature as being relevant demand shifters: the density of timber (timber volume over acres in the tract, which we label X^1); the government's appraisal value of the timber (which we label X^2); the estimated profit from manufacturing the timber (sales value minus manufacturing cost, X^3); and the species concentration (HHI computed as a function of the volume of various species present, X^4). Bids and monetary covariates (X^2 and X^3) are all measured in 1983 dollars. We let $X = (X^1, X^2, X^3, X^4)$ refer to the vector of covariates, and $X_i = (X_i^1, X_i^2, X_i^3, X_i^4)$ the data corresponding to the i^{th} auction.

4.3 First-Price Auctions

We drop auctions with one bidder, as there is no covariance relationship to test; and auctions with 12 bidders, as this appears to be top-coding for "more than 11". This provides us with 192 auctions with N between 2 and 11. A simple regression of B^I (the first randomly chosen bid) on B^{II} and the other covariates (X, N) reveals that B^{II} still has significant explanatory power. Our non-parametric test will essentially be checking whether that relationship persists when the other covariates are controlled for nonparametrically.

The results of the test of equation 1 on the first-price data are given in Table 4.

Table 4: Test Results on First-Price Auction Timber Data

Eq.	**Test of**	**Auctions**	N	**Bid assump**	$\widehat{t}$	**Outcome**
1	IPV	First-Price	Fixed	Equilibrium	**8.19**	Reject

Critical values to reject a model are 1.64 for $\alpha = 5\%$ and 2.33 for $\alpha = 1\%$.

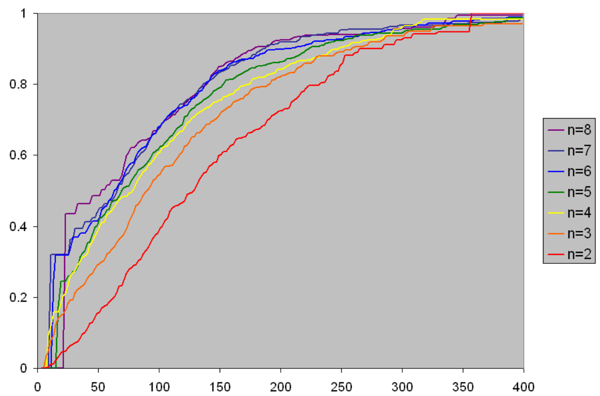
As shown in Appendix B.8, rejection of equation 1 is consistent across a wide range of bandwidth values. One might be concerned that with a small sample size, the nonparametric estimates of $E(B|X, N)$ could be unreliable, which could introduce spurious correlation among bids. However, at the median value of (X, N), where $E(B|X, N)$ should be most accurately estimated, we estimate the conditional correlation between two bids to be 0.62. Thus, while the sample size is quite small, the evidence nonetheless appears to suggest the presence of correlation.

4.4 English Auctions

We once again drop auctions with $N = 1$ (this time because there is no second-highest bidder to whose value we can link the transaction price) and $N = 12$, leaving us with 2,036 ascending auctions.

Before performing the formal tests, we can get a sense of what to expect by calculating nonparametric estimates of $\psi^{-1}_{n-1:n}(G_{n:n}(v|x))$ and checking visually how this varies with n. Equations 3, 4 and 5 are claims that this should be constant, increasing, and decreasing in n, respectively. Figure 2 shows these estimates, for x equal to the sample average and n varying from 2 to 8, with v on the x-axis.[23]

Figure 2: $\psi^{-1}_{n-1:n}\big(\widehat{G}_{n:n}(v|x)\big)$ at $x = \overline{X}$ for various values of N



These curves seem very clearly to increase with n; the results are similar if other values of x are used. Thus, it seems likely that formal testing will reject 3 and 5 (IPV with or without valuations varying with N) but fail to reject 4 (correlated values which are independent of N).

The results of the four tests on ascending auctions are summarized in Table 5 below. Appendix B.8 shows results of each test for a range of values of the bandwidths and tuning parameters involved; the results presented here are based on one choice of values. (Although the exact calculated value of each test statistic is sensitive to the choice, the qualitative

[23] $\widehat{G}_{n:n}$ is estimated using the same kernels and bandwidths as are used for testing, which are described in the appendix.

findings – in particular, whether each test rejects the model in question – are consistent across different values.)

Table 5: Test Results on Ascending Auction Timber Data

Eq.	Test of	Auctions	N	Bid assump	$\widehat{t}$	Outcome
2	IPV	English	Fixed	H&T bidding	**28.67**	Reject
3	IPV $\wedge$ $V \perp N$	English	Variable	$B_{n:n} = V_{n-1:n}$	**61.29**	Reject
4	$V \perp N$	English	Variable	$B_{n:n} = V_{n-1:n}$	**0.05**	Fail to Reject
5	IPV	English	Variable	$B_{n:n} = V_{n-1:n}$	**19.59**	Reject

Critical values to reject a model are 1.64 for $\alpha = 5\%$ and 2.33 for $\alpha = 1\%$.

These results paint a very consistent picture of the timber data. Both testing methods – comparing winning to losing bids in auctions of the same size, and comparing transaction prices across auctions of different sizes – allow us to reject independence of valuations, and instead give strong evidence of positive correlation among valuations. On the other hand, we fail to reject a model of correlated values which are independent of N; thus, the exclusion restriction appears plausible in the ascending auction data.

5 Conclusion

In this paper, we have presented a new strategy for testing implications of standard assumptions in auction theory, in particular, the assumption of independent private values which is the basis for much empirical work. Using widely-studied data from USFS timber auctions, we are able to reject independence in both first-price and ascending auctions. Due to the rich set of available covariates, the timber data might be considered nearly a "best-case scenario" for the plausibility of an IPV model; our results, however, suggest that conditioning on a seemingly large set of covariates is not always enough to ensure independence.

Of course, when the independence assumption is rejected, this does not automatically invalidate policy conclusions derived from IPV-based analysis; for a given environment, it remains an empirical question whether the correlation among values is economically important. In Aradillas-López, Gandhi, and Quint (2011), we find that for the English auctions in the timber data, ignoring correlation would lead to reserve prices between 15% and 47% higher than the optimal ones. At those reserve prices, expected profits would be overestimated by between 6% and 18%; the gains from setting the reserve price optimally, relative to simply setting it equal to the seller's valuation, would be overstated by between 86% and 202%.

This suggests that even in settings with rich observable covariates, independence of values may not be an innocuous assumption. The testable implications, and the nonparametric testing strategy, introduced in this paper provide researchers a starting point for assessing what empirical model is most appropriate for a particular setting. Our testing strategy could similarly be used to validate key modeling assumptions in other settings, such as strategic complementarities in simultaneous-move games; or to test economic hypotheses, such as heterogeneity of treatment effects or the presence of information asymmetries.[24]

[24]See de Paula and Tang (2011), Aradillas-López and Gandhi (2011), Imbens and Wooldridge (2009), Lee and Whang (2009), and Chiappori, Jullien, Salanié, and Salanié (2006), as cited earlier.

A Appendix – Extensions and Omitted Proofs

A.1 Proof of Proposition 2

Under the Haile-and-Tamer bidding assumptions, $B_{k:n} \leq V_{k:n}$, which implies $G_{k:n}(v|x) \geq F_{k:n}(v|x)$; and , $V_{n-1:n} \leq B_{n:n} + \Delta$, which implies $F_{n-1:n}(v|x) \geq G^{\Delta}_{n:n}(v|x)$. So under IPV,

$$\begin{aligned} \psi^{-1}_{k:n}\left(G_{k:n}(v|x)\right) &\geq \psi^{-1}_{k:n}\left(F_{k:n}(v|x)\right) = F_V(v|x) \\ &= \psi^{-1}_{n-1:n}\left(F_{n-1:n}(v|x)\right) \geq \psi^{-1}_{n-1:n}\left(G^{\Delta}_{n:n}(v|x)\right) \end{aligned}$$

For the second part, fix n, x, and v, and let $\Pr(v|m)$ denote the probability that $V_i < v$, conditional on exactly m of the other $n-1$ valuations being less than v. Suppress the dependence of value distributions on x. Let P_i denote the probability that exactly i valuations are greater than or equal to v, so $P_0 = F_{n:n}(v)$, $P_n = 1 - F_{1:n}(v)$, and $P_i = F_{n-i:n}(v) - F_{n-i+1:n}(v)$ for $1 \leq i < n$. Let $\Pr(m)$ be the probability that $V_1, \ldots, V_m \geq v$ and $V_{m+1}, \ldots, V_{n-1} < v$. By symmetry, $P_{i+1} = {}_nC_{i+1} \Pr(i)(1 - \Pr(v|n-1-i))$ and $P_i = {}_nC_i \Pr(i) \Pr(v|n-1-i)$; so

$$\frac{\frac{1}{{}_nC_{i+1}} P_{i+1}}{\frac{1}{{}_nC_i} P_i} = \frac{\Pr(i)(1 - \Pr(v|n-1-i))}{\Pr(i)\Pr(v|n-1-i)} = \frac{1 - \Pr(v|n-1-i)}{\Pr(v|n-1-i)}$$

By assumption, this is weakly increasing in i, and strictly increasing for some i.

Let $p = \psi^{-1}_{n-1:n}(F_{n-1:n}(v))$, and let $P^I_i = {}_nC_i p^{n-i}(1-p)^i$, and $F^I_{k:n}(v) = \psi_{k:n}(p) = \sum_{i=0}^{n-k} P^I_i$, so P^I_i and $F^I_{k:n}$ are what P_i and $F_{k:n}$ would be if valuations were independent draws from the distribution $F_V(\cdot) = \psi^{-1}_{n-1:n}(F_{n-1:n}(\cdot))$. By construction,

$$\frac{\frac{1}{{}_nC_{i+1}} P^I_{i+1}}{\frac{1}{{}_nC_i} P^I_i} = \frac{1-p}{p}$$

and therefore does not vary with i. Note that $P_0 + P_1 = F_{n-1:n}(v) = F^I_{n-1:n}(v) = P^I_0 + P^I_1$.

Claim 1. $P_0 > P^I_0$.

Proof is by contradiction. Since $P_0 + P_1 = P^I_0 + P^I_1$, if $P_0 \leq P^I_0$, then $P_1 \geq P^I_1$. Then

$$\frac{\frac{1}{{}_nC_2} P_2}{\frac{1}{{}_nC_1} P_1} \geq \frac{\frac{1}{{}_nC_1} P_1}{\frac{1}{{}_nC_0} P_0} \geq \frac{\frac{1}{{}_nC_1} P^I_1}{\frac{1}{{}_nC_0} P^I_0} = \frac{1-p}{p} = \frac{\frac{1}{{}_nC_2} P^I_2}{\frac{1}{{}_nC_1} P^I_1}$$

and so since $P_1 \geq P^I_1$ and $\frac{P_2}{P_1} \geq \frac{P^I_2}{P^I_1}$, then $P_2 \geq P^I_2$. Similarly, since $\frac{P_3}{P_2} \geq \frac{P_2}{P_1} \geq \frac{P^I_2}{P^I_1} = \frac{P^I_3}{P^I_2}$, $P_3 \geq P^I_3$; and likewise, $P_i \geq P^I_i$ for every $i > 3$, with at least one strict inequality due to the requirement that Assumption 2 holds strictly. This leads to $\sum_{i=0}^n P_i > \sum_{i=0}^n P^I_i$, which is a contradiction since both must be equal to 1.

Claim 2. $P_2 < P_2^I$.

Since $P_1 < P_1^I$, if $P_2 \geq P_2^I$, then $\frac{P_2}{P_1} > \frac{P_2^I}{P_1^I}$, and so $\frac{P_3}{P_2} > \frac{P_3^I}{P_2^I}$ giving $P_3 > P_3^I$ and so on; this would give $P_0 + P_1 = P_0^I + P_1^I$, $P_2 \geq P_2^I$, and $P_i > P_i^I$ for $i \geq 3$, yielding a contradiction.

(Note that if $\Pr(v|m)$ is only weakly increasing in m, everything up to here applies as weak inequalities and $P_2 \leq P_2^I$, which will be used below in the proof of Proposition 3.)

Claim 3. If $P_k > P_k^I$, then $P_{k'} > P_{k'}^I$ for all $k' > k$.

We know that $P_1 < P_1^I$. Let j denote the smallest $i > 0$ such that $P_i > P_i^I$. This means $P_j > P_j^I$ but $P_{j-1} \leq P_{j-1}^I$, and therefore

$$\frac{\frac{1}{{}_nC_j} P_j}{\frac{1}{{}_nC_{j-1}} P_{j-1}} > \frac{\frac{1}{{}_nC_j} P_j^I}{\frac{1}{{}_nC_{j-1}} P_{j-1}^I} = \frac{1-p}{p}$$

Which means that

$$\frac{\frac{1}{{}_nC_{j+1}} P_{j+1}}{\frac{1}{{}_nC_j} P_j} \geq \frac{\frac{1}{{}_nC_j} P_j}{\frac{1}{{}_nC_{j-1}} P_{j-1}} > \frac{1-p}{p} = \frac{\frac{1}{{}_nC_{j+1}} P_{j+1}^I}{\frac{1}{{}_nC_j} P_j^I}$$

and so $P_j > P_j^I$ and $\frac{P_{j+1}}{P_j} > \frac{P_{j+1}^I}{P_j^I}$, meaning $P_{j+1} > P_{j+1}^I$. Likewise,

$$\frac{\frac{1}{{}_nC_{j+2}} P_{j+2}}{\frac{1}{{}_nC_{j+1}} P_{j+1}} \geq \frac{\frac{1}{{}_nC_{j+1}} P_{j+1}}{\frac{1}{{}_nC_j} P_j} > \frac{1-p}{p} = \frac{\frac{1}{{}_nC_{j+2}} P_{j+2}^I}{\frac{1}{{}_nC_{j+1}} P_{j+1}^I}$$

and so $P_{j+2} > P_{j+2}^I$, and so on, proving the claim.

Claim 4. For $k > 1$, if $F_{n-k:n}(v) \geq F_{n-k:n}^I(v)$ then $P_k > P_k^I$.

By construction, $F_{n-k:n}(v) = \sum_{i=0}^k P_i$ and $F_{n-k:n}^I(v) = \sum_{i=0}^k P_i^I$. We know that $P_0 + P_1 = P_0^I + P_1^I$, and $P_2 < P_2^I$; so if $\sum_{i=0}^k P_k \geq \sum_{i=0}^k P_k^I$, there must be some j $(2 < j \leq k)$ such that $P_j > P_j^I$. But then by the previous claim, $P_k > P_k^I$.

Claim 5. For $1 < k < n$, $F_{n-k:n}(v) < F_{n-k:n}^I(v)$.

If $F_{n-k:n}(v) \geq F_{n-k:n}^I(v)$, then by the last claim, $P_k > P_k^I$. But then by the previous claim, $P_{k'} > P_{k'}^I$ for all $k' > k$. So

$$1 = F_{n-k:n}(v) + \sum_{k'>k} P_k > F_{n-k:n}^I(v) + \sum_{k'>k} P_k^I = 1$$

a contradiction. So it must be that $F_{n-k:n}(v) < F_{n-k:n}^I(v)$. But

$$F_{n-k:n}^I(v) = \psi_{n-k:n}(p) = \psi_{n-k:n}\left(\psi_{n-1:n}^{-1}(F_{n-1:n}(v))\right)$$

so the last claim is that $F_{n-k:n}(v) < \psi_{n-k:n}\left(\psi_{n-1:n}^{-1}(F_{n-1:n}(v))\right)$, proving the proposition.

A.2 Generalizing Prop. 3 and 4 to Haile-and-Tamer Bidding

Propositions 3 and 4 have direct analogs which rely on the bidding assumptions of Haile and Tamer rather than the requirement that $G_{n:n} = F_{n-1:n}$:

Proposition A1. *Assume bidding behavior satisfies the Haile-and-Tamer assumptions and valuations are independent of N.*

(a) Under IPV, for any (x, n, n', v),

$$\psi^{-1}_{n-1:n}\left(G_{n-1:n}(v|x)\right) \quad \geq \quad \psi^{-1}_{n'-1:n'}\left(G^{\Delta}_{n':n'}(v|x)\right) \tag{6}$$

(b) Under Assumption 2, for any (x, n, n', v),

$$n > n' \quad \longrightarrow \quad \psi^{-1}_{n-1:n}\left(G_{n-1:n}(v|x)\right) \quad \geq \quad \psi^{-1}_{n'-1:n'}\left(G^{\Delta}_{n':n'}(v|x)\right) \tag{7}$$

Proposition A2. *Assume bidding behavior satisfies the Haile-and-Tamer assumptions and Assumption 3 holds. Under IPV, for any (x, n, n', v),*

$$n > n' \quad \longrightarrow \quad \psi^{-1}_{n-1:n}\left(G^{\Delta}_{n:n}(v|x)\right) \quad \leq \quad \psi^{-1}_{n'-1:n'}\left(G_{n'-1:n'}(v|x)\right) \tag{8}$$

These are proved side-by-side with Propositions 3 and 4 below.

A.3 Proof of Propositions 3 and A1

Part a. Under IPV and the exclusion restriction,

$$\begin{aligned} \psi^{-1}_{n-1:n}\left(F_{n-1:n}(v|x)\right) &= F_V(v|x,n) \\ &= F_V(v|x,n') \quad = \quad \psi^{-1}_{n'-1:n'}\left(F_{n'-1:n'}(v|x)\right) \end{aligned}$$

If $B_{n:n} = V_{n-1:n}$, then $G_{n:n}(v|x) = F_{n-1:n}(v|x)$ and $G_{n':n'}(v|x) = F_{n'-1:n'}(v|x)$, giving (3). Under Haile-and-Tamer bidding, $G_{n-1:n}(v|x) \geq F_{n-1:n}(v|x)$ and $F_{n'-1:n'}(v|x) \geq G^{\Delta}_{n':n'}(v|x)$, giving (6).

Part b. Fix n, x, and v. As above, let $p = \psi^{-1}_{n-1:n}(F_{n-1:n}(v|x))$, and let P_i be the probability that exactly i (of n) valuations are at least v. Under Assumption 2, as noted above in the proof of Proposition 2, $P_2 \leq {}_nC_2 p^{n-2}(1-p)^2$. If valuations are independent of N, plugging

$r = n-2$ into equation 9 of Athey and Haile (2002) and rearranging gives

$$\begin{aligned} F_{n-2:n-1}(v|x) &= F_{n-1:n}(v|x) + \tfrac{2}{n}\left[F_{n-2:n}(v|x) - F_{n-1:n}(v|x)\right] \\ &= np^{n-1} - (n-1)p^n + \tfrac{2}{n}P_2 \\ &\leq np^{n-1} - (n-1)p^n + \tfrac{2}{n}\tfrac{n(n-1)}{2}p^{n-2}(1-p)^2 \\ &= (n-1)p^{n-2} - (n-2)p^{n-1} \\ &= \psi_{n-2:n-1}(p) \\ &= \psi_{n-2:n-1}\left(\psi^{-1}_{n-1:n}\left(F_{n-1:n}(v|x)\right)\right) \end{aligned}$$

or $\psi^{-1}_{n-2:n-1}(F_{n-2:n-1}(v|x)) \leq \psi^{-1}_{n-1:n}(F_{n-1:n}(v|x))$; if Assumption 2 holds strictly at (v,x,n), then (from the proof of Proposition 2 above) $P_2 < {}_nC_2 p^{n-2}(1-p)^2$ and this holds strictly. From there, $G_{n:n} = F_{n-1:n}$ and $G_{n':n'} = F_{n'-1:n'}$ establish (4), and $G_{n-1:n} \geq F_{n-1:n}$ and $F_{n'-1:n'} \geq G^{\Delta}_{n':n'}$ establish (7) under Haile-and-Tamer bidding.

A.4 Proof of Propositions 4 and A2

Let $n > n'$; under Assumption 3, $F_V(\,\cdot\,|x,n) \succsim_{FOSD} F_V(\,\cdot\,|x,n')$, so

$$\begin{aligned} \psi^{-1}_{n-1:n}\left(F_{n-1:n}(v|x)\right) &= F_V(v|x,n) \\ &\leq F_V(v|x,n') = \psi^{-1}_{n'-1:n'}\left(F_{n'-1:n'}(v|x)\right) \end{aligned}$$

Again, if $B_{n:n} = V_{n-1:n}$, then $G_{n:n} = F_{n-1:n}$ and $G_{n':n'} = F_{n'-1:n'}$, and (5) follows, with strict inequality whenever $F_V(v|x,n) < F_V(v|x,n')$; under Haile-and-Tamer bidding, $G^{\Delta}_{n:n} \leq F_{n-1:n}$ and $G_{n'-1:n'} \geq F_{n'-1:n'}$, and (8) follows.

A.5 Violations of Equation 4 Under Two Entry Models

Here, we show how dependence of valuations on N generated by two standard models of endogenous participation in auctions would lead to rejection of the exclusion restriction due to a violation of Equation 4.

Consider a model of independent private values with unobserved heterogeneity. There is a one-dimensional variable $\theta \in \Re$ which is observed by bidders but not the analyst. Valuations are *i.i.d.* $\sim F_V(\cdot \mid \theta)$, and $\theta > \theta'$ implies $F(\,\cdot\,|\theta) \succsim_{FOSD} F(\,\cdot\,|\theta')$. For each θ, assume $F_V(\cdot|\theta)$ is twice differentiable and has bounded support $[\underline{v}, \overline{v}]$. Let $f_V(\cdot \mid \theta)$ denote the density function.

Now, we apply two standard models of endogeous entry to this environment. In the first model, that of Levin and Smith (1994), there are $\overline{n}$ potential bidders, who each observe θ

but not their own valuations before deciding whether to enter (in which case they incur a cost c and participate in the auction) or not (earning a payoff of 0). Bidders play a different symmetric mixed strategy for each realization of θ, leading to a stochastic N with a different distribution for each θ.

In the second model, that of Samuelson (1985), bidders observe both θ and their own valuation before making their entry decision, and play a different pure-strategy symmetric equilibrium in cutoff strategies for each θ.

Proposition A3. *In the Levin-Smith entry game, if $f_V(\overline{v}|\theta)$ and the equilibrium entry probability are both strictly increasing in θ, then the valuations generated would violate Equation* (4) *over some range of v.*

In the Sameulson entry game, if valuations and θ are related via the Strict Monotone Likelihood Ratio Property, then the valuations generated would violate (4) *over some range of v.*

Proof. The Taylor expansion of $\psi^{-1}_{n-1:n}(F_{n-1:n}(v))$ around $v = \overline{v}$, after a lot of algebra, gives

$$\psi^{-1}_{n-1:n}(F_{n-1:n}(v)) = 1 - (\overline{v} - v)\sqrt{E_{\theta|n}\left(f_V(\overline{v}|\theta)\right)^2} + O\left((\overline{v} - v)^2\right) \tag{9}$$

Let $n > n'$. If $\theta|N = n \succ_{FOSD} \theta|N = n'$ and $f_V(\overline{v}|\theta)$ is increasing in θ, then $E_{\theta|n}\left(f_V(\overline{v}|\theta)\right)^2 > E_{\theta|n'}\left(f_V(\overline{v}|\theta)\right)^2$, in which case $\psi^{-1}_{n-1:n}(F_{n-1:n}(v)) < \psi^{-1}_{n'-1:n'}(F_{n'-1:n'}(v))$ for v sufficiently close to $\overline{v}$, violating (4).

For the Levin-Smith result, this is all we need. We showed in Aradillas-López, Gandhi, and Quint (2011) that if the entry probability is increasing in θ, then $n > n'$ implies $\theta|N = n \succsim_{FOSD} \theta|N = n'$; the same argument shows this is strict when the entry probability is strictly increasing.

In the Samuelson game, the entry cutoff $v^*(\theta)$ is the solution to $vF_V^{\bar{n}-1}(v|\theta) = c$. Under the strict MLRP, $F_V(v|\theta)$ is strictly decreasing in θ on $(\underline{v}, \overline{v})$, so $v^*(\theta)$ and $1 - F_V(v^*(\theta)|\theta)$ are both strictly increasing in θ; so $\theta|N = n \succ_{FOSD} \theta|N = n'$ when $n > n'$, as above. But now we require the density of valuations at $\overline{v}$ *conditional on entry* to be strictly increasing in θ. This density can be written as

$$\frac{f_V(\overline{v}|\theta)}{1 - F_V(v^*(\theta)|\theta)} = \frac{f_V(\overline{v}|\theta)}{\int_{v^*(\theta)}^{\overline{v}} f_V(v|\theta)dv} = \left(\int_{v^*(\theta)}^{\overline{v}} \frac{f_V(v|\theta)}{f_V(\overline{v}|\theta)}dv\right)^{-1}$$

Since $v^*(\theta)$ is increasing in θ, an increase in θ shrinks the interval $[v^*(\theta), \overline{v}]$ over which the integral is taken; and if v and θ are related by the strict MLRP, since $v < \overline{v}$, $\frac{f_V(v|\theta)}{f_V(\overline{v}|\theta)}$ is strictly decreasing in θ. So $\int_{v^*(\theta)}^{\overline{v}} \frac{f_V(v|\theta)}{f_V(\overline{v}|\theta)} dv$ is strictly decreasing in θ, meaning $\frac{f_V(\overline{v}|\theta)}{1-F_V(v^*(\theta)|\theta)}$ is strictly increasing in θ; so (9) gives $\psi_{n-1:n}^{-1}(F_{n-1:n}(v)) < \psi_{n'-1:n'}^{-1}(F_{n'-1:n'}(v))$ for v close to $\overline{v}$.

B Econometric Tests and Their Asymptotic Properties

Here we describe the statistical tests we propose, along with their asymptotic properties. The proofs to all results can be found in the Supplementary Appendix.

B.1 Preliminary Nonparametric Estimators

A maintained assumption throughout is that we observe an *i.i.d.* sample $(X_i, N_i, \boldsymbol{B}_i)_{i=1}^L$ of auction covariates, number of bidders, and bids from either a series of first-price, button, or ascending auctions. Let $U_i = (X_i, N_i, \boldsymbol{B}_i)$. As in the text, let $B_{k:N_i}$ denote the k^{th}-lowest element of $\boldsymbol{B}_i$, and $Z_i = B_{N_i:N_i}$. If the data comes from first-price auctions, then for each observation, we will also choose one bid at random and label it B_i^I, and a second bid at random out of the remaining bids and label it B_i^{II}. Let X_i^c and X_i^d denote the continuous and discrete elements of X_i, respectively, and let $q = \dim X^c$.

Kernels

Let $K : \mathbb{R}^q \to \mathbb{R}$ be a kernel function and let $h_{_L}$ be a nonnegative bandwidth sequence converging to zero.

Assumption B.1. The kernel $K : \mathbb{R}^q \to \mathbb{R}$ (where q is the number of continuous covariates) is Lipschitz-continuous, bounded, and symmetric around zero. It is also bias-reducing of order M (That is, letting $\psi \equiv (\psi_1, \ldots, \psi_q)'$, then K satisfies $\int K(\psi)d\psi = 1$, $\int \left(\psi_1^{c_1} \cdots \psi_q^{c_q}\right) K(\psi) d\psi_1 \cdots d\psi_q = 0$ for all $1 \leq c_1 + \cdots + c_q \leq M-1$ and $\int \|\psi\|^M \cdot |K(\psi)| d\psi < \infty$.) There exists a sufficiently small $\delta > 0$ such that $L^{1+\delta} \cdot h_{_L}^{M+q} \longrightarrow 0$ and $L^{1-\delta} \cdot h_{_L}^{2q} \longrightarrow \infty$.

For our empirical application, we use a multiplicative kernel of the form $K(\psi) = \prod_{\ell=1}^{q} k(\psi_\ell)$, where $k(z) = \sum_{j=1}^M b_j \cdot \left(s^2 - z^2\right)^{2j} \cdot \mathbb{1}\{|z| \leq s\}$ with the coefficients b_j chosen to satisfy the bias-reducing restrictions given the chosen support $[-s, s]$. We will impose additional assumptions on the rate of convergence of h_L as needed below.

"Leave-one-out" and "Leave-two-out" Estimators

The following "leave-one-out" estimators are estimates of a density, conditional expectation, or distribution function calculated separately for each observation using the other $L-1$ observations. The first set will be used in our test of Equation 1. For each $i \in \{1, \ldots, L\}$ and any $x = (x^c, x^d)$, n, and b, define

$$\begin{aligned}
\widehat{f}^{-i}_{X,N}(x,n) &= \frac{1}{(L-1)} \cdot \frac{1}{h_L^q} \sum_{j \neq i} K\left(\frac{X_j^c - x^c}{h_L}\right) \cdot \mathbb{1}\{X_j^d = x^d\} \cdot \mathbb{1}\{N_j = n\} \\
\widehat{\nu}^{I^{-i}}(x,n) &= \frac{1}{\widehat{f}^{-i}_{X,N}(x,n)} \frac{1}{(L-1)} \cdot \frac{1}{h_L^q} \sum_{j \neq i} B_j^I \cdot K\left(\frac{X_j^c - x^c}{h_L}\right) \cdot \mathbb{1}\{X_j^d = x^d\} \cdot \mathbb{1}\{N_j = n\} \\
\widehat{\nu}^{II^{-i}}(x,n) &= \frac{1}{\widehat{f}^{-i}_{X,N}(x,n)} \frac{1}{(L-1)} \cdot \frac{1}{h_L^q} \sum_{j \neq i} B_j^{II} \cdot K\left(\frac{X_j^c - x^c}{h_L}\right) \cdot \mathbb{1}\{X_j^d = x^d\} \cdot \mathbb{1}\{N_j = n\} \\
\widehat{E}^{-i}\left[B^I B^{II}|x,n\right] &= \frac{1}{\widehat{f}^{-i}_{X,N}(x,n)} \frac{1}{(L-1)} \cdot \frac{1}{h_L^q} \sum_{j \neq i} B_j^I \cdot B_j^{II} \cdot K\left(\frac{X_j^c - x^c}{h_L}\right) \cdot \mathbb{1}\{X_j^d = x^d\} \cdot \mathbb{1}\{N_j = n\}, \\
\widehat{cov}^{-i}\left(B^I, B^{II}|x,n\right) &= \widehat{E}^{-i}\left[B^I B^{II}|x,n\right] - \widehat{\nu}^{I^{-i}}(x,n)\widehat{\nu}^{II^{-i}}(x,n).
\end{aligned} \tag{10}$$

The rest will be used in our tests of Equations 2, 4 and 5. For each $i \in \{1, \ldots, L\}$ and any $x = (x^c, x^d)$, n, and $k \leq n$, define

$$\begin{aligned}
\widehat{f}^{-i}_{X,N}(x,n) &= \frac{1}{(L-1)} \cdot \frac{1}{h_L^q} \sum_{j \neq i} K\left(\frac{X_j^c - x^c}{h_L}\right) \cdot \mathbb{1}\{X_j^d = x^d\} \cdot \mathbb{1}\{N_j = n\} \\
\widehat{G}^{-i}_{k:n}(z|x) &= \frac{1}{\widehat{f}^{-i}_{X,N}(x,n)} \frac{1}{(L-1)} \cdot \frac{1}{h_L^q} \sum_{j \neq i} \mathbb{1}\{B_{k:N_j} \leq z\} \cdot K\left(\frac{X_j^c - x^c}{h_L}\right) \cdot \mathbb{1}\{X_j^d = x^d\} \cdot \mathbb{1}\{N_j = n\} \\
\widehat{G}^{\Delta^{-i}}_{n:n}(z|x) &= \frac{1}{\widehat{f}^{-i}_{X,N}(x,n)} \frac{1}{(L-1)} \cdot \frac{1}{h_L^q} \sum_{j \neq i} \mathbb{1}\{B_{N_j:N_j} \leq z - \Delta\} \cdot K\left(\frac{X_j^c - x^c}{h_L}\right) \cdot \mathbb{1}\{X_j^d = x^d\} \cdot \mathbb{1}\{N_j = n\}
\end{aligned} \tag{11}$$

Similarly, the following "leave-two-out" estimators are calculated separately for each pair of observations (i, j), and will be used in our test of Equation 3. For each $i \neq j$ and each $x = (x^c, x^d)$, n, and $k \leq n$, define

$$\begin{aligned}
\widehat{f}^{-i,j}_{X,N}(x,n) &= \frac{1}{(L-2)} \cdot \frac{1}{h_L^q} \sum_{\ell \neq i,j} K\left(\frac{X_\ell^c - x^c}{h_L}\right) \cdot \mathbb{1}\{X_\ell^d = x^d\} \cdot \mathbb{1}\{N_\ell = n\} \\
\widehat{G}^{-i,j}_{k:n}(z|x) &= \frac{1}{\widehat{f}^{-i,j}_{X,N}(x,n)} \frac{1}{(L-2)} \cdot \frac{1}{h_L^q} \sum_{\ell \neq i,j} \mathbb{1}\{B_{k:N_\ell} \leq z\} \cdot K\left(\frac{X_\ell^c - x^c}{h_L}\right) \cdot \mathbb{1}\{X_\ell^d = x^d\} \cdot \mathbb{1}\{N_\ell = n\}
\end{aligned} \tag{12}$$

B.2 A Test for Equation 1

Let $f_{X,N}$ denote the joint density function of (X, N). We begin by prespecifying a range $\mathcal{N} \subseteq Supp(N)$ which we will consider for n. Given $\mathcal{N}$, we will test equation 1 on a subset of $Supp(X, N)$ on which $f_{X,N}$ is bounded away from 0. Specifically, fix $\underline{f} > 0$, and define

$$\mathcal{T}_1 = \Big\{(x, n) \in Supp(X, N) : n \in \mathcal{N} \quad \text{and} \quad f_{X,N}(x, n) \geq \underline{f}\Big\}$$

Assumption B.2.
(i) $Pr[f_{X,N}(X, N) \geq \underline{f}] > 0$ and $Pr[f_{X,N}(X, N) = \underline{f}] = 0$.
(ii) Let M be as described in Assumption B.1. Then the following functionals are assumed to be M times differentiable with respect to x^c (the continuous elements in x) with bounded derivatives a.e in our testing range: $f_{X,N}(x, n)$, $\nu^I(x, n)$, $\nu^{II}(x, n)$ and $E[B^I \cdot B^{II}|x, n]$.
(iii) $E\left[(B^I)^4\right] < \infty$, $E\left[(B^{II})^4\right] < \infty$ and $E\left[(B^I \cdot B^{II})^4\right] < \infty$.

Define $\mathbb{I}_i = \mathbb{1}\left\{(X_i, N_i) \in \mathcal{T}_1\right\}$ and let

$$\begin{aligned}\mathcal{M}_1 &= E\left[\left(B_i^I B_i^{II} - \nu^I(X_i, N_i)\nu^{II}(X_i, N_i)\right) \cdot cov\left(B^I, B^{II}|X_i, N_i\right) \cdot \mathbb{I}_i\right] \\ &= E\left[cov(B^I, B^{II}|X_i, N_i)^2 \cdot \mathbb{I}_i\right]\end{aligned} \tag{13}$$

Note that $\mathcal{M}_1$ is the simply the expected value $1'$, but restricted to the testing range $\mathcal{T}_1$. By design, it is nonnegative, and equals zero only if equation 1 is satisfied w.p.1 on $\mathcal{T}_1$. Let

$$\widehat{\mathbb{I}}_i = \mathbb{1}\Big\{N_i \in \mathcal{N} \quad \text{and} \quad \widehat{f}^{-i}_{X,N}(X_i, N_i) \geq \underline{f}\Big\}.$$

In addition to the bandwidth h_L described above, in the construction of our test we will employ an additional bandwidth sequence $\widetilde{h}_L$ satisfying the bandwidth convergence restriction described in Assumption B.1; namely, $L \cdot \widetilde{h}_L^{2M} \longrightarrow 0$. We will let $\widetilde{\nu}^{I^{-i}}$ and $\widetilde{\nu}^{II^{-i}}$ denote the nonparametric estimators described in (10) after we replace h_L with $\widetilde{h}_L$.

Assumption B.3. Let $\delta > 0$ be as defined in Assumption B.1. The bandwidth sequences $h_L \to 0$ and $\widetilde{h}_L \to 0$ converge at different rates, but $\widetilde{h}_L$ also satisfied $L^{1-\delta} \cdot \widetilde{h}_L^{2q} \longrightarrow \infty$. If $q \geq 1$ (i.e, if there is at least one continuous covariate in X), the relative rates of convergence are such that

$$\exists\, \epsilon > 0 \quad \text{such that} \quad L^{\epsilon} \cdot \left(\frac{h_L}{\widetilde{h}_L}\right)^{\frac{q}{2}} \longrightarrow 0.$$

In addition, we strengthen the condition $L^{1+\delta} \cdot h_L^{2M} \longrightarrow 0$ (from Assumption B.1) to $L^{1+\delta} \cdot h_L^{\frac{q}{2}} \cdot h_L^M \longrightarrow 0$.

The conditions described in Assumption B.3 are sufficient to ensure that

$$\sup_{(x,n)\in\mathcal{T}_1} \left| \left(\widetilde{\nu}^{I^{-i}}(x,n) \cdot \widetilde{\nu}^{II^{-i}}(x,n) - \nu^I(x,n) \cdot \nu^{II}(x,n) \right) \cdot \left(\widehat{cov}^{-i}(B^I, B^{II}|x,n) - cov(B^I, B^{II}|x,n) \right) \right|$$
$$= o_p\left(\frac{1}{L \cdot h_L^{\frac{q}{2}}} \right).$$

This, in turn, will allow our proposed test-statistic (described below) to converge to a non-degenerate distribution at the rate of $L \cdot h_L^{\frac{q}{2}}$. The details can be found in the Supplementary Appendix. Our estimator for $\mathcal{M}_1$ will be given by

$$\widehat{\mathcal{M}}_1 = \frac{1}{L} \sum_{i=1}^{L} \left(B_i^I B_i^{II} - \widetilde{\nu}^{I^{-i}}(X_i, N_i) \cdot \widetilde{\nu}^{II^{-i}}(X_i, N_i) \right) \cdot \widehat{cov}^{-i}\left(B^I, B^{II} | X_i, N_i \right) \cdot \widehat{\mathbb{I}}_i$$

Proposition B1. Suppose $q \geq 1$ (at least one continuously distributed covariate in X). Let

$$H_{1_L}(U_j|x,n) =$$
$$\left\{ \frac{1}{f_{X,N}(x,n)} \times \left[\left(B_j^I B_j^{II} - E\left[B^I B^{II} | x, n \right] \right) - \nu^{II}(x,n) \cdot \left(B_j^I - \nu^I(x,n) \right) - \nu^I(x,n) \cdot \left(B_j^{II} - \nu^{II}(x,n) \right) \right] \right\}$$
$$\times \mathbb{1}\{N_j = n\} \cdot \mathbb{1}\left\{ X_j^d = x^d \right\} \cdot K\left(\frac{X_j^c - x^c}{h_L} \right)$$

and

$$\phi_{1_L}(U_i, U_j) = \left[\left(B_i^I B_i^{II} - \nu^I(X_i, N_i)\nu^{II}(X_i, N_i) \right) \cdot \mathbb{I}_i \cdot H_{1_L}(U_j|X_i, N_i) \right.$$
$$\left. + \left(B_j^I B_j^{II} - \nu^I(X_j, N_j)\nu^{II}(X_j, N_j) \right) \cdot \cdot \mathbb{I}_j \cdot H_{1_L}(U_i|X_j, N_j) \right]$$

Under Assumptions B.1, B.3 and B.2,

(i) If (1) is satisfied w.p.1 over our testing range, then

$$L \cdot h_L^{\frac{q}{2}} \cdot \widehat{\mathcal{M}}_1 \overset{d}{\longrightarrow} \mathcal{N}\left(0, \sigma_1^2\right)$$

where $\sigma_1^2 = \frac{1}{2} \cdot \lim_{L\to\infty} E\left[\frac{1}{h_L^q} \phi_{1_L}(U_i, U_j)^2 \right]$.

(ii) If (1) is violated with nonzero probability over our testing range, then

$$L \cdot h_L^{\frac{q}{2}} \cdot \widehat{\mathcal{M}}_1 \overset{p}{\longrightarrow} +\infty$$

Proof: See Supplemental Appendix[25].

A Rejection Rule for Equation 1

Consider the null hypothesis H_1 : *Equation 1 holds w.p.1 over our testing range* against the alternative H_1' : *Equation 1 is violated with positive probability over our testing range*, and let $\alpha \in (0,1)$ denote a target significance level. Let

$$\widehat{t}_1 = \frac{L \cdot h_L^{\frac{q}{2}} \cdot \widehat{\mathcal{M}}_1}{\widehat{\sigma}_1}$$

where $\widehat{\sigma}_1$ is a consistent estimator of σ_1. Suppose we reject H_1 if $\widehat{t}_1 > z_\alpha = \Phi^{-1}(1-\alpha)$, where $\Phi(\cdot)$ is the standard normal CDF. By Proposition B1, this rejection rule has the following asymptotic properties:

$$\lim_{L\to\infty} Pr(\text{Rejecting } H_1 \text{ when it is true}) = \alpha$$

$$\lim_{L\to\infty} Pr(\text{Rejecting } H_1 \text{ when it is false}) = 1$$

B.3 A Test for Equation 2

For a given n and $k \leq n-2$, let $\mathcal{T}_2^{n,k} \subseteq Supp(X,Z)$ be defined as

$$\mathcal{T}_2^{n,k} = \left\{(x,z) \subseteq Supp(X,Z) : f_{X,N}(x,n) \geq \underline{f} > 0 \quad \text{and} \quad \underline{z}^{n,k} \leq z \leq \overline{z}^{n,k}\right\}$$

where $\underline{f}$, $\underline{z}^{n,k}$ and $\overline{z}^{n,k}$ are chosen such that

$$\underline{c} \leq G^{\Delta}_{n:n}(z|x) \leq \overline{c} \quad \text{and} \quad \underline{c} \leq G_{k:n}(z|x) \leq \overline{c} \quad \forall\ (z,x) : \underline{z}^{n,k} \leq z \leq \overline{z}^{n,k},\ f_{X,N}(x,n) \geq \underline{f}$$

for some $0 < \underline{c} < \overline{c} < 1$. Define $\mathbb{I}_i^{n,k} = \mathbb{1}\left\{(X_i,Z_i) \in \mathcal{T}_2^{n,k}\right\}$. Again let $\mathcal{N}$ denote the range of values considered for n. Let

$$\mathcal{M}_2 = E\left[\sum_{\substack{n\in\mathcal{N}\\ k\leq n-2}} \max\left\{\eta(Z_i,X_i;k,n),0\right\} \cdot \mathbb{1}\{N_i = n\} \cdot \mathbb{I}_i^{n,k}\right] \tag{14}$$

[25] Proposition B1 maintains $q \geq 1$. If there are no continuously distributed elements in X, then if (1) is satisfied w.p.1 over our testing range, $L \cdot \widehat{\mathcal{M}}_1$ will converge in distribution to a random variable $\mathcal{Y}$ whose distribution is that of a transformation of χ_1^2 random variables (see Section 5.5.2 in Serfling (1980)). The reason why this asymptotic distribution differs from the case $q \geq 1$ stems from the contrast in asymptotic behavior between degenerate U-statistics whose summands are functions of the sample size L (our case through the presence of the bandwidth h_L if $q \geq 1$) and those which do not have this property (which would be the case if $q = 0$). See Hall (1984), Fan and Li (1996) and the references cited therein.

Note that $\mathcal{M}_2$ is the expectation $2'$, restricted to our testing range. By design, it is nonnegative and it equals zero only if 2 is satisfied w.p.1 there. For a given z, x, n and $k \leq n-2$, let

$$\widehat{\eta}^{-i}(z,x;k,n) = \psi^{-1}_{n-1:n}\left(\widehat{G}^{\Delta^{-i}}_{n:n}(z|x)\right) - \psi^{-1}_{k:n}\left(\widehat{G}^{-i}_{k:n}(z|x)\right)$$

and define

$$\widehat{\mathbb{I}}^{n,k}_i = \mathbb{1}\left\{\widehat{f}^{-i}_{X,N}(X_i,n) \geq \underline{f} \quad \text{and} \quad \underline{z}^{n,k} \leq Z_i \leq \overline{z}^{n,k}\right\}$$

Our estimator of $\mathcal{M}_2$ is given by

$$\widehat{\mathcal{M}}_2 = \frac{1}{L}\sum_{i=1}^{L}\left\{\sum_{\substack{n\in\mathcal{N}\\ k\leq n-2}} \widehat{\eta}^{-i}(Z_i,X_i;k,n)\cdot\mathbb{1}\left\{\widehat{\eta}^{-i}(Z_i,X_i;k,n)\geq -b_{_L}\right\}\cdot\mathbb{1}\{N_i=n\}\cdot\widehat{\mathbb{I}}^{n,k}_i\right\}$$

The sequence $b_{_L} \to 0$ plays a role analogous to the sequence β_n used in Jun, Pinkse, and Wan (2010). The following condition describes the convergence rate restrictions for $b_{_L}$ and $h_{_L}$.

Assumption B.4. Let $\delta > 0$ be as described in Assumption B.1. We will strengthen the conditions there to impose the additional restrictions, $L^{\frac{1}{2}-\delta}\cdot h_L^{\frac{q}{2}}\cdot b_{_L} \longrightarrow \infty$, $L^{\frac{1}{2}+\delta}\cdot b^2_{_L} \longrightarrow 0$, and $L^{\delta}\cdot b_{_L}\cdot h_{_L}^{-\frac{q}{2}} \longrightarrow 0$.

Assumption B.5.
(i) $Pr\left[f_{X,N}(X,n) \geq \underline{f}\right] > 0$ and $Pr\left[f_{X,N}(X,n) = \underline{f}\right] = 0 \quad \forall\, n \in \mathcal{N}$
(ii) For a given k, n, x and z let

$$\nabla\eta^a(z,x;n) = \frac{(n-2)!}{n!}\cdot\left[\psi^{-1}_{n-1:n}\left(G^{\Delta}_{n:n}(z|x)\right)\right]^{2-n}\cdot\left(1-\left[\psi^{-1}_{n-1:n}\left(G^{\Delta}_{n:n}(z|x)\right)\right]\right)^{-1}$$
$$\nabla\eta^b(z,x;k,n) = \frac{(n-k)!(k-1)!}{n!}\cdot\left[\psi^{-1}_{k:n}\left(G_{k:n}(z|x)\right)\right]^{1-k}\cdot\left(1-\left[\psi^{-1}_{k:n}\left(G_{k:n}(z|x)\right)\right]\right)^{k-n}$$

and a given b let

$$\theta^a_2(b,x;k,n) = \frac{E_{Z|X}\left[\left(\mathbb{1}\{b\leq Z-\Delta\}-G^{\Delta}_{n:n}(Z|x)\right)\cdot\nabla\eta^a(Z,x;n)\cdot\mathbb{1}\left\{\eta(Z,x;k,n)\geq 0\right\}\cdot\mathbb{1}\{N=n\}\cdot\mathbb{1}\left\{(x,Z)\in\mathcal{T}^{n,k}_2\right\}\Big|X=x\right]}{Pr(N=n|X=x)}$$

$$\theta^b_2(b,x;k,n) = \frac{E_{Z|X}\left[\left(\mathbb{1}\{b\leq Z\}-G_{k:n}(Z|x)\right)\cdot\nabla\eta^b(Z,x;k,n)\cdot\mathbb{1}\left\{\eta(Z,x;k,n)\geq 0\right\}\cdot\mathbb{1}\{N=n\}\cdot\mathbb{1}\left\{(x,Z)\in\mathcal{T}^{n,k}_2\right\}\Big|X=x\right]}{Pr(N=n|X=x)}$$

Let M be as described in Assumption B.1. Then the following functionals are assumed to be M times differentiable with respect to x^c (the continuous elements in x) with bounded derivatives a.e in our testing range and for all $z, b \in \mathbb{R}$: $f_{X,N}(x,n)$, $G_{n:n}(z|x)$, $G_{k:n}(z|x)$, $\theta_2^a(b,x;k,n)$ and $\theta_2^b(b,x;k,n)$.

Assumption B.6. Define

$$H_L^{\eta}(U_j|z,x;k,n) =$$
$$\left\{\nabla\eta^a(z,x;n)\cdot\left(\mathbb{1}\{B_{N_j:N_j}\le z-\Delta\}-G^{\Delta}_{n:n}(z|x)\right)-\nabla\eta^b(z,x;k,n)\cdot\left(\mathbb{1}\{B_{k:N_j}\le z\}-G_{k:n}(z|x)\right)\right\}$$
$$\times\frac{1}{f_{X,N}(x,n)}\cdot\mathbb{1}\{N_j=n\}\cdot\mathbb{1}\{X_i^d=x^d\}\cdot\frac{1}{h_L^q}K\left(\frac{X_j^c-x^c}{h_L}\right)$$

and denote

$$\xi_L^{-i}(z,x;k,n)=\widehat{\eta}^{-i}(z,x;k,n)-\eta(z,x;k,n)-\frac{1}{(L-1)}\sum_{j\neq i}H_L^{\eta}(U_j|z,x;k,n)$$

$\xi_L^{-i}(z,x;k,n)$ is the remainder of the asymptotic linear representation of $\widehat{\eta}^{-i}(z,x;k,n)$. There exists a bounded sequence $\{a_L\}$ such that,

(i) For $i\neq j$,

$$\left|E\left[H_L^{\eta}(U_j|z,x;k,n)\,\middle|\,\xi_L^{-i}(z,x;k,n)\right]-E\left[H_L^{\eta}(U_j|z,x;k,n)\right]\right|\le\left|\xi_L^{-i}(z,x;k,n)\right|\cdot a_L,$$

for all z, x, n in our testing range and any $k\le n-2$.

(ii) There exists a $\bar{t}>0$ such that, for all $n\in\mathcal{N}$ and $k\le n-2$,

$$Pr\left[-t\le\eta(Z_i,X_i;k,n)<0\middle|\,\xi_L^{-i}(Z_i,X_i;k,n)\right]\le t\cdot a_L\quad\forall\,0<t\le\bar{t}$$

Note that Assumption B.6(ii) allows for $\eta(Z_i,X_i;k,n)$ to have a point mass at zero, a feature that must be allowed for.

Proposition B2. Let

$$\phi_2(U_i)=\sum_{\substack{n\in\mathcal{N}\\k\le n-2}}\left\{\theta_2^a\left(B_{N_i:N_i},X_i;k,n\right)-\theta_2^b\left(B_{k:N_i},X_i;k,n\right)\right\}\cdot\mathbb{1}\{N_i=n\}$$

Under Assumptions B.1, B.4, B.5 and B.6,

(i) If (2) is satisfied as a **strict** inequality w.p.1 over our testing range, then

$$\sqrt{L}\cdot\widehat{\mathcal{M}}_2=o_p\left(L^{-\epsilon}\right)\quad\text{for some }\epsilon>0$$

(ii) If (2) is satisfied w.p.1 over our testing range and it holds as an **equality** over a subset of it with strictly positive probability measure, then

$$\sqrt{L} \cdot \widehat{\mathcal{M}}_2 \xrightarrow{d} \mathcal{N}\left(0, \sigma_2^2\right)$$

where $\sigma_2^2 = E\left[\phi_2(U_i)^2\right]$

(iii) If (2) is violated with positive probability over our testing range, then

$$\sqrt{L} \cdot \widehat{\mathcal{M}}_2 \xrightarrow{p} +\infty$$

Proof: See Supplemental Appendix.

A Rejection Rule for Equation 2

Consider the null hypothesis H_2 : *Equation 2 holds w.p.1 over our testing range* against the alternative H_2' : *Equation 2 is violated with positive probability over our testing range*, and let $\alpha \in (0,1)$ denote a target significance level. Let

$$\widehat{t}_2 = \frac{\sqrt{L} \cdot \widehat{\mathcal{M}}_2}{\max\{\widehat{\sigma}_2, c_L\}},$$

where $\widehat{\sigma}_2$ is a consistent estimator of σ_2 and c_L is a nonnegative sequence such that $\frac{L^{-\epsilon}}{c_L} \to 0$ for any $\epsilon > 0$. Suppose we reject H_2 if $\widehat{t}_2 > z_\alpha = \Phi^{-1}(1-\alpha)$. By Proposition B2, this rejection rule has the following asymptotic properties:

$$\lim_{L\to\infty} Pr(\text{Rejecting } H_2 \text{ when it is true}) \leq \alpha$$
$$\lim_{L\to\infty} Pr(\text{Rejecting } H_2 \text{ when it is false}) = 1$$

B.4 A Test for Equation 3

We will assume that $Supp(Z,X,N) = Supp(Z)\times Supp(X)\times Supp(N)$. (This is not a crucial assumption, but it will help simplify the expression and computation of our test-statistic.) For a given n and n', define $\mathcal{T}_3^{n,n'} \subseteq Supp(X,Z)$ as

$$\mathcal{T}_3^{n,n'} = \left\{(x,z) \subseteq Supp(X,Z) : \ \min\left\{f_{X,N}(x,n), f_{X,N}(x,n')\right\} \geq \underline{f} > 0 \quad \text{and} \quad \underline{z}^{n,n'} \leq z \leq \overline{z}^{n,n'}\right\}$$

where $\underline{f}$, $\underline{z}^{n,n'}$ and $\overline{z}^{n,n'}$ are chosen such that

$$\underline{c} \leq G^{\Delta}_{n:n}(z|x) \leq \overline{c} \quad \text{and} \quad \underline{c} \leq G_{n':n'}(z|x) \leq \overline{c} \quad \forall\ (z,x) : \underline{z}^{n,n'} \leq z \leq \overline{z}^{n,n'},\ f_{X,N}(x,n) \geq \underline{f}$$

for some $0 < \underline{c} < \overline{c} < 1$. Define $\mathbb{I}_{i,j}^{n,n'} = \mathbb{1}\left\{(X_i, Z_j) \in \mathcal{T}_3^{n,n'}\right\}$. Again let $\mathcal{N}$ denote the range of values considered for n. Let

$$\begin{aligned}\mathcal{M}_3 &= E\left[\sum_{n,n'\in\mathcal{N}} \mathbb{1}\{N_i = n\} \cdot (\mathbb{1}\{B_{N_i:N_i} \leq Z_j\} - \lambda_{n,n'}(Z_j|X_i)) \cdot (G_{n:n}(Z_j|X_i) - \lambda_{n,n'}(Z_j|X_i)) \cdot \mathbb{I}_{i,j}^{n,n'}\right] \\ &= E\left[\sum_{n,n'\in\mathcal{N}} \mathbb{1}\{N_i = n\} \cdot \left(G_{n:n}(Z_j|X_i) - \lambda_{n,n'}(Z_j|X_i)\right)^2 \cdot \mathbb{I}_{i,j}^{n,n'}\right]\end{aligned} \tag{15}$$

$\mathcal{M}_3$ is the expectation $3'$ restricted to our testing range. Again, by design it is nonnegative, and is zero only if equation 3 holds w.p.1 there. Let

$$\widehat{\lambda}_{n,n'}^{-i,j}(z|x) = \psi_{n-1:n}\left(\psi_{n'-1:n'}^{-1}\left(\widehat{G}_{n':n'}^{-i,j}(z|x)\right)\right)$$

and

$$\widehat{\mathbb{I}}_{i,j}^{n,n'} = \mathbb{1}\left\{\min\left\{\widehat{f}_{X,N}^{-i,j}(X_i,n), \widehat{f}_{X,N}^{-i,j}(X_i,n')\right\} \geq \underline{f} \quad \text{and} \quad \underline{z}^{n,n'} \leq Z_j \leq \overline{z}^{n,n'}\right\}.$$

As we did in the tests of Equation (1), we make use of the additional bandwidth $\widetilde{h}_L$, which satisfies Assumption B.3. Correspondingly, we denote

$$\widetilde{\lambda}_{n,n'}^{-i,j}(z|x) = \psi_{n-1:n}\left(\psi_{n'-1:n'}^{-1}\left(\widetilde{G}_{n':n'}^{-i,j}(z|x)\right)\right).$$

Our estimator for $\mathcal{M}_3$ is

$$\widehat{\mathcal{M}}_3 = \frac{1}{L(L-1)}\sum_{i\neq j}\left\{\sum_{n,n'\in\mathcal{N}} \mathbb{1}\{N_i = n\} \cdot \left(\mathbb{1}\{B_{N_i:N_i} \leq Z_j\} - \widetilde{\lambda}_{n,n'}^{-i,j}(Z_j|X_i)\right) \cdot \left(\widehat{G}_{n:n}^{-i,j}(Z_j|X_i) - \widehat{\lambda}_{n,n'}^{-i,j}(Z_j|X_i)\right) \cdot \widehat{\mathbb{I}}_{i,j}^{n,n'}\right\}$$

Assumption B.7.

(i) $Pr\left[f_{X,N}(X,n) \geq \underline{f}\right] > 0$ and $Pr\left[f_{X,N}(X,n) = \underline{f}\right] = 0 \quad \forall\, n \in \mathcal{N}$.

(ii) Let

$$\nabla\lambda_{n,n'}(z|x) = \frac{n\cdot(n-1)}{n'\cdot(n'-1)} \cdot \left[\psi_{n'-1:n'}\left(G_{n':n'}(z|x)\right)\right]^{n-n'}$$

and for a given b_1, b_2 and x let

$$\rho_{n,n'}^{a}(b_1,b_2,x) = E_Z\left[(\mathbb{1}\{b_1 \leq Z\} - \lambda_{n,n'}(Z|x)) \cdot (\mathbb{1}\{b_2 \leq Z\} - G_{n:n}(Z|x)) \cdot \mathbb{1}\left\{(x,Z) \in \mathcal{T}_3^{n,n'}\right\}\right]$$

$$\rho_{n,n'}^{b}(b_1,b_2,x) = E_Z\left[(\mathbb{1}\{b_1 \leq Z\} - \lambda_{n,n'}(Z|x)) \cdot (\mathbb{1}\{b_2 \leq Z\} - G_{n':n'}(Z|x)) \cdot \nabla\lambda_{n,n'}(Z|x) \cdot \mathbb{1}\left\{(x,Z) \in \mathcal{T}_3^{n,n'}\right\}\right]$$

Let M be as described in Assumption B.1. Then the following functionals are assumed to be M times differentiable with respect to x^c (the continuous elements in x) with bounded derivatives a.e in our testing range and for all $z, b_1, b_2 \in \mathbb{R}$: $f_{X,N}(x,n)$, $G_{n:n}(z|x)$, $\rho^a_{n,n'}(b_1,b_2,x)$ and $\rho^b_{n,n'}(b_1,b_2,x)$.

Proposition B3. Suppose $q \geq 1$ (at least one continuously distributed covariate in X). For any pair of observations $i \neq j$, let

$$\phi^a_{3_L}(U_i,U_j) = \sum_{n,n' \in \mathcal{N}} \left\{ \frac{\rho^a_{n,n'}(B_{N_i:N_i}, B_{N_j:N_j}, X_i)}{f_{X,N}(X_i,n)} + \frac{\rho^a_{n,n'}(B_{N_j:N_j}, B_{N_i:N_i}, X_j)}{f_{X,N}(X_j,n)} \right\}$$
$$\times \mathbb{1}\{N_i = n, N_j = n\} \cdot \mathbb{1}\left\{X_i^d = X_j^d\right\} \cdot K\left(\frac{X_i^c - X_j^c}{h_L}\right)$$
$$\phi^b_{3_L}(U_i,U_j) = \sum_{n,n' \in \mathcal{N}} \left\{ \frac{\rho^b_{n,n'}(B_{N_i:N_i}, B_{N_j:N_j}, X_i)}{f_{X,N}(X_i,n)} \cdot \mathbb{1}\{N_i = n, N_j = n'\} \right.$$
$$\left. + \frac{\rho^b_{n,n'}(B_{N_j:N_j}, B_{N_i:N_i}, X_j)}{f_{X,N}(X_j,n)} \cdot \mathbb{1}\{N_j = n, N_i = n'\} \right\} \times \mathbb{1}\left\{X_i^d = X_j^d\right\} \cdot K\left(\frac{X_i^c - X_j^c}{h_L}\right)$$
$$\phi_{3_L}(U_i,U_j) = \phi^a_{3_L}(U_i,U_j) - \phi^b_{3_L}(U_i,U_j)$$

Under Assumptions B.1, B.3 and B.7,

(i) If (3) is satisfied w.p.1 over our testing range, then

$$L \cdot h_L^{\frac{q}{2}} \cdot \widehat{\mathcal{M}}_3 \xrightarrow{d} \mathcal{N}\left(0, \sigma_3^2\right)$$

where $\sigma_3^2 = \frac{1}{2} \cdot \lim_{L \to \infty} E\left[\frac{1}{h_L^q} \phi_{3_L}(U_i,U_j)^2\right]$

(ii) If (3) is violated with nonzero probability over our testing range, then

$$L \cdot h_L^{\frac{q}{2}} \cdot \widehat{\mathcal{M}}_3 \xrightarrow{p} +\infty$$

Proof: See Supplemental Appendix.

If $q = 0$ (X contains no continuously distributed covariate), the discussion in Footnote 25 applies here, too.

A Rejection Rule for Equation 3

Consider the null hypothesis H_3 : *Equation 3 holds w.p.1 over our testing range* against the alternative H_3' : *Equation 3 is violated with positive probability over our testing range*, and let $\alpha \in (0,1)$ denote a target significance level. Let

$$\widehat{t}_3 = \frac{L \cdot h_L^{\frac{q}{2}} \cdot \widehat{\mathcal{M}}_3}{\widehat{\sigma}_3}$$

where $\widehat{\sigma}_3$ is a consistent estimator of σ_3. Suppose we reject H_3 if $\widehat{t}_3 > z_\alpha = \Phi^{-1}(1-\alpha)$. By Proposition B3, this rejection rule has the following asymptotic properties:

$$\lim_{L\to\infty} Pr(\text{Rejecting } H_3 \text{ when it is true}) = \alpha$$

$$\lim_{L\to\infty} Pr(\text{Rejecting } H_3 \text{ when it is false}) = 1$$

B.5 A Test for Equations 4 and 5

Let $\mathcal{T}_3^{n,n'}$ be defined as above. Define $\mathbb{I}_i^{n,n'} = \mathbb{1}\left\{(X_i, Z_i) \in \mathcal{T}_3^{n,n'}\right\}$. Again let $\mathcal{N}$ denote the range of values considered for n. Let

$$\tau(z,x;n,n') = \psi_{n-1:n}^{-1}\left(G_{n:n}(z|x)\right) - \psi_{n'-1:n'}^{-1}\left(G_{n':n'}(z|x)\right)$$

and

$$\begin{aligned}
\mathcal{M}_4 &= -E\left[\sum_{n>n'\in\mathcal{N}} \min\left\{\tau(Z_i, X_i; n, n')\ ,\ 0\right\}\cdot \mathbb{1}\{N_i = n, n'\} \cdot \mathbb{I}_i^{n,n'}\right] \\
\mathcal{M}_5 &= E\left[\sum_{n>n'\in\mathcal{N}} \max\left\{\tau(Z_i, X_i; n, n')\ ,\ 0\right\}\cdot \mathbb{1}\{N_i = n, n'\} \cdot \mathbb{I}_i^{n,n'}\right]
\end{aligned} \tag{16}$$

$\mathcal{M}_4$ and $\mathcal{M}_5$ are the expectations $4'$ and $5'$ restricted to our testing range. Once again, they are both nonnegative and they equal zero only if 4 and 5 are satisfied w.p.1, respectively, over our testing range. Let

$$\widehat{\tau}^{-i}(z,x;n,n') = \psi_{n-1:n}^{-1}\left(\widehat{G}_{n:n}^{-i}(z|x)\right) - \psi_{n'-1:n'}^{-1}\left(\widehat{G}_{n':n'}^{-i}(z|x)\right)$$

and

$$\widehat{\mathbb{I}}_i^{n,n'} = \mathbb{1}\left\{\min\left\{\widehat{f}_{X,N}^{-i}(X_i,n), \widehat{f}_{X,N}^{-i}(X_i,n')\right\} \geq \underline{f} \quad \text{and} \quad \underline{z}^{n,n'} \leq Z_i \leq \overline{z}^{n,n'}\right\}$$

Our estimators for $\mathcal{M}_4$ and $\mathcal{M}_5$ are

$$\widehat{\mathcal{M}}_4 = -\frac{1}{L}\sum_{i=1}^{L}\left\{\sum_{n>n'\in\mathcal{N}} \widehat{\tau}^{-i}(Z_i, X_i; n, n') \cdot \mathbb{1}\left\{\widehat{\tau}^{-i}(Z_i, X_i; n, n') \leq b_L\right\} \cdot \mathbb{1}\{N_i = n, n'\} \cdot \widehat{\mathbb{I}}_i^{n,n'}\right\}$$

$$\widehat{\mathcal{M}}_5 = \frac{1}{L}\sum_{i=1}^{L}\left\{\sum_{n>n'\in\mathcal{N}} \widehat{\tau}^{-i}(Z_i, X_i; n, n') \cdot \mathbb{1}\left\{\widehat{\tau}^{-i}(Z_i, X_i; n, n') \geq -b_L\right\} \cdot \mathbb{1}\{N_i = n, n'\} \cdot \widehat{\mathbb{I}}_i^{n,n'}\right\}$$

Assumption B.8.

(i) $Pr\left[f_{X,N}(X,n) \geq \underline{f}\right] > 0$ and $Pr\left[f_{X,N}(X,n) = \underline{f}\right] = 0 \quad \forall\ n \in \mathcal{N}$

(ii) Let

$$\nabla\psi^{-1}_{n-1:n}(s) = \Big\{n \cdot (n-1) \cdot \left[\psi^{-1}_{n-1:n}(s)\right]^{n-2} \cdot \left(1 - \left[\psi^{-1}_{n-1:n}(s)\right]\right)\Big\}^{-1}$$

and for a given b let

$$\theta^a_4(b,x;n,n') = \frac{E_{Z|X}\left[(\mathbb{1}\{b \leq Z\} - G_{n':n'}(Z|x)) \cdot \nabla\psi^{-1}_{n'-1:n'}\left(G_{n':n'}(Z|x)\right) \cdot \mathbb{1}\left\{\tau(Z,x;n,n') \leq 0\right\} \cdot \mathbb{1}\{N = n, n'\} \cdot \mathbb{1}\big\{(x,Z) \in \mathcal{T}^{n,n'}_3\big\} \Big| X = x\right]}{Pr(N = n'|X = x)}$$

$$\theta^b_4(b,x;n,n') = \frac{E_{Z|X}\left[(\mathbb{1}\{b \leq Z\} - G_{n:n}(Z|x)) \cdot \nabla\psi^{-1}_{n-1:n}\left(G_{n:n}(Z|x)\right) \cdot \mathbb{1}\left\{\tau(Z,x;n,n') \leq 0\right\} \cdot \mathbb{1}\{N = n, n'\} \cdot \mathbb{1}\big\{(x,Z) \in \mathcal{T}^{n,n'}_3\big\} \Big| X = x\right]}{Pr(N = n|X = x)}$$

and

$$\theta^a_5(b,x;n,n') = \frac{E_{Z|X}\left[(\mathbb{1}\{b \leq Z\} - G_{n:n}(Z|x)) \cdot \nabla\psi^{-1}_{n-1:n}\left(G_{n:n}(Z|x)\right) \cdot \mathbb{1}\left\{\tau(Z,x;n,n') \geq 0\right\} \cdot \mathbb{1}\{N = n, n'\} \cdot \mathbb{1}\big\{(x,Z) \in \mathcal{T}^{n,n'}_3\big\} \Big| X = x\right]}{Pr(N = n|X = x)}$$

$$\theta^b_5(b,x;n,n') = \frac{E_{Z|X}\left[(\mathbb{1}\{b \leq Z\} - G_{n':n'}(Z|x)) \cdot \nabla\psi^{-1}_{n'-1:n'}\left(G_{n':n'}(Z|x)\right) \cdot \mathbb{1}\left\{\tau(Z,x;n,n') \geq 0\right\} \cdot \mathbb{1}\{N = n, n'\} \cdot \mathbb{1}\big\{(x,Z) \in \mathcal{T}^{n,n'}_3\big\} \Big| X = x\right]}{Pr(N = n'|X = x)}$$

Let M be as described in Assumption B.1. Then the following functionals are assumed to be M times differentiable with respect to x^c (the continuous elements in x) with bounded derivatives a.e in our testing range and for all $z, b \in \mathbb{R}$: $f_{X,N}(x,n)$, $G_{n:n}(z|x)$, $\theta^a_4(b,x;n,n')$, $\theta^b_4(b,x;n,n')$, $\theta^a_5(b,x;n,n')$ and $\theta^b_5(b,x;n,n')$.

Assumption B.9. Define

$$\begin{aligned} H^\tau_L\left(U_j|z,x;n,n'\right) = \Bigg\{&\nabla\psi^{-1}_{n-1:n}\left(G_{n:n}(z|x)\right) \cdot \frac{\left(\mathbb{1}\{B_{N_j:N_j} \leq z\} - G_{n:n}(z|x)\right)}{f_{X,N}(x,n)} \cdot \mathbb{1}\{N_j = n\} \\ &- \nabla\psi^{-1}_{n'-1:n'}\left(G_{n':n'}(z|x)\right) \cdot \frac{\left(\mathbb{1}\{B_{N_j:N_j} \leq z\} - G_{n':n'}(z|x)\right)}{f_{X,N}(x,n')} \cdot \mathbb{1}\{N_j = n'\}\Bigg\} \times \mathbb{1}\{X^d_j = x^d\} \\ &\times \frac{1}{h^q_L} K\left(\frac{X^c_j - x^c}{h_L}\right) \end{aligned}$$

and define

$$\varepsilon^{-i}_L(z,x;n,n') = \widehat{\tau}^{-i}(z,x;n,n') - \tau(z,x;n,n') - \frac{1}{(L-1)}\sum_{j \neq i} H^\tau_L\left(U_j|z,x;n,n'\right)$$

$\varepsilon_L^{-i}(z,x;n,n')$ is the remainder of the asymptotic linear representation of $\widehat{\tau}^{-i}(z,x;n,n')$. Analogously to Assumption B.6, we impose two Lipschitz-type conditions:

(i) For $i \neq j$, and any z,x,n in our testing range and any $k<n$,

$$\left|E\left[H_L^\tau\left(U_j|z,x;n,n'\right) \mid \beta \cdot \varepsilon_L^{-i}(z,x;n,n')\right] - E\left[H_L^\tau\left(U_j|z,x;n,n'\right) \mid \beta' \cdot \varepsilon_L^{-i}(z,x;n,n')\right]\right|$$
$$\leq |\beta-\beta'| \cdot O_p\left(\varepsilon_L^{-i}(z,x;n,n')\right) \quad \forall\, \beta,\beta' \in [0,1]$$

(ii) There exists a $\bar{t}>0$ such that, for all $n \in \mathcal{N}$ and $k \leq n-2$,

$$Pr\big[-t \leq \tau(Z_i,X_i;n,n') < 0 \big|\, \varepsilon_L^{-i}(Z_i,X_i;n,n')\big] \leq O\big(t\big) \quad \forall\, 0<t\leq \bar{t}$$
$$Pr\big[0 < \tau(Z_i,X_i;n,n') \leq t \big|\, \varepsilon_L^{-i}(Z_i,X_i;n,n')\big] \leq O\big(t\big) \quad \forall\, 0<t\leq \bar{t}$$

Proposition B4. Let

$$\phi_4(U_i) = \sum_{n>n' \in \mathcal{N}} \Big(\theta_4^a\left(B_{N_i:N_i}, X_i; n, n'\right) \cdot \mathbb{1}\{N_i = n'\} - \theta_4^b\left(B_{N_i:N_i}, X_i; n, n'\right) \cdot \mathbb{1}\{N_i = n\}\Big)$$
$$\phi_5(U_i) = \sum_{n>n' \in \mathcal{N}} \Big(\theta_5^a\left(B_{N_i:N_i}, X_i; n, n'\right) \cdot \mathbb{1}\{N_i = n\} - \theta_5^b\left(B_{N_i:N_i}, X_i; n, n'\right) \cdot \mathbb{1}\{N_i = n'\}\Big)$$

Under Assumptions B.1, B.5, B.8 and B.9,

(i) If (4) is satisfied as a **strict** inequality w.p.1 over our testing range, then

$$\sqrt{L} \cdot \widehat{\mathcal{M}}_4 = o_p\left(L^{-\epsilon}\right) \quad \text{for some } \epsilon > 0.$$

(ii) If (4) is satisfied w.p.1 over our testing range and it holds as an **equality** over a subset of it with strictly positive probability measure, then

$$\sqrt{L} \cdot \widehat{\mathcal{M}}_4 \xrightarrow{d} \mathcal{N}\left(0, \sigma_4^2\right), \quad \text{where} \quad \sigma_4^2 = E\left[\phi_4(U_i)^2\right].$$

(iii) If (4) is violated with positive probability over our testing range, then

$$\sqrt{L} \cdot \widehat{\mathcal{M}}_4 \xrightarrow{p} +\infty.$$

(iv) If (5) is satisfied as a **strict** inequality w.p.1 over our testing range, then

$$\sqrt{L} \cdot \widehat{\mathcal{M}}_5 = o_p\left(L^{-\epsilon}\right) \quad \text{for some } \epsilon > 0.$$

(v) If (5) is satisfied w.p.1 over our testing range and it holds as an **equality** over a subset of it with strictly positive probability measure, then

$$\sqrt{L} \cdot \widehat{\mathcal{M}}_5 \xrightarrow{d} \mathcal{N}\left(0, \sigma_5^2\right), \quad \text{where} \quad \sigma_5^2 = E\left[\phi_5(U_i)^2\right].$$

(vi) If (5) is violated with positive probability over our testing range, then

$$\sqrt{L} \cdot \widehat{\mathcal{M}}_5 \xrightarrow{p} +\infty.$$

Proof: See Supplemental Appendix.

A Rejection Rule for Equations 4 and 5

Consider the null hypothesis H_4 : *Equation 4 holds w.p.1 over our testing range* against the alternative H_4' : *Equation 4 is violated with positive probability over our testing range*, and let $\alpha \in (0,1)$ denote a target significance level. Let

$$\widehat{t}_4 = \frac{\sqrt{L} \cdot \widehat{\mathcal{M}}_4}{\max\{\widehat{\sigma}_4, c_L\}}$$

where $\widehat{\sigma}_4$ is a consistent estimator of σ_4 and c_L is a nonnegative sequence such that $\frac{L^{-\epsilon}}{c_L} \to 0$ for any $\epsilon > 0$.. Suppose we reject H_4 if $\widehat{t}_4 > z_\alpha = \Phi^{-1}(1-\alpha)$. By Proposition B4, this rejection rule has the following asymptotic properties:

$$\lim_{L\to\infty} Pr(\text{Rejecting } H_4 \text{ when it is true}) \leq \alpha$$

$$\lim_{L\to\infty} Pr(\text{Rejecting } H_4 \text{ when it is false}) = 1$$

Identical results hold if we let

$$\widehat{t}_5 = \frac{\sqrt{L} \cdot \widehat{\mathcal{M}}_5}{\max\{\widehat{\sigma}_5, c_L\}}$$

and if we reject H_5 : *"Equation 5 holds w.p.1 over our testing range"* if $\widehat{t}_5 > z_\alpha$.

B.6 On the Choice of Tuning Parameters

As is customary in many nonparametric and semiparametric settings, our asymptotic results hold for generic families of bandwidth sequences and kernel functions, without providing a guide as to how these tuning parameters should be chosen. As with many complex nonparametric estimation problems, characterizing data-driven, *optimal* ways to choose the various bandwidths involved appears to be an unassailable problem. The intuitive arguments provided in Section 5 of Jun, Pinkse, and Wan (2010) could be helpful for coming up with a heuristic guidance for choosing b_L. However, the computational feasibility of our various tests allows practitioners to evaluate the robustness of the results obtained under different values of the different tuning parameters (bandwidths) involved. We follow this route in our Monte Carlo analysis, as well as in our empirical results using the USFS data.

B.7 Monte Carlo Experiments

To investigate the finite-sample properties of our tests we revisit the four DGPs described in Section 2.2.2 and shown in Figure 1. Bidder values are drawn from a log-normal distribution, with $\log(V_i) \sim N(\mu, \sigma^2)$ with $\sigma^2 = 0.5$. The four cases are as follows:

DGP1: Values are IPV and F_V does not depend on N: specifically, $\mu = 2.25$ for every N.

DGP2: Values are independent of N, but correlated with each other via conditional independence: regardless of N, with probability $\frac{1}{2}$, $\mu = 2.0$ for all bidders, and with probability $\frac{1}{2}$, $\mu = 2.5$.

DGP3: Values are IPV, but the distribution varies with N: specifically, $\mu = 2 + 0.05N$.

DGP4: Values are correlated with each other, and with N. $\mu = 2.5$ or 1.5 with probabilities 1/3 and 2/3 respectively, and N is determined endogenously via equilibrium play of the entry game described in Samuelson (1985). There are 12 potential bidders, each of whom learns μ and his own valuation before deciding whether to pay a cost of \$10 to participate in the auction. Bidders play a symmetric, cutoff-strategy equilibrium, with the cutoff value varying with μ (see Footnote 9); this induces a positive relationship between N and μ, and therefore between N and valuations.

For computational simplicity, our experimental design does not include observable covariates X. As a result, we do not have to choose a kernel or a bandwidth h_L. The remaining tuning parameters are the bandwidth sequence b_L and c_L. We set c_L at 10^{-8} in all cases, and we computed our simulation results for various values of b_L. For DGPs 1-3, we generated N randomly as described there, with $N \geq 2$ in every auction simulated. In DGP4, N is determined endogenously. For our simulations, we only considered auctions where $N \geq 2$. Thus, in our results, when we say "sample size $L = 600$", we refer to simulated samples of size $L = 600$ where $N \geq 2$ for *every* observation in the sample. In our testing range, we trimmed N above at the value of 8 in all cases. Finally, the target asymptotic significance level was set to 5%; thus the critical value used was 1.645.

Looking at Tables B.1-B.2, we see that our tests displayed remarkably good power properties for DGPs 1-3. The asymptotic size properties described in Proposition B4 are also reasonably good approximations for the finite sample properties of the tests. Overall, for a sample size of $L = 1000$, the asymptotic features of our tests appear to be already in display. For the case of DGP4, as we could anticipate from Figure 1, it takes relatively larger sample sizes to achieve good power, and the latter is significantly higher for (4) than for (5). This was also something we could anticipate from Figure 1, where the evidence against (4) appears to be stronger than that against (5). To investigate further the power properties of our tests for the case of DGP4, we ran additional simulations with samples of size $L = 2000$. As the results in Table B.4 show in this case, our test has significant power against (4) for this sample size, while larger ones still appear to be needed in order to have power against (5). In all cases, choosing the sequence b_L such that $b_L \leq 0.005$ for moderately large sample sizes (i.e, $L \geq 500$) appeared to achieve an adequate balance between size and power.

Table B.1: Proportion of rejections for DGP1†. 1000 Samples of Size 'L' Simulated in Each Case.

	$L = 200$		$L = 600$		$L = 1000$	
b_L	Eq. 4	Eq. 5	Eq. 4	Eq. 5	Eq. 4	Eq. 5
0.05	0.086	0.188	0.012	0.007	0.001	0.002
0.01	0.300	0.502	0.081	0.117	0.027	0.034
0.005	0.381	0.541	0.099	0.154	0.047	0.055
0.001	0.346	0.573	0.119	0.179	0.062	0.081
0.0005	0.347	0.573	0.121	0.182	0.065	0.085
0.0001	0.348	0.575	0.122	0.183	0.065	0.086

(†) Asymptotic rejection probabilities: Eq. 4: 5%. Eq. 5: 5%.

Table B.2: Proportion of rejections for DGP2◇. 1000 Samples of Size 'L' Simulated in Each Case.

	$L = 200$		$L = 600$		$L = 1000$	
b_L	Eq. 4	Eq. 5	Eq. 4	Eq. 5	Eq. 4	Eq. 5
0.05	0.014	0.530	0.000	0.368	0.000	0.379
0.01	0.073	0.814	0.004	0.733	0.000	0.770
0.005	0.093	0.839	0.005	0.780	0.000	0.812
0.001	0.110	0.854	0.009	0.818	0.001	0.847
0.0005	0.111	0.855	0.009	0.819	0.001	0.852
0.0001	0.113	0.856	0.009	0.821	0.001	0.857

(◇) Asymptotic rejection probabilities: Eq. 4: 0%. Eq. 5: 100%.

Table B.3: Proportion of rejections for DGP3[‡]. 1000 Samples of Size 'L' Simulated in Each Case.

	$L = 200$		$L = 600$		$L = 1000$	
b_L	Eq. 4	Eq. 5	Eq. 4	Eq. 5	Eq. 4	Eq. 5
0.05	0.757	0.002	0.965	0.000	0.997	0.000
0.01	0.903	0.059	0.994	0.000	1.000	0.000
0.005	0.914	0.076	0.997	0.001	1.000	0.000
0.001	0.925	0.094	0.997	0.003	1.000	0.000
0.0005	0.925	0.097	0.997	0.003	1.000	0.000
0.0001	0.926	0.098	0.997	0.003	1.000	0.000

(‡) Asymptotic rejection probabilities: Eq. 4: 100%. Eq. 5: 0%.

Table B.4: Proportion of rejections for DGP4[§]. 1000 Samples of Size 'L' Simulated in Each Case.

	$L = 200$		$L = 600$		$L = 1000$		$L = 2000$	
b_L	Eq. 4	Eq. 5	Eq. 4	Eq. 5	Eq. 4	Eq. 5	Eq. 4	Eq. 5
0.05	0.049	0.109	0.112	0.099	0.161	0.088	0.253	0.050
0.01	0.100	0.188	0.265	0.156	0.380	0.202	0.647	0.203
0.005	0.107	0.198	0.302	0.167	0.427	0.212	0.703	0.232
0.001	0.117	0.212	0.326	0.176	0.454	0.218	0.740	0.257
0.0005	0.118	0.212	0.328	0.177	0.456	0.218	0.741	0.258
0.0001	0.120	0.212	0.330	0.177	0.458	0.218	0.743	0.259

(§) Asymptotic rejection probabilities: Eq. 4: 100%. Eq. 4: 100%.

B.8 Empirical Results for USFS Timber Data

We present the results of our tests for Equations 1, 2, 3, 4 and 5 applied to the USFS Timber Data described in Section 4, where we describe the vector of observable covariates $X = (X^1, X^2, X^3, X^4)$ included in the analysis.

B.8.1 Kernels employed

The functionals we estimate in our counterfactual analysis are highly demanding of the data since they entail the estimation of nonparametric functionals for each observation in our sample. Furthermore, these functionals are all functions of N, which in turn means that for the estimation of the relevant functionals evaluated at the i^{th} observation we only utilize the subsample of observations with auction size N_i. Not surprisingly this, along with the fact that we are utilizing four continuously distributed covariates in X implies that, even though our asymptotic results are invariant to the choice of kernel and bandwidth (as long as Assumption B.1 is satisfied), our choice of these tuning parameters matters for the estimates obtained in the sample at hand. Assumption B.1 requires the use of a bias-reducing kernel with compact support. We used a multiplicative kernel of the form $K(X) = \prod_{\ell=1}^{4} k(x_\ell)$, where k is a bias-reducing kernel of the form

$$k(\psi) = \sum_{j=1}^{6} b_j \cdot \left(s^2 - \psi^2\right)^{2j} \cdot \mathbb{1}\{|\psi| \leq s\}.$$

By construction, $k(\psi)$ is symmetric around zero and therefore $\int_{-s}^{s} \psi^\ell k(\psi) d\psi = 0$ for all odd ℓ. Given s, the coefficients $(b_j)_{j=1}^{5}$ are chosen such that

$$\int_{-s}^{s} k(\psi) d\psi = 1, \quad and \quad \int_{-s}^{s} \psi^\ell k(\psi) d\psi = 0 \quad \text{for } \ell = 2, 4, 6, 8, 10.$$

This gives rise to a bias-reducing kernel of order 12 which is compatible with Assumptions B.5, B.8 and B.9. In an effort to be able to extract as much information as possible from our data -see the discussion above- we used a kernel with compact, but relatively large support; specifically, we used $s = 60$.

B.8.2 Test of Equation 1

Bandwidths and testing range

What follows pertains to the first-price auction data described in Section 4.2. We chose $\mathcal{N} = \{2, \ldots, 11\}$ as our testing range for N. Let

$$\underline{f}^1 = \left\{ 5^{th} \text{ smallest value of } \left\{ \widehat{f}_{X,N}^{-i}(X_i, N_i) \right\}_{i=1}^{L} : N_i \in \mathcal{N} \text{ and } \widehat{f}_{X,N}^{-i}(X_i, N_i) > 0 \right\}.$$

For our testing range we used

$$\widehat{\mathbb{I}}_i = \mathbb{1}\left\{ N_i \in \mathcal{N} \quad \text{and} \quad \widehat{f}_{X,N}^{-i}(X_i, N_i) \geq \underline{f}^1 \right\}.$$

Prior to testing, we took the natural log of B^I and B^{II}, de-meaned each, and divided each by its standard deviation. Finally, in order to investigate the sensitivity of our findings to the choice of the bandwidth h_L, we computed the test-statistic for three different values of h_L (given our fixed sample size). These were, $h_L = 0.1 \cdot \widehat{\sigma}(X)$, $h_L = 0.4 \cdot \widehat{\sigma}(X)$ and $h_L = 0.7 \cdot \widehat{\sigma}(X)$. The bandwidth $\widetilde{h}_L$ was set to $\widetilde{h}_L = 1.25 \cdot h_L$ in all cases.

Results

Not surprisingly given our relatively small sample size ($L = 192$) for first-price auction data, our numerical results are sensitive to the choice of bandwidths. However, as Table B.5 shows, rejection of Equation 1 (and therefore of IPV) is a robust finding for the relatively wide range of bandwidth values examined.

Table B.5: Results for our test of Equation 1.

	$h_L = 0.1 \cdot \widehat{\sigma}(X)$	$h_L = 0.4 \cdot \widehat{\sigma}(X)$	$h_L = 0.7 \cdot \widehat{\sigma}(X)$
$\widehat{t}_1$	8.1906	42.3939	49.3395

$\widetilde{h}_L = 1.25 \cdot h_L$ in all cases.

In all cases, the p-value of our test (given by $1 - \Phi(\widehat{t}_1)$, where Φ is the $N(0,1)$ cdf.) was effectively zero, leading us to reject Equation 1 (and therefore of IPV) at any significance value.

B.8.3 Test of Equation 2

Bandwidths and testing range

What follows pertains to the English auction data described in Section 4.2. We chose $\mathcal{N} = 3, \ldots, 11$ as our testing range for N. Given the nature of our data, for each n we used only $k = n - 2$ in our test. For each given $n \in \mathcal{N}$, let

$$\underline{z}^{n,k} = \min\{Z_i : N_i = n\} \quad \text{and} \quad \overline{z}^{n,k} = \max\{Z_i : N_i = n\},$$
$$\underline{f}_n^2 = \left\{ 5^{th} \text{ smallest value of } \left\{\widehat{f}_{X,N}^{-i}(X_i, n)\right\}_{i=1}^{L} : N_i = n \text{ and } \widehat{f}_{X,N}^{-i}(X_i, n) > 0 \right\}.$$

For our testing range we used

$$\widehat{\mathbb{I}}_i^{n,k} = \mathbb{1}\left\{\widehat{f}_{X,N}^{-i}(X_i, n) \geq \underline{f}_n^2 \quad \text{and} \quad \underline{z}^{n,k} \leq Z_i \leq \overline{z}^{n,k}\right\}.$$

The bandwidth h_L was set so that $h_L = 0.4 \cdot \widehat{\sigma}(X)$ for the sample size L of our data[26], and we chose c_L such that $c_L = 10^{-12}$ at our sample size. Given these tuning parameters, we computed our test statistics for different values of the sequence b_L.

Results

As Table B.6 illustrates, our results reject Equation 2 (and therefore IPV under the "incomplete model" assumptions in Haile and Tamer (2003)) for all the different values of tuning parameters employed.

Table B.6: Results for our test of Equation 2.

	$b_L = 0.05$	$b_L = 0.01$	$b_L = 0.005$	$b_L = 0.001$	$b_L = 0.0005$	$b_L = 0.0001$
$\widehat{t}_2$	22.4685	28.5372	28.6733	28.8303	28.8554	28.8678

$h_L = 0.4 \cdot \widehat{\sigma}(X)$ and $c_L = 10^{-12}$ in all cases.

The p-value (given by $1 - \Phi(\widehat{t}_2)$) was effectively zero in all cases, leading us to reject Equation 2 for any significance level.

[26] The bandwidth employed in the empirical analysis in Haile and Tamer (2003) for the USFS Timber data was equal to $0.4 \cdot \widehat{\sigma}(X)$.

B.8.4 Test of Equation 3

Bandwidths and testing range

This test involves our English auction data. We chose $\mathcal{N} = 2, \ldots, 11$ as our testing range for N. For each pair n, n' in $\mathcal{N}$ let

$$\underline{z}^{n,n'} = \min\{Z_i : N_i = n, n'\} \quad \text{and} \quad \overline{z}^{n,n'} = \max\{Z_i : N_i = n, n'\},$$
$$\underline{f}^3_n = \left\{5^{th} \text{ smallest value of } \left\{\widehat{f}^{-i,j}_{X,N}(X_i, n)\right\}_{i=1}^{L} : N_i = n \text{ and } \widehat{f}^{-i,j}_{X,N}(X_i, n) > 0\right\},$$
$$\underline{f}^3_{n'} = \left\{5^{th} \text{ smallest value of } \left\{\widehat{f}^{-i,j}_{X,N}(X_i, n')\right\}_{i=1}^{L} : N_i = n' \text{ and } \widehat{f}^{-i,j}_{X,N}(X_i, n') > 0\right\}$$

and $\underline{f}^3_{n,n'} = \max\left\{\underline{f}_n \,,\, \underline{f}_{n'}\right\}$. For our testing range we used

$$\widehat{\mathbb{I}}^{n,n'}_{i,j} = \mathbb{1}\left\{\min\left\{\widehat{f}^{-i,j}_{X,N}(X_i, n) \,,\, \widehat{f}^{-i,j}_{X,N}(X_i, n')\right\} \geq \underline{f}^3_{n,n'} \quad \text{and} \quad \underline{z}^{n,n'} \leq Z_i \leq \overline{z}^{n,n'}\right\}.$$

As we did in our test of Equation 1, we computed our test for three alternative values of the bandwidth h_L. Namely, $h_L = 0.1 \cdot \widehat{\sigma}(X)$, $h_L = 0.4 \cdot \widehat{\sigma}(X)$ and $h_L = 0.7 \cdot \widehat{\sigma}(X)$. The bandwidth $\widetilde{h}_L$ was set to $\widetilde{h}_L = 1.25 \cdot h_L$ in all cases.

Results

As we discussed in Section 4.2, the sample size of our English auction data ($L = 2036$) is considerably larger than the one for first-price auctions ($L = 192$). Given the asymptotic properties of the tests for equations 1 and 3 (both of which explode to $+\infty$ at a rate faster than $\sqrt{L}$ if the corresponding equations are violated), we would expect to observe considerably large, positive values of $\widehat{t}_3$ if Equation 3 is violated. This is in fact what we observe in Table B.7 for the range of bandwidths studied. Our results provide very strong evidence against Equation 3 and consequently against IPV in a setting where the distribution of valuations is independent of N and transaction price corresponds to the second highest valuation among bidders. The p-value of our test-statistic is zero in all cases presented.

B.8.5 Test of Equations 4 and 5

Bandwidths and testing ranges

We employed the same testing range definition as in our test for Equation 3 (replacing leave-two-out nonparametric estimators with leave-one-out ones). As we did in the test of

Table B.7: Results for our test of Equation 3.

	$h_L = 0.1 \cdot \widehat{\sigma}(X)$	$h_L = 0.4 \cdot \widehat{\sigma}(X)$	$h_L = 0.7 \cdot \widehat{\sigma}(X)$
$\widehat{t}_3$	61.2945	414.3434	406.2894

$\widetilde{h}_L = 1.25 \cdot h_L$ in all cases.

(2), the bandwidth h_L was set so that $h_L = 0.4 \cdot \widehat{\sigma}(X)$ for the sample size L of our data, and we chose c_L such that $c_L = 10^{-12}$ at our sample size. Given these tuning parameters, we computed our test statistics for different values of the sequence b_L.

Results

The results in Table B.8 support the visual evidence in Figure 2. As our p-values show, Equation 5 can be rejected at any significance level, while we would fail to reject Equation 4, e.g., at a significance level of 5% or 1%. Thus, even allowing for the distribution of valuations to depend on N, our results still reject IPV. However, allowing for correlation in values, the assertion that their distribution is independent of N cannot be rejected in the data. Going over all our results, *Equation 4 was the only one we failed to reject.*

Table B.8: Test Results for Equations 4 and 5

	Equation 4		Equation 5	
b_L	$\widehat{t}_4$	p-value	$\widehat{t}_5$	p-value
0.05	-1.0924	0.8627	19.4442	0.0000
0.01	0.0198	0.4921	19.5307	0.0000
0.005	0.0482	0.4808	19.5871	0.0000
0.001	0.0561	0.4776	19.6387	0.0000
0.0005	0.0563	0.4775	19.6452	0.0000
0.0001	0.0564	0.4775	19.6490	0.0000

$h_L = 0.4 \cdot \widehat{\sigma}(X)$ and $c_L = 10^{-12}$ in all cases.

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