Supplementary Appendix for Aradillas-López, Gandhi and Quint (2011), "Testing Auction Models: Are Private Values Really Independent?"

December 22, 2011

Equations in this appendix are numbered in a format **(A.xx)**. Equation numbers without this format refer to those in the main paper.

1 Proof of Propositions B1 and B3

we begin by proving the results that involve testing conditional moment equalities in the paper. These are the results described in Propositions B1 and B3.

1.1 Proposition B1

Lemma A1. Under Assumptions B.1, B.3 and B.2,

$$Pr\Big[\mathbb{1}\left\{\widehat{f}^{-i}_{X,N}(X_i,N_i)\geq \underline{f}\right\}\neq \mathbb{1}\left\{f_{X,N}(X_i,N_i)\geq \underline{f}\right\} \text{ for at least one } i=1,\ldots,L \text{ over our testing range}\Big]\longrightarrow 0$$

Proof: First, note that the probability in Lemma A1 is bounded above by

$$\sum_{i=1}^{L} Pr\left[\mathbb{1}\left\{\widehat{f}^{-i}_{X,N}(X_i,N_i)\geq \underline{f}\right\}\neq \mathbb{1}\left\{f_{X,N}(X_i,N_i)\geq \underline{f}\right\}\right].$$

By Assumption B.2(i), we have $Pr\left[f_{X,N}(X_i,N_i)=\underline{f}\right]=0$. Therefore,

$$\begin{aligned}Pr\Big[\mathbb{1}\left\{\widehat{f}^{-i}_{X,N}(X_i,N_i)\geq \underline{f}\right\}\neq \mathbb{1}\left\{f_{X,N}(X_i,N_i)\geq \underline{f}\right\}\Big] = \\ E\left[Pr\left[\widehat{f}^{-i}_{X,N}(X_i,N_i)\geq \underline{f}\Big|X_i,N_i\right]\cdot \mathbb{1}\left\{f_{X,N}(X_i,N_i)<\underline{f}\right\}\right] \\ +E\left[Pr\left[\widehat{f}^{-i}_{X,N}(X_i,N_i)<\underline{f}\Big|X_i,N_i\right]\cdot \mathbb{1}\left\{f_{X,N}(X_i,N_i)>\underline{f}\right\}\right]\end{aligned}$$

Next note that our smoothness assumptions imply that

$$\sup_{(x,n)\in\mathcal{X}\times\mathcal{N}}\left|E\left[\frac{1}{h_L^q}K\left(\frac{X_j^c-x^c}{h_L}\right)\cdot\mathbb{1}\left\{X_j^d=x^d\right\}\cdot\mathbb{1}\left\{N_j=n\right\}\right]-f_{X,N}(x,n)\right|=O\left(h_L^M\right)$$

for any compact sets $\mathcal{X}$ and $\mathcal{N}$ with the features of our testing range. Let

$$\widetilde{s}_{L}(X_i, N_i) = \underline{f} - E\left[\frac{1}{h_L^q} K\left(\frac{X_j^c - X_i^c}{h_L^q}\right) \mathbb{1}\{X_j^d = X_i^d\}\mathbb{1}\{N_j = N_i\}\middle| X_i, N_i\right].$$

Then by Bernstein's inequality (see Section 2.2.2 in van der Vaart and Wellner (1996)), for sufficiently large L, there exist positive constants C_1 and C_2 such that

$$\begin{aligned} &Pr\left[\widehat{f}_{X,N}^{-i}(X_i, N_i) \geq \underline{f}\middle| X_i, N_i\right] \cdot \mathbb{1}\left\{f_{X,N}(X_i, N_i) < \underline{f}\right\} \\ &\qquad\qquad \leq \exp\left\{-\frac{(L-1)h_L^q \widetilde{s}_L(X_i, N_i)^2}{C_1 + C_2 \widetilde{s}_L(X_i, N_i)}\right\} \mathbb{1}\left\{f_{X,N}(X_i, N_i) < \underline{f}\right\} \quad \forall\ i. \end{aligned}$$

and

$$\begin{aligned} &Pr\left[\widehat{f}_{X,N}^{-i}(X_i, N_i) < \underline{f}\middle| X_i, N_i\right] \cdot \mathbb{1}\left\{f_{X,N}(X_i, n) > \underline{f}\right\} \\ &\qquad\qquad \leq \exp\left\{-\frac{(L-1)h_L^q \widetilde{s}_L(X_i, N_i)^2}{C_1 - C_2 \widetilde{s}_L(X_i, N_i)}\right\} \mathbb{1}\left\{f_{X,N}(X_i, N_i) > \underline{f}\right\} \quad \forall\ i. \end{aligned}$$

From here, invoking the Lebesgue dominating convergence theorem yields

$$\sum_{i=1}^{L} Pr\left[\mathbb{1}\left\{\widehat{f}_{X,N}^{-i}(X_i, N_i) \geq \underline{f}\right\} \neq \mathbb{1}\left\{f_{X,N}(X_i, N_i) \geq \underline{f}\right\}\right] \longrightarrow 0,$$

proving the claim. $\square$

As we defined in the statement of Proposition B1, let

$$\begin{aligned} \mathbb{I}_i &= \mathbb{1}\left\{N_i \in \mathcal{N} \quad \text{and} \quad f_{X,N}(X_i, N_i) \geq \underline{f}\right\}, \\ \widehat{\mathbb{I}}_i &= \mathbb{1}\left\{N_i \in \mathcal{N} \quad \text{and} \quad \widehat{f}_{X,N}^{-i}(X_i, N_i) \geq \underline{f}\right\}. \end{aligned}$$

From Lemma A1 we have

$$Pr\left[\mathbb{I}_i \neq \widehat{\mathbb{I}}_i \text{ for at least one } i = 1, \ldots, L \text{ over our testing range}\right] \longrightarrow 0 \qquad \text{(A.1)}$$

A linear representation for $\widetilde{\nu}^{I^{-i}}$, $\widetilde{\nu}^{II^{-i}}$, $\widehat{cov}^{-i}$

Let

$$\begin{aligned} &H_{1_L}(U_j|x, n) = \\ &\left\{\frac{1}{f_{X,N}(x,n)} \times \left[\left(B_j^I B_j^{II} - E\left[B^I B^{II}|x, n\right]\right) - \nu^{II}(x,n) \cdot \left(B_j^I - \nu^I(x,n)\right) - \nu^I(x,n) \cdot \left(B_j^{II} - \nu^{II}(x,n)\right)\right]\right\} \\ &\times \mathbb{1}\left\{N_j = n\right\} \cdot \mathbb{1}\left\{X_j^d = x^d\right\} \cdot K\left(\frac{X_j^c - x^c}{h_L}\right) \end{aligned}$$

Under the assumptions of Proposition B1, we can show (see, e.g, the Appendix to Aradillas-Lopez (2010)) that

$$\widehat{cov}^{-i}\left(B^I, B^{II}|x,n\right) = cov\left(B^I, B^{II}|x,n\right) + \frac{1}{(L-1)}\sum_{j\neq i}\frac{1}{h_L^q}H_{1_L}(U_j|x,n) + \varepsilon_L^{-i}(x,n),$$
$$\text{where} \quad \underset{(x,n)\in\mathcal{T}_1}{sup}\left|\varepsilon_L^{-i}(x,n)\right| = O_p\left(\frac{1}{L^{1-\delta}h_L^q}\right) \quad \forall\ \delta > 0 \tag{A.2}$$

and

$$\widetilde{\nu}^{I^{-i}}(x,n) =$$
$$\nu^I(x,n) + \frac{1}{(L-1)}\sum_{j\neq i}\frac{\left[B_j^I - \nu^I(x,n)\right]}{f_{X,N}(x,n)}\cdot\mathbb{1}\left\{N_j = n\right\}\cdot\mathbb{1}\left\{X_j^d = x^d\right\}\cdot\frac{1}{\widetilde{h}_L^q}K\left(\frac{X_j^c - x^c}{\widetilde{h}_L}\right) + \widetilde{\varepsilon}_L^{I^{-i}}(x,n),$$
$$\widetilde{\nu}^{II^{-i}}(x,n) =$$
$$\nu^{II}(x,n) + \frac{1}{(L-1)}\sum_{j\neq i}\frac{\left[B_j^{II} - \nu^{II}(x,n)\right]}{f_{X,N}(x,n)}\cdot\mathbb{1}\left\{N_j = n\right\}\cdot\mathbb{1}\left\{X_j^d = x^d\right\}\cdot\frac{1}{\widetilde{h}_L^q}K\left(\frac{X_j^c - x^c}{\widetilde{h}_L}\right) + \widetilde{\varepsilon}_L^{II^{-i}}(x,n),$$
$$\text{where} \quad \underset{(x,n)\in\mathcal{T}_1}{sup}\left|\widetilde{\varepsilon}_L^{I^{-i}}(x,n)\right| = O_p\left(\frac{1}{L^{1-\delta}\widetilde{h}_L^q}\right), \quad \underset{(x,n)\in\mathcal{T}_1}{sup}\left|\widetilde{\varepsilon}_L^{II^{-i}}(x,n)\right| = O_p\left(\frac{1}{L^{1-\delta}\widetilde{h}_L^q}\right) \quad \forall\ \delta > 0. \tag{A.3}$$

Note, in particular, that (A.2) and (A.3) imply that

$$\begin{aligned}
&\left|\frac{1}{L}\sum_{i=1}^{L}\left\{\left(\widetilde{\nu}^{I^{-i}}(X_i,N_i)\cdot\widetilde{\nu}^{II^{-i}}(X_i,N_i) - \nu^I(X_i,N_i)\cdot\widetilde{\nu}^{II}(X_i,N_i)\right)\right.\right. \\
&\qquad\qquad \left.\left.\times\left(\widehat{cov}^{-i}\left(B^I, B^{II}|X_i,N_i\right) - cov\left(B^I, B^{II}|X_i,N_i\right)\right)\cdot\mathbb{I}_i\right\}\right| \\
&\qquad\qquad \le \underset{(x,n)\in\mathcal{T}_1}{sup}\left|\left(\widetilde{\nu}^{I^{-i}}(x,n)\cdot\widetilde{\nu}^{II^{-i}}(x,n) - \nu^I(x,n)\cdot\widetilde{\nu}^{II}(x,n)\right)\right| \\
&\qquad\qquad \times \underset{(x,n)\in\mathcal{T}_1}{sup}\left|\widehat{cov}^{-i}\left(B^I, B^{II}|x,n\right) - cov\left(B^I, B^{II}|x,n\right)\right| \\
&\qquad\qquad = O_p\left(\frac{1}{L^{1-\delta}\cdot h_L^{\frac{q}{2}}\cdot\widetilde{h}_L^{\frac{q}{2}}}\right) \quad \forall\ \delta > 0 \\
&\qquad\qquad = o_p\left(\frac{1}{L\cdot h_L^{\frac{q}{2}}}\right),
\end{aligned} \tag{A.4}$$

where the last line follows if we use the δ described in Assumption B.3.

The result in Equation (A.1) allows us to study $\widehat{\mathcal{M}}_1$ as

$$\widehat{\mathcal{M}}_1 = \frac{1}{L}\sum_{i=1}^{L}\left(B_i^I B_i^{II} - \widetilde{\nu}^{I^{-i}}(X_i,N_i)\cdot\widetilde{\nu}^{II^{-i}}(X_i,N_i)\right)\cdot\widehat{cov}^{-i}\left(B^I, B^{II}|X_i,N_i\right)\cdot\mathbb{I}_i.$$

From here, it will be convenient to decompose $\widehat{\mathcal{M}}_1$ in the following way,

$$\widehat{\mathcal{M}}_1 = \overline{\mathcal{M}}_1 + \widehat{\mathcal{Q}}_1 + \widetilde{\mathcal{S}}_1^a + \widehat{\mathcal{S}}_1^b,$$

where

$$\overline{\mathcal{M}}_1 = \frac{1}{L}\sum_{i=1}^{L}\left(B_i^I B_i^{II} - \nu^I(X_i,N_i)\cdot\nu^{II}(X_i,N_i)\right)\cdot cov\left(B^I, B^{II}|X_i,N_i\right)\cdot \mathbb{I}_i,$$

$$\widehat{\mathcal{Q}}_1 = \frac{1}{L}\sum_{i=1}^{L}\left(B_i^I B_i^{II} - \nu^I(X_i,N_i)\cdot\nu^{II}(X_i,N_i)\right)\cdot\left(\widehat{cov}^{-i}\left(B^I, B^{II}|X_i,N_i\right) - cov\left(B^I, B^{II}|X_i,N_i\right)\right)\cdot \mathbb{I}_i,$$

$$\widetilde{\mathcal{S}}_1^a = \frac{1}{L}\sum_{i=1}^{L}\left(\nu^I(X_i,N_i)\cdot\nu^{II}(X_i,N_i) - \widetilde{\nu}^{I^{-i}}(X_i,N_i)\cdot\widetilde{\nu}^{II^{-i}}(X_i,N_i)\right)\cdot cov\left(B^I, B^{II}|X_i,N_i\right)\cdot \mathbb{I}_i,$$

$$\widehat{\mathcal{S}}_1^b = \frac{1}{L}\sum_{i=1}^{L}\Bigg\{\left(\nu^I(X_i,N_i)\cdot\nu^{II}(X_i,N_i) - \widetilde{\nu}^{I^{-i}}(X_i,N_i)\cdot\widetilde{\nu}^{II^{-i}}(X_i,N_i)\right)$$
$$\times\left(\widehat{cov}^{-i}\left(B^I, B^{II}|X_i,N_i\right) - cov\left(B^I, B^{II}|X_i,N_i\right)\right)\cdot \mathbb{I}_i\Bigg\} = o_p\left(\frac{1}{L\cdot h_L^{\frac{q}{2}}}\right)$$

where the last result follows directly from (A.4). Thus,

$$\widehat{\mathcal{M}}_1 = \overline{\mathcal{M}}_1 + \widehat{\mathcal{Q}}_1 + \widetilde{\mathcal{S}}_1^a + o_p\left(\frac{1}{L\cdot h_L^{\frac{q}{2}}}\right). \tag{A.5}$$

Let

$$\phi_{1_L}(U_i,U_j) = \Bigg[\left(B_i^I B_i^{II} - \nu^I(X_i,N_i)\nu^{II}(X_i,N_i)\right)\cdot \mathbb{I}_i\cdot H_{1_L}(U_j|X_i,N_i)$$
$$+\left(B_j^I B_j^{II} - \nu^I(X_j,N_j)\nu^{II}(X_j,N_j)\right)\cdot\cdot \mathbb{I}_j\cdot H_{1_L}(U_i|X_j,N_j)\Bigg]$$

By (A.2) we have

$$\widehat{\mathcal{Q}}_1 = \binom{L}{2}^{-1}\sum_{i<j}\frac{1}{h_L^q}\cdot\frac{1}{2}\cdot\phi_{1_L}(U_i,U_j) + o_p\left(\frac{1}{L\cdot h_L^{\frac{q}{2}}}\right) \equiv V_{1_L} + o_p\left(\frac{1}{L\cdot h_L^{\frac{q}{2}}}\right). \tag{A.6}$$

Next note that $E\left[B_j^I B_j^{II} - \nu^I(X_j,N_j)\nu^{II}(X_j,N_j)\big|U_i,X_j,N_j\right] = 0$. In addition, under the smoothness conditions described in Assumption B.1 we have

$$E\left[\frac{1}{h_L^q}\cdot H_{1_L}(U_j|X_i,N_i)\big|U_i,X_j,N_j\right] = O(h_L^M).$$

This, along with iterated expectations and a dominated convergence argument yield

$$E\left[\frac{1}{h_L^q}\cdot\phi_{1_L}(U_i,U_j)\big|U_i\right] = O(h_L^M) = o_p\left(\frac{1}{L\cdot h_L^{\frac{q}{2}}}\right),$$

where the last equality follows from the bandwidth convergence restrictions described in Assumptions B.1 and B.3. That is, up to a term of order $o_p\left(\frac{1}{L\cdot h_L^{\frac{q}{2}}}\right)$, the term $\widehat{\mathcal{Q}}_1$ is a *degenerate* U-statistic of order two. Applying Theorem 1 of Hall (1984) we can obtain its asymptotic distribution.

Lemma A2. Consider a second order U-statistic of the form

$$V_L = \binom{L}{2}^{-1} \sum_{i<j} H_L(U_i, U_j),$$

where $\{U_i\}_{i=1}^L$ is an iid random sample and H_L is a symmetric function, i.e., $H_L(U_i,U_j) = H_L(U_j,U_i)$. Assume $E\left[H_L(U_i,U_j)\middle|U_i\right] = 0$ almost surely and $E\left[H_L^2(U_i,U_j)\right] < \infty$. Define

$$G_L(U_i,U_j) = E\left[H_L(U_k,U_i)\cdot H_L(U_k,U_j)\middle|U_i,U_j\right].$$

If

$$\frac{E\left[G_L^2(U_i,U_j)\right] + L^{-1}E\left[H_L^4(U_i,U_j)\right]}{\left\{E\left[H_L^2(U_i,U_j)\right]\right\}^2} \longrightarrow 0 \quad \text{as } L\to\infty, \tag{A.7}$$

then

$$\frac{L\cdot V_L}{\sqrt{2\cdot E\left[H_L^2(U_i,U_j)\right]}} \xrightarrow{d} \mathcal{N}(0,1).$$

Proof: This is Theorem 1 in Hall (1984). □

Lemma A3. Under the assumptions of Proposition B1, we have

$$L\cdot h_L^{\frac{q}{2}}\cdot V_{1_L} \xrightarrow{d} \mathcal{N}\left(0,\sigma_1^2\right)$$

where $\sigma_1^2 = \frac{1}{2}\cdot\lim_{L\to\infty} E\left[\frac{1}{h_L^q}\phi_{1_L}^2(U_i,U_j)\right]$. Therefore, by (A.6) we also have

$$L\cdot h_L^{\frac{q}{2}}\cdot \widehat{\mathcal{Q}}_1 \xrightarrow{d} \mathcal{N}\left(0,\sigma_1^2\right)$$

Proof: Under the existence-of-moment condition in Assumption B.2, it is not hard to show that

$$E\left[\left\{E\left[\frac{1}{h_L^{2q}}\cdot\phi_{1_L}(U_k,U_i)\cdot\phi_{1_L}(U_k,U_j)\middle|U_i,U_j\right]\right\}^2\right] = O\left(\frac{1}{h_L^q}\right),$$

$$E\left[\frac{1}{h_L^{2q}}\cdot\phi_{1_L}^2(U_i,U_j)\right] = O\left(\frac{1}{h_L^q}\right),\quad E\left[\frac{1}{h_L^{4q}}\cdot\phi_{1_L}^4(U_i,U_j)\right] = O\left(\frac{1}{h_L^{3q}}\right).$$

And therefore the criterion in (A.7) in the case of the U-statistic V_{1_L} in (A.6) is of the form

$$\frac{O\left(\frac{1}{h_L^q}\right) + O\left(\frac{1}{L\cdot h_L^{3q}}\right)}{O\left(\frac{1}{L\cdot h_L^{2q}}\right)} = \frac{O(h_L^q) + O\left(\frac{1}{L\cdot h_L^q}\right)}{O(1)} \longrightarrow 0,$$

where the last result follows from the bandwidth convergence restriction $L \cdot h_L^q \to 0$. Therefore, the conditions of Lemma A2 are satisfied for the U-statistic V_{1_L} in (A.6) and we obtain

$$L \cdot h_L^{\frac{q}{2}} \cdot V_{1_L} \xrightarrow{d} \mathcal{N}\left(0, \sigma_1^2\right)$$

where $\sigma_1^2 = \frac{1}{2} \cdot \lim_{L\to\infty} E\left[\frac{1}{h_L^q}\phi_{1_L}^2(U_i, U_j)\right]$. $\square$

Lemma A4. Under the conditions of Proposition B1, we have

$$\widetilde{\mathcal{S}}_1^a = o_p\left(\frac{1}{L \cdot h_L^{\frac{q}{2}}}\right).$$

Proof: Note that one of the implications of Lemma A3 is that $\widehat{\mathcal{Q}}_1 = O_p\left(\frac{1}{L\cdot h_L^{\frac{q}{2}}}\right)$. Following the same steps, we can show the analogous result for $\widetilde{\mathcal{S}}_1^a$. Namely,

$$\widetilde{\mathcal{S}}_1^a = O_p\left(\frac{1}{L \cdot \widetilde{h}_L^{\frac{q}{2}}}\right) = o_p\left(\frac{1}{L \cdot h_L^{\frac{q}{2}}}\right),$$

where the last equality follows from the bandwidth convergence conditions in Assumption B.3. $\square$

Lemma A5. Under Assumptions B.1, B.3 and B.2,

(i) If (1) is satisfied w.p.1 over our testing range, then

$$L \cdot h_L^{\frac{q}{2}} \cdot \widehat{\mathcal{M}}_1 \xrightarrow{d} \mathcal{N}\left(0, \sigma_1^2\right)$$

where $\sigma_1^2 = \frac{1}{2} \cdot \lim_{L\to\infty} E\left[\frac{1}{h_L^{q+1}}\phi_{1_L}(U_i, U_j)^2\right]$.

(ii) If (1) is violated with nonzero probability over our testing range, then

$$L \cdot h_L^{\frac{q}{2}} \cdot \widehat{\mathcal{M}}_1 \xrightarrow{p} +\infty$$

Proof: If (1) is satisfied w.p.1 over our testing range, we have $cov(B^I, B^{II}|X_i, N_i) \cdot \mathbb{I}_i = 0$ almost surely. Therefore, $\overline{\mathcal{M}}_1 = 0$ almost surely and part (i) of Lemma A5 follows from Eq. (A.5) and Lemmas A3-A4. For part (ii), notice that the uniform convergence results in (A.2)-(A.3) along with Lemma A1 imply that $\widehat{\mathcal{M}}_1 \xrightarrow{p} \mathcal{M}_1$. If (1) is violated with positive probability over our testing range, we have $\mathcal{M}_1 > 0$. Since $L \cdot h_L^{\frac{q}{2}} \to \infty$, the result in part (ii) of Lemma A5 follows. $\square$

Lemma A5 proves Proposition B1. $\square$

1.2 Proposition B3

Following parallel steps to the proof of Lemma A1, we can show that under Assumptions B.1, B.3 and B.7, we have

$$Pr\Big[\mathbb{I}_{i,j}^{n,n'} \neq \widehat{\mathbb{I}}_{i,j}^{n,n'} \text{ for at least one } i,j \in 1,\ldots,L \text{ over our testing range}\Big] \longrightarrow 0 \quad \forall\, n,n' \in \mathcal{N}. \tag{A.8}$$

From here on we will abbreviate

$$(L)_2 \equiv L \cdot (L-1).$$

(A.8) will allow us to study $\widehat{\mathcal{M}}_3$ as

$$\widehat{\mathcal{M}}_3 =$$

$$\frac{1}{(L)_2}\sum_{i\neq j}\left\{\sum_{n,n'\in\mathcal{N}} \mathbb{1}\{N_i = n\}\cdot\Big(\mathbb{1}\{B_{N_i:N_i} \leq Z_j\} - \widetilde{\lambda}_{n,n'}^{-i,j}(Z_j|X_i)\Big)\cdot\Big(\widehat{G}_{n:n}^{-i,j}(Z_j|X_i) - \widehat{\lambda}_{n,n'}^{-i,j}(Z_j|X_i)\Big)\cdot \mathbb{I}_{i,j}^{n,n'}\right\}$$

A linear representation for $\widetilde{\lambda}_{n,n'}^{-i,j}$ and $\widehat{G}_{n:n}^{-i,j} - \widehat{\lambda}_{n,n'}^{-i,j}$

Let

$$H_{3_L}(U_\ell|z,x,n,n') = \left\{\frac{(\mathbb{1}\{B_{N_\ell:N_\ell} \leq z\} - G_{n:n}(z|x))}{f_{X,N}(x,n)}\cdot \mathbb{1}\{N_\ell = n\}\right.$$
$$\left. - \frac{(\mathbb{1}\{B_{N_\ell:N_\ell} \leq z\} - G_{n':n'}(z|x))}{f_{X,N}(x,n')}\cdot \nabla\lambda_{n,n'}(z|x)\cdot \mathbb{1}\{N_\ell = n'\}\right\}\cdot \mathbb{1}\left\{X_\ell^d = x^d\right\}\cdot K\left(\frac{X_\ell^c - x^c}{h_L}\right)$$

Under the assumptions of Proposition B3 we can show (see, e.g, the Appendix to Aradillas-Lopez (2010)) that

$$\widehat{G}_{n:n}^{-i,j}(z|x) - \widehat{\lambda}^{-i,j}(z|x) = G_{n:n}(z|x) - \lambda(z|x) + \frac{1}{(L-2)}\sum_{\ell\neq i,j}\frac{1}{h_L^q}\cdot H_{3_L}(U_\ell|z,x,n,n') + \varsigma_L^{-i,j}(x,z,n,n'),$$
$$\text{where} \quad \sup_{\substack{(x,z)\in\mathcal{T}_3^{n,n'}\\ (n,n')\in\mathcal{N}}}\left|\varsigma_L^{-i,j}(x,z,n,n')\right| = O_p\left(\frac{1}{L^{1-\delta}h_L^q}\right) \quad \forall\, \delta > 0. \tag{A.9}$$

and

$$\widetilde{\lambda}^{-i,j}(z|x) = \lambda(z|x) + \frac{1}{(L-2)} \sum_{\ell \neq i,j} \left\{ \frac{1}{h_L^q} \cdot \frac{(\mathbb{1}\{B_{N_\ell:N_\ell} \leq z\} - G_{n':n'}(z|x))}{f_{X,N}(x,n')} \cdot \nabla \lambda_{n,n'}(z|x) \right.$$
$$\left. \times \mathbb{1}\{N_\ell = n'\} \cdot \mathbb{1}\left\{X_\ell^d = x^d\right\} \cdot K\left(\frac{X_\ell^c - x^c}{h_L}\right) \right\} + \widetilde{\varsigma}_L^{-i,j}(x,z,n,n'),$$
$$\text{where} \sup_{\substack{(x,z)\in\mathcal{T}_3^{n,n'} \\ (n,n')\in\mathcal{N}}} \left|\widetilde{\varsigma}_L^{-i,j}(x,z,n,n')\right| = O_p\left(\frac{1}{L^{1-\delta} h_L^q}\right) \quad \forall\, \delta > 0. \tag{A.10}$$

Parallel to Equation (A.4), from (A.9) and (A.10) we have in particular,

$$\begin{aligned}
&\left| \frac{1}{(L)_2} \sum_{i\neq j} \left\{ \sum_{n,n'\in\mathcal{N}} \mathbb{1}\{N_i = n\} \cdot \left(\widetilde{\lambda}^{-i,j}(Z_j|X_i) - \lambda(Z_j|X_i)\right) \right.\right. \\
&\qquad \left.\left. \times \left(\left[\widehat{G}_{n:n}^{-i,j}(Z_j|X_i) - \widehat{\lambda}^{-i,j}(Z_j|X_i)\right] - \left[G_{n:n}(Z_j|X_i) - \lambda(Z_j|X_i)\right]\right) \cdot \mathbb{I}_{i,j}^{n,n'} \right\} \right| \\
&\leq \sup_{\substack{(x,z)\in\mathcal{T}_3^{n,n'} \\ (n,n')\in\mathcal{N}}} \left|\widetilde{\lambda}^{-i,j}(z|x) - \lambda(z|x)\right| \times \sup_{\substack{(x,z)\in\mathcal{T}_3^{n,n'} \\ (n,n')\in\mathcal{N}}} \left|\left[\widehat{G}_{n:n}^{-i,j}(z|x) - \widehat{\lambda}^{-i,j}(z|x)\right] - \left[G_{n:n}(z|x) - \lambda(z|x)\right]\right| \\
&= O_p\left(\frac{1}{L^{1-\delta} \cdot h_L^{\frac{q}{2}} \cdot \widetilde{h}_L^{\frac{q}{2}}}\right) \quad \forall\, \delta > 0 \\
&= o_p\left(\frac{1}{L \cdot h_L^{\frac{q}{2}}}\right),
\end{aligned} \tag{A.11}$$

where the last equality follows if we use the δ described in Assumption B.3. Using (A.8)-(A.11), we can decompose $\widehat{\mathcal{M}}_3$ as

$$\widehat{\mathcal{M}}_3 = \overline{\mathcal{M}}_3 + \widehat{\mathcal{Q}}_3 + \widetilde{\mathcal{S}}_3 + o_p\left(\frac{1}{L \cdot h_L^{\frac{q}{2}}}\right), \tag{A.12}$$

where

$$\overline{\mathcal{M}}_3 =$$
$$\frac{1}{(L)_2} \sum_{i\neq j} \left\{ \sum_{n,n'\in\mathcal{N}} \mathbb{1}\{N_i = n\} \cdot (\mathbb{1}\{B_{N_i:N_i} \leq Z_j\} - \lambda_{n,n'}(Z_j|X_i)) \cdot (G_{n:n}(Z_j|X_i) - \lambda_{n,n'}(Z_j|X_i)) \cdot \mathbb{I}_{i,j}^{n,n'} \right\},$$

$$\widehat{\mathcal{Q}}_3 = \frac{1}{(L)_2} \sum_{i\neq j} \left\{ \sum_{n,n'\in\mathcal{N}} \mathbb{1}\{N_i = n\} \cdot \left(\mathbb{1}\{B_{N_i:N_i} \leq Z_j\} - \lambda_{n,n'}(Z_j|X_i)\right) \right.$$
$$\left. \times \left(\left[\widehat{G}_{n:n}^{-i,j}(Z_j|X_i) - \widehat{\lambda}_{n,n'}^{-i,j}(Z_j|X_i)\right] - \left[G_{n:n}(Z_j|X_i) - \lambda_{n,n'}(Z_j|X_i)\right]\right) \cdot \mathbb{I}_{i,j}^{n,n'} \right\},$$

$$\widetilde{\mathcal{S}}_3 =$$
$$\frac{1}{(L)_2} \sum_{i\neq j} \left\{ \sum_{n,n'\in\mathcal{N}} \mathbb{1}\{N_i = n\} \cdot \left(\lambda_{n,n'}(Z_j|X_i) - \widetilde{\lambda}_{n,n'}^{-i,j}(Z_j|X_i)\right) \cdot (G_{n:n}(Z_j|X_i) - \lambda_{n,n'}(Z_j|X_i)) \cdot \mathbb{I}_{i,j}^{n,n'} \right\}.$$

Define

$$T_{3_L}(U_i, U_j, U_\ell) = \sum_{n,n' \in \mathcal{N}} \mathbb{1}\{N_i = n\} \cdot \Big(\mathbb{1}\left\{B_{N_i:N_i} \leq Z_j\right\} - \lambda_{n,n'}(Z_j|X_i) \Big) \cdot H_{3_L}(U_\ell | Z_j, X_i, n, n') \cdot \mathbb{I}_{i,j}^{n,n'}$$

and

$$\mu_{3_L}(U_i, U_j, U_\ell) = \frac{1}{3!} \sum_{\substack{c_3(i,j,\ell) \in \\ (s_1,s_2,s_3)}} T_{3_L}(U_{s_1}, U_{s_2}, U_{s_3})$$

where $c_3(i, j, \ell)$ denotes the 3! permutations (s_1, s_2, s_3) of (i, j, ℓ). By construction, ϕ_{3_L} is symmetric in its three arguments. From (A.9)-(A.10), and the bandwidth convergence conditions in Assumptions B.1 and B.3 we can express

$$\widehat{\mathcal{Q}}_3 = \binom{L}{3}^{-1} \sum_{i<j<\ell} \frac{1}{h_L^q} \cdot \mu_{3_L}(U_i, U_j, U_\ell) + o_p\left(\frac{1}{L \cdot h_L^{\frac{q}{2}}}\right) \equiv V_{3_L} + o_p\left(\frac{1}{L \cdot h_L^{\frac{q}{2}}}\right) \tag{A.13}$$

Under the smoothness conditions in Assumption B.7, it is easy to show that

$$E\left[\frac{1}{h_L^q} T_{3_L}(U_i, U_j, U_\ell) \middle| U_i, U_j\right] = O(h_L^M).$$

In addition, if Equation (3) is satisfied w.p.1 over our testing range, we have

$$\begin{aligned} & E\left[\mathbb{1}\{N_i = n\} \cdot (\mathbb{1}\{B_{N_i:N_i} \leq Z_j\} - \lambda_{n,n'}(Z_j|X_i)) \,\middle|\, N_i, X_i, U_j, U_\ell, \mathbb{I}_{i,j} = 1\right] = 0 \\ \implies \quad & E\left[T_{3_L}(U_i, U_j, U_\ell) \middle| U_j, U_\ell\right] = 0. \end{aligned}$$

Thus, if Equation (3) is satisfied w.p.1 over our testing range and Assumptions B.1, B.3 and B.7 are satisfied, iterated expectations and a dominated convergence argument show that

$$\begin{aligned} & E\left[\frac{1}{h_L^q} T_{3_L}(U_i, U_j, U_\ell) \middle| U_i\right] = O(h_L^M), \\ & E\left[\frac{1}{h_L^q} T_{3_L}(U_i, U_j, U_\ell) \middle| U_j\right] = E\left[\frac{1}{h_L^q} T_{3_L}(U_i, U_j, U_\ell) \middle| U_\ell\right] = 0. \end{aligned} \tag{A.14}$$

From (A.14), if Equation (3) is satisfied w.p.1 over our testing range and Assumptions B.1, B.3 and B.7 are satisfied, we have

$$E\left[\mu_{3_L}(U_i, U_j, U_\ell) \middle| U_i\right] = O(h_L^M) = o\left(\frac{1}{L \cdot h_L^{\frac{q}{2}}}\right),$$

where the last equality follows from the bandwidth convergence restrictions in our assumptions. Thus, up to a term of order $o_p\left(\frac{1}{L \cdot h_L^{\frac{q}{2}}}\right)$, the term $\widehat{\mathcal{Q}}_3$ (as shown in (A.13)) is a *degenerate U-statistic of order three*. We can analyze its asymptotic distribution by using the results in Fan and Li (1996), who extended the theory developed in Hall (1984) to degenerate U-statistics of order higher than two.

Lemma A6. Consider a third order U-statistic of the form

$$V_L = \binom{L}{3}^{-1} \sum_{i<j<\ell} H_L(U_i, U_j, U_\ell),$$

where $\{U_i\}_{i=1}^L$ is an iid random sample and H_L is a symmetric function, i.e, $H_L(U_i, U_j, U_\ell) = H_L(U_i, U_\ell, U_j) = H_L(U_j, U_i, U_\ell) = H_L(U_j, U_\ell, U_i) = H_L(U_\ell, U_i, U_j) = H_L(U_\ell, U_j, U_i)$. Suppose $E\left[H_L(U_i, U_j, U_\ell)\middle|U_i\right] = 0$ almost surely and $E\left[H_L^2(U_i, U_j, U_\ell)\right] < \infty$. Denote

$$\begin{aligned}
\overline{H}_L(U_i, U_j) &= E\left[H_L(U_i, U_j, U_\ell)\middle|U_i, U_j\right], \\
\overline{G}_L(U_i, U_j) &= E\left[\overline{H}_L(U_k, U_i) \cdot \overline{H}_L(U_k, U_j)\middle|U_i, U_j\right].
\end{aligned}$$

If

$$\frac{E\left[\overline{G}_L^2(U_i, U_j)\right] + L^{-1} E\left[\overline{H}_L^4(U_i, U_j)\right]}{\left\{E\left[\overline{H}_L^2(U_i, U_j)\right]\right\}^2} \longrightarrow 0 \quad \text{as } L \to \infty, \tag{A.15}$$

then

$$\frac{L \cdot V_L}{\sqrt{18 \cdot E\left[\overline{H}_L^2(U_i, U_j)\right]}} \xrightarrow{d} \mathcal{N}(0, 1).$$

Proof: This is a special case of Lemma B.4 in Fan and Li (1996). □

Lemma A7. Let $\phi_{3_L}^a$ and $\phi_{3_L}^b$ as we did in the statement of Proposition B3. If Equation (3) is satisfied w.p.1 over our testing range and Assumptions B.1, B.3 and B.7 are satisfied, then

$$L \cdot h_L^{\frac{q}{2}} \cdot \widehat{\mathcal{Q}}_3 \xrightarrow{d} \mathcal{N}\left(0, \sigma_3^2\right), \quad \text{where} \quad \sigma_3^2 = \frac{1}{2} \cdot \lim_{L\to\infty} E\left[\frac{1}{h_L^q} \phi_{3_L}(U_i, U_j)^2\right]. \tag{A.16}$$

Proof: Let ρ^a and ρ^b be as defined in Assumption B.7(ii). Then we have

$$\begin{aligned}
&E\left[\frac{1}{h_L^q} \cdot T_{3_L}(U_i, U_j, U_\ell)\middle|U_i, U_\ell\right] = \\
&\sum_{n,n'\in\mathcal{N}} \left\{\frac{\rho_{n,n'}^a(B_{N_i:N_i}, B_{N_\ell:N_\ell}, X_i)}{f_{X,N}(X_i, n)} \cdot \mathbb{1}\left\{N_i = n, N_\ell = n\right\} - \frac{\rho_{n,n'}^b(B_{N_i:N_i}, B_{N_\ell:N_\ell}, X_i)}{f_{X,N}(X_i, n')} \cdot \mathbb{1}\left\{N_i = n, N_\ell = n'\right\}\right\} \\
&\qquad \times \mathbb{1}\left\{X_i^d = X_\ell^d\right\} \cdot K\left(\frac{X_\ell^c - X_i^c}{h_L}\right)
\end{aligned}$$

And so if we define $\phi^a_{3_L}$ and $\phi^b_{3_L}$ as we did in the statement of Proposition B3, namely

$$\phi^a_{3_L}(U_i,U_j) = \sum_{n,n'\in\mathcal{N}} \left\{ \frac{\rho^a_{n,n'}(B_{N_i:N_i}, B_{N_j:N_j}, X_i)}{f_{X,N}(X_i,n)} + \frac{\rho^a_{n,n'}(B_{N_j:N_j}, B_{N_i:N_i}, X_j)}{f_{X,N}(X_j,n)} \right\}$$
$$\times \mathbb{1}\{N_i = n, N_j = n\} \cdot \mathbb{1}\left\{X_i^d = X_j^d\right\} \cdot K\left(\frac{X_i^c - X_j^c}{h_L}\right)$$
$$\phi^b_{3_L}(U_i,U_j) = \sum_{n,n'\in\mathcal{N}} \left\{ \frac{\rho^b_{n,n'}(B_{N_i:N_i}, B_{N_j:N_j}, X_i)}{f_{X,N}(X_i,n)} \cdot \mathbb{1}\{N_i = n, N_j = n'\} \right.$$
$$\left. + \frac{\rho^b_{n,n'}(B_{N_j:N_j}, B_{N_i:N_i}, X_j)}{f_{X,N}(X_j,n)} \cdot \mathbb{1}\{N_j = n, N_i = n'\} \right\} \times \mathbb{1}\left\{X_i^d = X_j^d\right\} \cdot K\left(\frac{X_i^c - X_j^c}{h_L}\right)$$
$$\phi_{3_L}(U_i,U_j) = \phi^a_{3_L}(U_i,U_j) - \phi^b_{3_L}(U_i,U_j)$$

Then our previous results imply that, if Equation (3) is satisfied w.p.1 over our testing range and Assumptions B.1, B.3 and B.7 are satisfied, we have

$$E\left[\frac{1}{h_L^q} \cdot \mu_{3_L}(U_i,U_j,U_\ell)\big|U_i,U_j\right] = \frac{1}{3!}\frac{1}{h_L^q} \cdot \phi_{3_L}(U_i,U_j).$$

In order to verify if (A.15) is satisfied in the case of the U-statistic V_{3_L}, note that under our assumptions we have

$$E\left[\left\{E\left[\frac{1}{h_L^{2q}} \cdot \phi_{3_L}(U_k,U_i) \cdot \phi_{3_L}(U_k,U_j)\big|U_i,U_j\right]\right\}^2\right] = O\left(\frac{1}{h_L^q}\right),$$

$$E\left[\frac{1}{h_L^{2q}} \cdot \phi^2_{3_L}(U_i,U_j)\right] = O\left(\frac{1}{h_L^q}\right), \quad E\left[\frac{1}{h_L^{4q}} \cdot \phi^4_{3_L}(U_i,U_j)\right] = O\left(\frac{1}{h_L^{3q}}\right).$$

And therefore the criterion in (A.15) in the case of V_{3_L} is of the form

$$\frac{O\left(\frac{1}{h_L^q}\right) + O\left(\frac{1}{L\cdot h_L^{3q}}\right)}{O\left(\frac{1}{L\cdot h_L^{2q}}\right)} = \frac{O(h_L^q) + O\left(\frac{1}{L\cdot h_L^q}\right)}{O(1)} \longrightarrow 0,$$

where the last result follows from our bandwidth convergence conditions. Therefore, using (A.13) and Lemma A6 we have that, if Equation (3) is satisfied w.p.1 over our testing range and Assumptions B.1, B.3 and B.7 are satisfied, then

$$L \cdot h_L^{\frac{q}{2}} \cdot \widehat{\mathcal{Q}}_3 \xrightarrow{d} \mathcal{N}\left(0,\sigma_3^2\right), \quad \text{where} \quad \sigma_3^2 = \frac{1}{2} \cdot \lim_{L\to\infty} E\left[\frac{1}{h_L^q}\phi_{3_L}(U_i,U_j)^2\right]. \tag{A.17}$$

□

Lemma A8. If Equation (3) is satisfied w.p.1 over our testing range and Assumptions B.1, B.3 and B.7 are satisfied, then

$$\widetilde{\mathcal{S}}_3 = o_p\left(\frac{1}{L \cdot h_L}\right).$$

Proof: One of the implications of Lemma A7 was that $\widehat{\mathcal{Q}}_3 = O_p\left(\frac{1}{L \cdot h_L^{\frac{q}{2}}}\right)$. Following parallel steps, we can show the analogous result for $\widetilde{\mathcal{S}}_3$. Namely,

$$\widetilde{\mathcal{S}}_3 = O_p\left(\frac{1}{L \cdot \widetilde{h}_L^{\frac{q}{2}}}\right)$$

Using the bandwidth convergence rate restrictions in Assumption B.3, this immediately yields

$$\widetilde{\mathcal{S}}_3 = o_p\left(\frac{1}{L \cdot h_L^{\frac{q}{2}}}\right),$$

as claimed. $\square$

Using (A.12) and our previous lemmas, we have that if Equation (3) is satisfied w.p.1 over our testing range and Assumptions B.1, B.3 and B.7 are satisfied, then

$$L \cdot h_L^{\frac{q}{2}} \cdot \widehat{\mathcal{M}}_3 \xrightarrow{d} \mathcal{N}\left(0, \sigma_3^2\right), \quad \text{where} \quad \sigma_3^2 = \frac{1}{2} \cdot \lim_{L \to \infty} E\left[\frac{1}{h_L^q} \phi_{3_L}(U_i, U_j)^2\right].$$

This proves part (i) of Proposition B3. To prove part (ii), note that the type of uniform convergence results described in (A.9)-(A.10) along with the result in (A.8) imply that $\widehat{\mathcal{M}}_3 \xrightarrow{p} \mathcal{M}_3$. If (3) is violated with positive probability over our testing range, we have $\mathcal{M}_3 > 0$. Since $L \cdot h_L^{\frac{q}{2}} \to \infty$, the result in part (ii) of Proposition B3 follows, and its proof is complete. $\square$

2 Proof of Propositions B2 and B4

We now move on to the results that involve testing conditional moment *inequalities* in the paper. These are the results described in Propositions B2 and B4. Since the steps and arguments for proving both propositions are entirely analogous, we focus solely on the proof of Proposition B2.

Lemma A9. Under Assumptions B.1 and B.4 and B.5,

$$Pr\Big[\mathbb{1}\left\{\widehat{f}_{X,N}^{-i}(X_i, n) \geq \underline{f}\right\} \neq \mathbb{1}\left\{f_{X,N}(X_i, n) \geq \underline{f}\right\} \text{ for at least one } i = 1, \ldots, L \text{ over our testing range}\Big] \longrightarrow 0$$

for each $n \in \mathcal{N}$.

Proof: First, note that the probability in Lemma A9 is bounded above by

$$\sum_{i=1}^{L} Pr\left[\mathbb{1}\left\{\widehat{f}_{X,N}^{-i}(X_i,n) \geq \underline{f}\right\} \neq \mathbb{1}\left\{f_{X,N}(X_i,n) \geq \underline{f}\right\}\right].$$

By Assumption B.5, we have $Pr\left[f_{X,N}(X_i,n) = \underline{f}\right] = 0$. Therefore,

$$Pr\left[\mathbb{1}\left\{\widehat{f}_{X,N}^{-i}(X_i,n) \geq \underline{f}\right\} \neq \mathbb{1}\left\{f_{X,N}(X_i,n) \geq \underline{f}\right\}\right] =$$
$$E\left[Pr\left[\widehat{f}_{X,N}^{-i}(X_i,n) \geq \underline{f}\middle|X_i\right]\cdot \mathbb{1}\left\{f_{X,N}(X_i,n) < \underline{f}\right\}\right] + E\left[Pr\left[\widehat{f}_{X,N}^{-i}(X_i,n) < \underline{f}\middle|X_i\right]\cdot \mathbb{1}\left\{f_{X,N}(X_i,n) > \underline{f}\right\}\right]$$

Next note that our smoothness assumptions imply that

$$\underset{(x,n)\in\mathcal{X}\times\mathcal{N}}{sup}\left|E\left[\frac{1}{h_L^q}K\left(\frac{X_j^c - x^c}{h_L}\right)\cdot \mathbb{1}\left\{X_j^d = x^d\right\}\cdot \mathbb{1}\left\{N_j = n\right\}\right] - f_{X,N}(x,n)\right| = O\left(h_L^M\right)$$

for any compact sets $\mathcal{X}$ and $\mathcal{N}$ with the features of our testing range. Let

$$\widetilde{r}_L(X_i,n) = \underline{f} - E\left[\frac{1}{h_L^q}K\left(\frac{X_j^c - X_i^c}{h_L^q}\right)\mathbb{1}\{X_j^d = X_i^d\}\mathbb{1}\{N_j = n\}\middle|X_i\right].$$

Then by Bernstein's inequality, for sufficiently large L, there exist positive constants C_1 and C_2 such that

$$Pr\left[\widehat{f}_{X,N}^{-i}(X_i,n) \geq \underline{f}\middle|X_i\right]\cdot \mathbb{1}\left\{f_{X,N}(X_i,n) < \underline{f}\right\}$$
$$\leq \exp\left\{-\frac{(L-1)h_L^q\widetilde{r}_L(X_i,n)^2}{C_1 + C_2\widetilde{r}_L(X_i,n)}\right\}\mathbb{1}\left\{f_{X,N}(X_i,n) < \underline{f}\right\} \quad \forall\ i.$$

and

$$Pr\left[\widehat{f}_{X,N}^{-i}(X_i,n) < \underline{f}\middle|X_i\right]\cdot \mathbb{1}\left\{f_{X,N}(X_i,n) > \underline{f}\right\}$$
$$\leq \exp\left\{-\frac{(L-1)h_L^q\widetilde{r}_L(X_i,n)^2}{C_1 - C_2\widetilde{r}_L(X_i,n)}\right\}\mathbb{1}\left\{f_{X,N}(X_i,n) > \underline{f}\right\} \quad \forall\ i.$$

From here, invoking the Lebesgue dominating convergence theorem yields

$$\sum_{i=1}^{L} Pr\left[\mathbb{1}\left\{\widehat{f}_{X,N}^{-i}(X_i,n) \geq \underline{f}\right\} \neq \mathbb{1}\left\{f_{X,N}(X_i,n) \geq \underline{f}\right\}\right] \longrightarrow 0,$$

proving the claim. $\square$

A linear representation for $\widehat{\eta}^{-i}$

Letting

$$H_L^\eta(U_j|z,x;k,n) =$$
$$\left\{\nabla\eta^a(z,x;n)\cdot\left(\mathbb{1}\{B_{N_j:N_j} \leq z-\Delta\} - G_{n:n}^{\Delta}(z|x)\right) - \nabla\eta^b(z,x;k,n)\cdot\left(\mathbb{1}\{B_{k:N_j} \leq z\} - G_{k:n}(z|x)\right)\right\}$$
$$\times\frac{1}{f_{X,N}(x,n)}\cdot\mathbb{1}\{N_j = n\}\cdot\mathbb{1}\{X_i^d = x^d\}\cdot\frac{1}{h_L^q}K\left(\frac{X_j^c - x^c}{h_L}\right),$$

then under the assumptions of Proposition B2 we have (see, e.g, the Appendix to Aradillas-Lopez (2010))

$$\widehat{\eta}^{-i}(z,x;k,n) = \eta(z,x;k,n) + \frac{1}{(L-1)}\sum_{j\neq i} H_L^{\eta}\left(U_j|z,x;k,n\right) + \xi_{_L}^{-i}(z,x;k,n),$$

$$\text{where} \quad \sup_{\substack{(x,z)\in\mathcal{T}_2^{n,k}\\ k\leq n\in\mathcal{N}}}\left|\xi_{_L}^{-i}(z,x;k,n)\right| = O_p\left(\frac{1}{L^{1-\delta}h_L^q}\right) \quad \forall\ \delta>0 \tag{A.18}$$

$$= o_p(L^{-1/2-\epsilon}) \quad \text{for some } \epsilon>0,$$

where the last line follows if we use the $\delta>0$ described in Assumption B.4.

Lemma A10. Let $\mathbb{I}_i^{n,k} = \mathbb{1}\left\{f_{X,N}(X_i,n)\geq \underline{f} \quad \text{and} \quad \underline{z}^{n,k}\leq Z_i\leq \overline{z}^{n,k}\right\}$. Under the assumptions of Proposition B2, we have

$$\left|\xi_{_L}^{-i}(Z_i,X_i;k,n)\right| \leq \overline{\xi}_{_L} \quad \forall\ i:\ \mathbb{I}_i^{n,k}=1,$$

where

$$\mathbb{1}\left\{\overline{\xi}_{_L}\geq b_{_L}\right\} = o_p\left(L^{-1/2-\epsilon}\right) \quad \text{for some } \epsilon>0.$$

Proof: From (A.18), having $\mathbb{I}_i^{n,k}=1$ implies

$$\left|\xi_{_L}^{-i}(Z_i,X_i;k,n)\right| \leq \sup_{\substack{(x,z)\in\mathcal{T}_2^{n,k}\\ k\leq n\in\mathcal{N}}}\left|\xi_{_L}^{-i}(z,x;k,n)\right| = O_p\left(\frac{1}{L^{1-\delta}h_L^q}\right) \quad \forall\ \delta>0.$$

Next note that, under the assumptions of Proposition B2 and given the result in Lemma A9, with probability approaching one, for large enough L we have

$$\left|\left\{\widehat{\eta}^{-i}(Z_i,X_i;k,n) - \eta(Z_i,X_i;k,n) - \frac{1}{(L-1)}\sum_{j\neq i} H_L^{\eta}\left(U_j|Z_i,X_i;k,n\right)\right\}\cdot\mathbb{I}_i^{n,k}\right| \leq \mathcal{R}_{_L},$$

where $\overline{\lim\limits_{L\to\infty}}\, E\left[\mathcal{R}_{_L}^4\right]<\infty$. As a result, for any $\delta>0$ we can express $\overline{\xi}_{_L}$ as

$$\overline{\xi}_{_L} = \frac{\mathcal{D}_{_L}}{L^{1-\delta}h_L^q}, \quad \text{where} \quad \overline{\lim_{L\to\infty}}\, E\left[\mathcal{D}_{_L}^4\right]<\infty.$$

From this expression, we have $Pr\left(\overline{\xi}_{_L}\geq b_{_L}\right) = Pr\left(\mathcal{D}_{_L}\geq b_{_L}\cdot L^{1-\delta}\cdot h_{_L}^q\right)$. The condition $\overline{\lim\limits_{L\to\infty}}\, E\left[\mathcal{D}_{_L}^4\right]<\infty$ is sufficient (but *not necessary*) to have

$$\left(b_{_L}\cdot L^{1-\delta}\cdot h_{_L}^q\right)^2\cdot Pr\left(\mathcal{D}_{_L}\geq b_{_L}\cdot L^{1-\delta}\cdot h_{_L}^q\right) = O(1).$$

Since this can be done for any $\delta > 0$, if we use the one described in Assumption B.4, we obtain

$$Pr\left(\overline{\xi}_{L} \geq b_{L}\right) = O\left(\frac{1}{\left(b_{L} \cdot L^{1-\delta} \cdot h_{L}^{d}\right)^{2}}\right) = o\left(\frac{1}{L^{1+\epsilon}}\right) \quad \text{for some } \epsilon > 0.$$

From here, Chebyshev's inequality yields $\mathbb{1}\left\{\overline{\xi}_{L} \geq b_{L}\right\} = o_p\left(L^{-1/2-\epsilon}\right)$ for some $\epsilon > 0$, as claimed in Lemma A10. $\square$

Let $\mathbb{I}_i^{n,k}$ be as defined in Lemma A10. The result in Lemma A9 allows us to study $\widehat{\mathcal{M}}_2$ as

$$\widehat{\mathcal{M}}_2 = \frac{1}{L}\sum_{i=1}^{L}\left\{\sum_{\substack{n\in\mathcal{N}\\k\leq n-2}} \widehat{\eta}^{-i}(Z_i, X_i; k, n) \cdot \mathbb{1}\left\{\widehat{\eta}^{-i}(Z_i, X_i; k, n) \geq -b_{L}\right\} \cdot \mathbb{1}\{N_i = n\} \cdot \mathbb{I}_i^{n,k}\right\}.$$

Let ξ_{L}^{-i} be as described in (A.18) and define

$$\varepsilon_{L}^{-i}(k,n) \equiv 2 \cdot \left(\left|\xi_{L}^{-i}(Z_i, X_i; k, n)\right| + b_{L}\right), \quad \text{with} \quad \overline{\varepsilon}_{L} = 2 \cdot \left(\overline{\xi}_{L} + b_{L}\right),$$

where $\overline{\xi}_{L}$ is as defined in Lemma A10. We will decompose $\widehat{\mathcal{M}}_2$ as

$$\widehat{\mathcal{M}}_2 = \overline{\mathcal{M}}_2 + \widehat{Q}_2 + \widehat{\mathcal{S}}_2^a + \widehat{\mathcal{S}}_2^b + \widehat{\mathcal{S}}_2^c + \widehat{\mathcal{S}}_2^d,$$

where

$$\overline{\mathcal{M}}_2 = \frac{1}{L}\sum_{i=1}^{L}\left\{\sum_{\substack{n\in\mathcal{N}\\k\leq n-2}} \eta(Z_i, X_i; k, n) \cdot \mathbb{1}\left\{\eta(Z_i, X_i; k, n) \geq 0\right\} \cdot \mathbb{1}\{N_i = n\} \cdot \mathbb{I}_i^{n,k}\right\},$$

$$\widehat{Q}_2 = \frac{1}{L}\sum_{i=1}^{L}\left\{\sum_{\substack{n\in\mathcal{N}\\k\leq n-2}} \left(\widehat{\eta}^{-i}(Z_i, X_i; k, n) - \eta(Z_i, X_i; k, n)\right) \cdot \mathbb{1}\left\{\eta(Z_i, X_i; k, n) \geq 0\right\} \cdot \mathbb{1}\{N_i = n\} \cdot \mathbb{I}_i^{n,k}\right\},$$

$$\widehat{\mathcal{S}}_2^a =$$

$$\frac{1}{L}\sum_{i=1}^{L}\left\{\sum_{\substack{n\in\mathcal{N}\\k\leq n-2}} \eta(Z_i, X_i; k, n) \cdot \mathbb{1}\left\{\widehat{\eta}^{-i}(Z_i, X_i; k, n) \geq -b_{L}\right\} \cdot \mathbb{1}\left\{\eta(Z_i, X_i; k, n) < -\varepsilon_{L}^{-i}(k,n)\right\} \cdot \mathbb{1}\{N_i = n\} \cdot \mathbb{I}_i^{n,k}\right\},$$

$$\widehat{\mathcal{S}}_2^b =$$

$$\frac{1}{L}\sum_{i=1}^{L}\left\{\sum_{\substack{n\in\mathcal{N}\\k\leq n-2}} \eta(Z_i, X_i; k, n) \cdot \mathbb{1}\left\{\widehat{\eta}^{-i}(Z_i, X_i; k, n) \geq -b_{L}\right\} \cdot \mathbb{1}\left\{-\varepsilon_{L}^{-i}(k,n) \leq \eta(Z_i, X_i; k, n) < 0\right\} \cdot \mathbb{1}\{N_i = n\} \cdot \mathbb{I}_i^{n,k}\right\},$$

$\widehat{\mathcal{S}}_2^c =$

$$\frac{1}{L}\sum_{i=1}^{L}\left\{\sum_{\substack{n\in\mathcal{N}\\k\leq n-2}}\eta(Z_i,X_i;k,n)\cdot\mathbb{1}\left\{\widehat{\eta}^{-i}(Z_i,X_i;k,n)<-b_{_L}\right\}\cdot\mathbb{1}\left\{\eta(Z_i,X_i;k,n)\geq 0\right\}\cdot\mathbb{1}\{N_i=n\}\cdot\mathbb{I}_i^{n,k}\right\},$$

and

$\widehat{\mathcal{S}}_2^d =$

$$\frac{1}{L}\sum_{i=1}^{L}\Bigg\{\sum_{\substack{n\in\mathcal{N}\\k\leq n-2}}\left(\widehat{\eta}^{-i}(Z_i,X_i;k,n)-\eta(Z_i,X_i;k,n)\right)\cdot\left(\mathbb{1}\left\{\widehat{\eta}^{-i}(Z_i,X_i;k,n)\geq -b_{_L}\right\}-\mathbb{1}\left\{\eta(Z_i,X_i;k,n)\geq 0\right\}\right)$$
$$\times\mathbb{1}\{N_i=n\}\cdot\mathbb{I}_i^{n,k}\Bigg\},$$

Lemma A11. If the assumptions of Proposition B2 are satisfied, then $\widehat{\mathcal{S}}_2^a = o_p(L^{-1/2-\epsilon})$ for some $\epsilon>0$.

Proof: Note first that, by (A.18), we can have $\widehat{\eta}^{-i}(Z_i,X_i;k,n)\geq -b_{_L}$ and $\eta(Z_i,X_i;k,n)<-\varepsilon_{_L}^{-i}(k,n)$ only if

$$\frac{1}{(L-1)}\sum_{j\neq i}H_L^{\eta}(U_j|Z_i,X_i;k,n)\geq b_{_L}+\left|\xi_{_L}^{-i}(Z_i,X_i;k,n)\right|.$$

Letting $\overline{\eta}=\underset{\substack{(x,z)\in\mathcal{T}_2^{n,k}\\k\leq n\in\mathcal{N}}}{sup}|\eta(z,x;k,n)|$, we have

$$\widehat{\mathcal{S}}_2^a\leq\overline{\eta}\times\frac{1}{L}\sum_{i=1}^{L}\mathbb{1}\left\{\frac{1}{(L-1)}\sum_{j\neq i}H_L^{\eta}(U_j|Z_i,X_i;k,n)\geq b_{_L}+\left|\xi_{_L}^{-i}(Z_i,X_i;k,n)\right|\right\}\cdot\mathbb{1}\{N_i=n\}\cdot\mathbb{I}_i^{n,k}$$

By Assumption B.6(i), we have

$$\left|E\left[H_L^{\eta}\left(U_j|Z_i,X_i;k,n\right)\,\middle|\,\xi_{_L}^{-i}(Z_i,X_i;k,n),U_i\right]-E\left[H_L^{\eta}\left(U_j|Z_i,X_i;k,n\right)\,\middle|\,U_i\right]\right|=O_p\left(\xi_{_L}^{-i}(Z_i,X_i;k,n)\right)$$

But under our smoothness assumptions, we have

$$E\left[H_L^{\eta}\left(U_j|Z_i,X_i;k,n\right)\,\middle|\,U_i\right]=O\left(h_{_L}^{M}\right)$$

where M is as described in Assumption B.1. Therefore,

$$E\left[H_L^{\eta}\left(U_j|Z_i,X_i;k,n\right)\,\middle|\,\xi_{_L}^{-i}(Z_i,X_i;k,n),U_i\right]=O_p\left(\xi_{_L}^{-i}(Z_i,X_i;k,n)\right)+O\left(h_{_L}^{M}\right).\quad\text{(A.19)}$$

From here, by Bernstein's inequality there exist positive constants C_1 and C_2 such that

$$Pr\left(\frac{1}{(L-1)}\sum_{j\neq i}H_L^{\eta}(U_j|Z_i,X_i;k,n)\geq b_{_L}+\left|\xi_{_L}^{-i}(Z_i,X_i;k,n)\right|\,\middle|\,\xi_{_L}^{-i}(Z_i,X_i;k,n),Z_i,X_i\right)$$
$$\leq\exp\left\{-\frac{(L-1)h_{_L}^q\left\{b_{_L}+O_p\left(\xi_{_L}^{-i}(Z_i,X_i;k,n)\right)\right\}^2}{C_1+C_2\cdot\left\{b_{_L}+O_p\left(\xi_{_L}^{-i}(Z_i,X_i;k,n)\right)\right\}}\right\}$$

From here, using iterated expectations and dominated convergence arguments together with our bandwidth convergence rate restrictions we obtain, in particular,

$$\frac{1}{L}\sum_{i=1}^{L}\mathbb{1}\left\{\frac{1}{(L-1)}\sum_{j\neq i}H_L^{\eta}(U_j|Z_i,X_i;k,n)\geq b_{_L}+\left|\xi_{_L}^{-i}(Z_i,X_i;k,n)\right|\right\}\cdot\mathbb{1}\{N_i=n\}\cdot\mathbb{I}_i^{n,k}=o_p(L^{-1/2-\epsilon}),$$

for some $\epsilon>0$, and therefore $\widehat{\mathcal{S}}_2^a=o_p(L^{-1/2-\epsilon})$ for some $\epsilon>0$, as claimed. $\square$

Lemma A12. If the assumptions of Proposition B2 are satisfied, then $\widehat{\mathcal{S}}_2^b=o_p(L^{-1/2-\epsilon})$ for some $\epsilon>0$.

Proof: Note that

$$\widehat{\mathcal{S}}_2^b\leq\frac{1}{L}\sum_{i=1}^{L}\left\{\sum_{\substack{n\in\mathcal{N}\\k\leq n-2}}\varepsilon_{_L}^{-i}(k,n)\cdot\mathbb{1}\left\{-\varepsilon_{_L}^{-i}(k,n)\leq\eta(Z_i,X_i;k,n)<0\right\}\cdot\mathbb{1}\{N_i=n\}\cdot\mathbb{I}_i^{n,k}\right\}$$
$$\leq\overline{\varepsilon}_{_L}\times\frac{1}{L}\sum_{i=1}^{L}\left\{\sum_{\substack{n\in\mathcal{N}\\k\leq n-2}}\mathbb{1}\left\{-\varepsilon_{_L}^{-i}(k,n)\leq\eta(Z_i,X_i;k,n)<0\right\}\cdot\mathbb{1}\{N_i=n\}\cdot\mathbb{I}_i^{n,k}\right\}.$$

Recall that $\varepsilon_{_L}^{-i}(k,n)\equiv 2\cdot\left(\left|\xi_{_L}^{-i}(Z_i,X_i;k,n)\right|+b_{_L}\right)$. Let $\overline{t}$ be the constant defined in Assumption B.6(ii). We have

$$\begin{aligned}E\left[\mathbb{1}\left\{-\varepsilon_{_L}^{-i}(k,n)\leq\eta(Z_i,X_i;k,n)<0\right\}\big|\xi_{_L}^{-i}(Z_i,X_i;k,n),U_i\right]&\leq O\left(\varepsilon_{_L}^{-i}(k,n)\right)+\mathbb{1}\left\{\overline{\varepsilon}_L>\overline{t}\right\}\\&\leq O\left(\varepsilon_{_L}^{-i}(k,n)\right)+\mathbb{1}\left\{\overline{\xi}_L>\frac{\overline{t}}{2}-b_{_L}\right\}\\&=O\left(\varepsilon_{_L}^{-i}(k,n)\right)+o_p(L^{-1/2-\epsilon}),\end{aligned}$$

for some $\epsilon>0$, where the last equality follows from the same arguments we used in Lemma A10. From here, iterated expectations along with dominated convergence arguments and Chebyshev's inequality yield

$$\frac{1}{L}\sum_{i=1}^{L}\left\{\sum_{\substack{n\in\mathcal{N}\\k\leq n-2}}\mathbb{1}\left\{-\varepsilon_{_L}^{-i}(k,n)\leq\eta(Z_i,X_i;k,n)<0\right\}\cdot\mathbb{1}\{N_i=n\}\cdot\mathbb{I}_i^{n,k}\right\}=O_p\left(\frac{b_L^{1/2}}{\sqrt{L}}\right)+O(b_L)\tag{A.20}$$

and therefore

$$\widehat{\mathcal{S}}_2^b \leq \bar{\varepsilon}_L \times \left(O_p\left(\frac{b_L^{1/2}}{\sqrt{L}}\right) + O(b_L) \right) = O_p(b_{_L}^2) = o_p(L^{-1/2-\epsilon}) \quad \text{for some } \epsilon > 0,$$

where the last equality follows from the bandwidth convergence restriction for $b_{_L}$ described in Assumption B.4. $\square$

Lemma A13. If the assumptions of Proposition B2 are satisfied, then $\widehat{\mathcal{S}}_2^c = o_p(L^{-1/2-\epsilon})$ for some $\epsilon > 0$.

Proof: By (A.18), we can have $\widehat{\eta}^{-i}(Z_i, X_i; k, n) < -b_{_L}$ and $\eta(Z_i, X_i; k, n) \geq 0$ only if

$$\frac{1}{(L-1)} \sum_{j \neq i} H_L^\eta(U_j | Z_i, X_i; k, n) < -b_{_L} - \xi_{_L}^{-i}(Z_i, X_i; k, n).$$

And therefore

$$\begin{aligned}
\widehat{\mathcal{S}}_2^c &\leq \overline{\eta} \times \frac{1}{L} \sum_{i=1}^{L} \mathbb{1}\left\{ \frac{1}{(L-1)} \sum_{j \neq i} H_L^\eta(U_j | Z_i, X_i; k, n) < -b_{_L} - \xi_{_L}^{-i}(Z_i, X_i; k, n) \right\} \cdot \mathbb{1}\{N_i = n\} \cdot \mathbb{I}_i^{n,k} \\
&\leq \overline{\eta} \times \frac{1}{L} \sum_{i=1}^{L} \mathbb{1}\left\{ \frac{1}{(L-1)} \sum_{j \neq i} H_L^\eta(U_j | Z_i, X_i; k, n) < -b_{_L} - \xi_{_L}^{-i}(Z_i, X_i; k, n) \right\} \cdot \mathbb{1}\left\{ b_{_L} + \xi_{_L}^{-i}(Z_i, X_i; k, n) > 0 \right\} \\
&+ \overline{\eta} \cdot \mathbb{1}\left\{ \overline{\xi}_{_L} \geq b_{_L} \right\} \\
&= \overline{\eta} \times \frac{1}{L} \sum_{i=1}^{L} \mathbb{1}\left\{ \frac{1}{(L-1)} \sum_{j \neq i} H_L^\eta(U_j | Z_i, X_i; k, n) < -b_{_L} - \xi_{_L}^{-i}(Z_i, X_i; k, n) \right\} \cdot \mathbb{1}\left\{ b_{_L} + \xi_{_L}^{-i}(Z_i, X_i; k, n) > 0 \right\} \\
&+ o_p(L^{-1/2}),
\end{aligned} \tag{A.21}$$

where the last equality follows from Lemma A10. Suppose $b_{_L} + \xi_{_L}^{-i}(Z_i, X_i; k, n) > 0$. Using (A.19) and Bernstein's inequality, there exist constants C_1 and C_2 such that

$$\begin{aligned}
&Pr\left(\frac{1}{(L-1)} \sum_{j \neq i} H_L^\eta(U_j | Z_i, X_i; k, n) < -b_{_L} - \xi_{_L}^{-i}(Z_i, X_i; k, n) \middle| \xi_{_L}^{-i}(Z_i, X_i; k, n), U_i \right) \\
&\leq \exp\left\{ -\frac{(L-1) h_{_L}^q \left\{ b_{_L} + O_p\left(\xi_{_L}^{-i}(Z_i, X_i; k, n)\right) \right\}^2}{C_1 + C_2 \cdot \left\{ b_{_L} + O_p\left(\xi_{_L}^{-i}(Z_i, X_i; k, n)\right) \right\}} \right\}.
\end{aligned}$$

In particular, this exponential inequality yields

$$\begin{aligned}
&\frac{1}{L} \sum_{i=1}^{L} \mathbb{1}\left\{ \frac{1}{(L-1)} \sum_{j \neq i} H_L^\eta(U_j | Z_i, X_i; k, n) < -b_{_L} - \xi_{_L}^{-i}(Z_i, X_i; k, n) \right\} \cdot \mathbb{1}\left\{ b_{_L} + \xi_{_L}^{-i}(Z_i, X_i; k, n) > 0 \right\} \\
&= o_p(L^{-1/2-\epsilon}) \quad \text{for some } \epsilon > 0,
\end{aligned}$$

from here and (A.21) we obtain $\widehat{\mathcal{S}}_2^c = o_p(L^{-1/2-\epsilon})$ for some $\epsilon > 0$, as claimed. $\square$

Lemma A14. If the assumptions of Proposition B2 are satisfied, then $\widehat{\mathcal{S}}_2^d = o_p(L^{-1/2-\epsilon})$ for some $\epsilon > 0$.

Proof: We have

$$
\begin{aligned}
\widehat{\mathcal{S}}_2^b \leq & \sup_{\substack{(x,z)\in\mathcal{T}_2^{n,k} \\ k\leq n\in\mathcal{N}}} \left| \widehat{\eta}^{-i}(z,x;k,n) - \eta(z,x;k,n) \right| \\
& \times \frac{1}{L}\sum_{i=1}^{L} \left(\mathbb{1}\left\{ \widehat{\eta}^{-i}(Z_i,X_i;k,n) \geq -b_{_L} \right\} - \mathbb{1}\left\{ \eta(Z_i,X_i;k,n) \geq 0 \right\} \right) \\
& = O_p\left(\frac{1}{\sqrt{L^{1-\delta}\cdot h_{_L}^q}} \right) \times \frac{1}{L}\sum_{i=1}^{L} \left(\mathbb{1}\left\{ \widehat{\eta}^{-i}(Z_i,X_i;k,n) \geq -b_{_L} \right\} - \mathbb{1}\left\{ \eta(Z_i,X_i;k,n) \geq 0 \right\} \right) \quad \forall\, \delta > 0,
\end{aligned}
$$

where the last equality follows from the arguments leading to (A.18) under our assumptions. Repeating some of the steps used in our proofs of Lemmas A11-A13 (the exponential inequalities established in Lemmas A11 and A13, along with Equation A.20 in Lemma A12) we can show that

$$
\frac{1}{L}\sum_{i=1}^{L} \left(\mathbb{1}\left\{ \widehat{\eta}^{-i}(Z_i,X_i;k,n) \geq -b_{_L} \right\} - \mathbb{1}\left\{ \eta(Z_i,X_i;k,n) \geq 0 \right\} \right) = O_p(b_{_L}).
$$

And from here we obtain

$$
\begin{aligned}
\widehat{\mathcal{S}}_2^b & \leq O_p\left(\frac{b_{_L}}{\sqrt{L^{1-\delta}\cdot h_{_L}^q}} \right) \quad \forall\, \delta > 0 \\
& = o_p(L^{-1/2-\epsilon}) \quad \text{for some } \epsilon > 0,
\end{aligned}
$$

where the last equality follows if we use the $\delta > 0$ described in Assumption B.4. $\square$

Combining Lemmas A11-A14, we have

$$
\widehat{\mathcal{M}}_2 = \overline{\mathcal{M}}_2 + \widehat{Q}_2 + o_p(L^{-1/2-\epsilon}) \quad \text{for some } \epsilon > 0,
$$

where

$$
\begin{aligned}
\overline{\mathcal{M}}_2 &= \frac{1}{L}\sum_{i=1}^{L} \left\{ \sum_{\substack{n\in\mathcal{N} \\ k\leq n-2}} \eta(Z_i,X_i;k,n)\cdot \mathbb{1}\left\{ \eta(Z_i,X_i;k,n) \geq 0 \right\} \cdot \mathbb{1}\{N_i = n\} \cdot \mathbb{I}_i^{n,k} \right\}, \\
\widehat{Q}_2 &= \frac{1}{L}\sum_{i=1}^{L} \left\{ \sum_{\substack{n\in\mathcal{N} \\ k\leq n-2}} \left(\widehat{\eta}^{-i}(Z_i,X_i;k,n) - \eta(Z_i,X_i;k,n) \right) \cdot \mathbb{1}\left\{ \eta(Z_i,X_i;k,n) \geq 0 \right\} \cdot \mathbb{1}\{N_i = n\} \cdot \mathbb{I}_i^{n,k} \right\}.
\end{aligned}
$$

Let

$$\mathcal{H}_L^\eta(U_i,U_j;k,n) = H_L^\eta\left(U_j|Z_i,X_i;k,n\right)\cdot \mathbb{1}\left\{\eta(Z_i,X_i;k,n)\geq 0\right\}\cdot \mathbb{1}\{N_i=n\}\cdot \mathbb{I}_i^{n,k},$$
$$\mathcal{G}_L^\eta(U_i,U_j;k,n) = \frac{1}{2}\cdot\left(\mathcal{H}_L^\eta(U_i,U_j;k,n)+\mathcal{H}_L^\eta(U_j,U_i;k,n)\right).$$

Using (A.18), we can express

$$\widehat{Q}_2 = \binom{L}{2}^{-1}\sum_{i<j}\left\{\sum_{\substack{n\in\mathcal{N}\\ k\leq n-2}}\mathcal{G}_L^\eta(U_i,U_j;k,n)\right\}+o_p(L^{-1/2-\epsilon}) \quad \text{for some } \epsilon>0.$$

A Hoeffding decomposition (see,Hoeffding (1961) and Lemma 5.1.5.A in Serfling (1980)) for the second-order U-statistic on the right hand side states that

$$\binom{L}{2}^{-1}\sum_{i<j}\left\{\sum_{\substack{n\in\mathcal{N}\\ k\leq n-2}}\mathcal{G}_L^\eta(U_i,U_j;k,n)\right\}=$$
$$\sum_{\substack{n\in\mathcal{N}\\ k\leq n-2}}E\left[\mathcal{G}_L^\eta(U_i,U_j;k,n)\right]+\frac{2}{L}\sum_{i=1}^{L}\left\{\sum_{\substack{n\in\mathcal{N}\\ k\leq n-2}}\Big(E\left[\mathcal{G}_L^\eta(U_i,U_j;k,n)\middle|U_i\right]-E\left[\mathcal{G}_L^\eta(U_i,U_j;k,n)\right]\Big)\right\}+O_p\left(\frac{1}{Lh_L^q}\right)$$

Letting θ_2^a and θ_2^b be as defined in Assumption B.5(ii), then under the smoothness assumptions described there, we have

$$E\left[\mathcal{G}_L^\eta(U_i,U_j;k,n)\middle|U_i\right]=\frac{1}{2}\Big\{\theta_2^a\left(B_{N_i:N_i},X_i;k,n\right)-\theta_2^b\left(B_{k:N_i},X_i;k,n\right)\Big\}\cdot\mathbb{1}\{N_i=n\}+O(h_L^M),$$
$$E\left[\mathcal{G}_L^\eta(U_i,U_j;k,n)\right]=O(h_L^M).$$

From here we have

$$\binom{L}{2}^{-1}\sum_{i<j}\left\{\sum_{\substack{n\in\mathcal{N}\\ k\leq n-2}}\mathcal{G}_L^\eta(U_i,U_j;k,n)\right\}$$
$$=\frac{1}{L}\sum_{i=1}^{L}\left\{\sum_{\substack{n\in\mathcal{N}\\ k\leq n-2}}\Big(\theta_2^a\left(B_{N_i:N_i},X_i;k,n\right)-\theta_2^b\left(B_{k:N_i},X_i;k,n\right)\Big)\cdot\mathbb{1}\{N_i=n\}\right\}+O(h_L^M)+O_p\left(\frac{1}{Lh_L^q}\right)$$
$$=\frac{1}{L}\sum_{i=1}^{L}\left\{\sum_{\substack{n\in\mathcal{N}\\ k\leq n-2}}\Big(\theta_2^a\left(B_{N_i:N_i},X_i;k,n\right)-\theta_2^b\left(B_{k:N_i},X_i;k,n\right)\Big)\cdot\mathbb{1}\{N_i=n\}\right\}+o_p(L^{-1/2-\epsilon})\quad\text{for some } \epsilon>0,$$

where the last result follows from the bandwidth convergence restrictions described in Assumptions B.1 and B.4. Denote

$$\phi_2(U_i)=\sum_{\substack{n\in\mathcal{N}\\ k\leq n-2}}\Big\{\theta_2^a\left(B_{N_i:N_i},X_i;k,n\right)-\theta_2^b\left(B_{k:N_i},X_i;k,n\right)\Big\}\cdot\mathbb{1}\{N_i=n\}$$

Iterated expectations show that $E\left[\phi_2(U_i)\right] = 0$. As we defined in (14), let

$$\mathcal{M}_2 = E\left[\sum_{\substack{n\in\mathcal{N}\\k\leq n-2}} \max\left\{\eta(Z_i, X_i; k, n), 0\right\} \cdot \mathbb{1}\{N_i = n\} \cdot \mathbb{I}_i^{n,k}\right]$$

Combining our previous results, we obtain

$$\begin{aligned}\widehat{\mathcal{M}}_2 = \mathcal{M}_2 + \frac{1}{L}\sum_{i=1}^{L}&\left\{\sum_{\substack{n\in\mathcal{N}\\k\leq n-2}} \eta(Z_i, X_i; k, n) \cdot \mathbb{1}\left\{\eta(Z_i, X_i; k, n) \geq 0\right\} \cdot \mathbb{1}\{N_i = n\} \cdot \mathbb{I}_i^{n,k} - \mathcal{M}_2\right\}\\ &+ \frac{1}{L}\sum_{i=1}^{L}\phi_2(U_i) + o_p(L^{-1/2-\epsilon}) \quad \text{for some } \epsilon > 0.\end{aligned}$$

As a result,

(i) If (2) is satisfied as a **strict** inequality w.p.1 over our testing range, then $\mathcal{M}_2 = 0$ and we also have $\mathbb{1}\left\{\eta(Z_i, X_i; k, n) \geq 0\right\} = 0$ almost surely over our testing range. This yields,

$$\sqrt{L} \cdot \widehat{\mathcal{M}}_2 = o_p\left(L^{-\epsilon}\right) \quad \text{for some } \epsilon > 0$$

(ii) If (2) is satisfied w.p.1 over our testing range and it holds as an **equality** over a subset of it with strictly positive probability measure, we still have $\mathcal{M}_2 = 0$, but $\mathbb{1}\left\{\eta(Z_i, X_i; k, n) \geq 0\right\} = 1$ with strictly positive probability. Then, $\widehat{\mathcal{M}}_2 = \frac{1}{L}\sum_{i=1}^{L}\phi_2(U_i) + o_p(L^{-1/2-\epsilon})$ for some $\epsilon > 0$. From here we have

$$\sqrt{L} \cdot \widehat{\mathcal{M}}_2 \xrightarrow{d} \mathcal{N}\left(0, \sigma_2^2\right), \quad \text{where } \sigma_2^2 = E\left[\phi_2(U_i)^2\right].$$

(iii) If (2) is violated with positive probability over our testing range, then $\mathcal{M}_2 > 0$ and since $\widehat{\mathcal{M}}_2 = \mathcal{M}_2 + O_p(L^{-1/2})$, we obtain

$$\sqrt{L} \cdot \widehat{\mathcal{M}}_2 \xrightarrow{p} +\infty$$

This proves Proposition B2. $\square$

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