Supplementary Appendix for Aradillas-López, Gandhi and Quint (2011), Identification and Inference in Ascending Auctions with Correlated Private Values

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Equation numbers with parentheses refer to equations in this appendix; equation numbers without parentheses refer to equations in the main paper.

Asymptotic theory results for the policy experiments in Section 4.2

This appendix supplements the results presented in Appendix A.4 in order to fully characterize the econometric theory behind the empirical results described in Section 4.2. Take a given reserve price policy $r(\cdot)$, where $r(X,N)$ denotes the reserve price prescribed for (X,N). Let $\mathcal{S}$ denote a compact subset of $Supp(X,N)$ and consider the functional

$$E_{X,N}\left[\pi_N(r(X,N)|X)\big|X,N\in\mathcal{S}\right]\equiv\mathcal{T}_r.$$

Under the conditions of Theorem 1, if we allow for correlation in bidders' values, $\mathcal{T}_r$ is bounded between $\mathcal{T}_r^\ell$ and $\mathcal{T}_r^u$, where

$$\mathcal{T}_r^\ell=E_{X,N}\left[\underline{\pi}_N(r(X,N)|X)\big|X,N\in\mathcal{S}\right]\quad and\quad \mathcal{T}_r^u=E_{X,N}\left[\overline{\pi}_N(r(X,N)|X)\big|X,N\in\mathcal{S}\right],$$

where $\underline{\pi}$ and $\overline{\pi}$ are the lower and upper bounds described in Theorem 1. We will begin by exploring the asymptotic properties of nonparametric estimators for $\mathcal{T}_r^\ell$ and $\mathcal{T}_r^u$. As before, denote $U_i\equiv(B_i,N_i,X_i)$ and define $\mathbb{I}(U_i)\equiv\mathbb{1}\{X_i,N_i\in\mathcal{S}\}$; we will abbreviate $r_i\equiv r(X_i,N_i)$ whenever it is notationally convenient. Our estimators for $\mathcal{T}_r^\ell$ and $\mathcal{T}_r^u$ are of the form

$$\widehat{\mathcal{T}}_r^\ell=\frac{1}{L}\sum_{i=1}^{L}\frac{\widehat{\underline{\pi}}_{N_i}(r_i|X_i)}{\widehat{Pr}(X,N\in\mathcal{S})}\cdot\mathbb{I}(U_i),\quad and\quad \widehat{\mathcal{T}}_r^u=\frac{1}{L}\sum_{i=1}^{L}\frac{\widehat{\overline{\pi}}_{N_i}(r_i|X_i)}{\widehat{Pr}(X,N\in\mathcal{S})}\cdot\mathbb{I}(U_i)\tag{1}$$

where $\widehat{Pr}(X,N\in\mathcal{S})=\frac{1}{L}\sum_{i=1}^{L}\mathbb{I}(U_i)$ and $\widehat{\underline{\pi}}$, $\widehat{\overline{\pi}}$ are as described in equation 6 in the paper. However, we replace Assumption 4 in Appendix A.4 with the following (stronger) restrictions:[1]

Assumption 4′ *1. The set $\mathcal{S}$ is such that the conditions in Assumption A.4 parts 1(i)-(iii) are satisfied for each (x,n,r) such that $(x,n)\in\mathcal{S}$ and $r=r(x,n)$. We strengthen part (iv) to*

[1] As in Appendix A.4, our results assume that the relevant range of reserve prices studied is a compact subset of the interior of the support of valuations. Menzel and Morganti (2011) describe the econometric irregularity issues that arise in ascending price auctions when the reserve price is allowed to be arbitrarily close to the boundary of the support of valuations.

assume now that $f_X(x)$, $f_{X|N}(x|n)$, $F_{n-1:n}(r|x)$, and $T_{n-1:n}(r|x)$ are $M-$times differentiable with respect to x^c (the continuous elements in x) with bounded derivatives, where M satisfies the restrictions to be described below.

2. *Let M be the constant mentioned above. The kernel K satisfies $\int K(\xi)d\xi = 1$, has compact support, and is Lipschitz-continuous, bounded, and symmetric around zero. If is also a bias-reducing kernel of order M. That is, if we denote $\xi \equiv (\xi_1, \ldots, \xi_z)$, then we have $\int (\xi_1^{q_1} \cdots \xi_z^{q_z}) K(\xi) d\xi_1 \cdots d\xi_z = 0 \quad \forall\, 0 < q_1 + \cdots + q_z < M$, and $\int \|\xi\|^M |K(\xi)| d\xi < \infty$.*

3. *The bandwidth sequence $h_L \longrightarrow 0$ is such that $\exists\, \overline{\delta} > 0$ for which $L^{1-\overline{\delta}} h_L^{2z} \longrightarrow \infty$, and $L^{1+\overline{\delta}} h_L^{M+z} \longrightarrow 0$*

Compared with Assumption 4 in Appendix A.4, we are now imposing a higher degree of smoothness (as indicated by M) on the relevant functionals, and we employ a bias-reducing kernel K which, may take on negative values. To be precise, the bandwidth convergence rates described above will be satisfied only if $M \geq z+1$. Thus, if $z \geq 2$ (i.e, if X^c includes at least two elements), we have $M \geq 3$ and our kernel K must take on negative values in order to satisfy the restrictions in Assumption $4'$. Let $\widehat{\underline{\pi}}_n(r|x)$ and $\widehat{\overline{\pi}}_n(r|x)$ be the estimators described in equation 6 of the paper, using the kernel K and bandwidth b_L as described in Assumption $4'$. Let $\underline{\psi}(r, U_i|x, n)$ and $\overline{\psi}(r, U_i|x, n)$ be as described in equation 9 of the paper. Under the conditions in Assumption $4'$, we have

$$\begin{aligned}
\widehat{\underline{\pi}}_n(r(x,n)|x) &= \underline{\pi}_n(r(x,n)|x) + \frac{1}{Lh_L^z}\sum_{i=1}^{L} \underline{\psi}(r(x,n), U_i|x,n)\mathbb{1}\{X_i^d = x^d\}K\left(\frac{X_i^c - x^c}{h_L}\right) + \underline{\varepsilon}_L(x,n), \\
\widehat{\overline{\pi}}_n(r(x,n)|x) &= \overline{\pi}_n(r(x,n)|x) + \frac{1}{Lh_L^z}\sum_{i=1}^{L} \overline{\psi}(r(x,n), U_i|x,n)\mathbb{1}\{X_i^d = x^d\}K\left(\frac{X_i^c - x^c}{h_L}\right) + \overline{\varepsilon}_L(x,n)
\end{aligned} \tag{2}$$

where

$$\begin{aligned}
\sup_{(x,n)\in\mathcal{S}} \left\| \left(\underline{\varepsilon}_L(x,n)\ ,\ \overline{\varepsilon}_L(x,n)\right)' \right\| &= O_p\left(\frac{1}{L^{1-\delta}h_L^z}\right) \quad \forall\, \delta > 0, \\
&= o_p(L^{-1/2}),
\end{aligned}$$

where the last equality follows from the properties of the constant $\overline{\delta}$ described in Assumption 4′. From Assumption 4′ and (2), we have

$$
\begin{aligned}
&\frac{1}{L}\sum_{i=1}^{L}\widehat{\underline{\pi}}_{N_i}(r_i|X_i)\cdot\mathbb{I}(U_i)=\frac{1}{L}\sum_{i=1}^{L}\underline{\pi}_{N_i}(r_i|X_i)\cdot\mathbb{I}(U_i)\\
&+\binom{L}{2}^{-1}\sum_{i<j}\frac{1}{h_L^z}\left(\frac{\underline{\psi}(r_i,U_j|X_i,N_i)\cdot\mathbb{I}(U_i)+\underline{\psi}(r_j,U_i|X_j,N_j)\cdot\mathbb{I}(U_j)}{2}\right)\cdot\mathbb{1}\{X_i^d=X_j^d\}K\left(\frac{X_i^c-X_j^c}{h_L}\right)\\
&+o_p(L^{-1/2}),\\
&\frac{1}{L}\sum_{i=1}^{L}\widehat{\overline{\pi}}_{N_i}(r_i|X_i)\cdot\mathbb{I}(U_i)=\frac{1}{L}\sum_{i=1}^{L}\overline{\pi}_{N_i}(r_i|X_i)\cdot\mathbb{I}(U_i)\\
&+\binom{L}{2}^{-1}\sum_{i<j}\frac{1}{h_L^z}\left(\frac{\overline{\psi}(r_i,U_j|X_i,N_i)\cdot\mathbb{I}(U_i)+\overline{\psi}(r_j,U_i|X_j,N_j)\cdot\mathbb{I}(U_j)}{2}\right)\cdot\mathbb{1}\{X_i^d=X_j^d\}K\left(\frac{X_i^c-X_j^c}{h_L}\right)\\
&+o_p(L^{-1/2})
\end{aligned}
\tag{3}
$$

Let

$$
\begin{aligned}
\underline{\varphi}_r(U_i|x)&=E\Big[\underline{\psi}(r(X_j,N_j),U_i|X_j,N_j)\cdot\mathbb{1}\{X_j,N_j\in\mathcal{S}\}\;\Big|\;U_i\;,\;X_j=x\Big]\cdot f_X(x),\\
\overline{\varphi}_r(U_i|x)&=E\Big[\overline{\psi}(r(X_j,N_j),U_i|X_j,N_j)\cdot\mathbb{1}\{X_j,N_j\in\mathcal{S}\}\;\Big|\;U_i\;,\;X_j=x\Big]\cdot f_X(x)
\end{aligned}
$$

The smoothness conditions in Assumption 4′ imply that, w.p.1 in U_i, both $\underline{\varphi}_r(U_i|x)$ and $\overline{\varphi}_r(U_i|x)$ are M times differentiable with respect to x^c with bounded derivatives everywhere on our trimming set $\mathcal{S}$. This, together with the Hoeffding decomposition (see Serfling (1980)) of the U-statistics in (3) yield

$$
\begin{aligned}
\frac{1}{L}\sum_{i=1}^{L}\widehat{\underline{\pi}}_{N_i}(r_i|X_i)&=\frac{1}{L}\sum_{i=1}^{L}\underline{\pi}_{N_i}(r_i|X_i)+\frac{1}{L}\sum_{i=1}^{L}\underline{\varphi}_r(U_i|X_i)+o_p(L^{-1/2}),\\
\frac{1}{L}\sum_{i=1}^{L}\widehat{\overline{\pi}}_{N_i}(r_i|X_i)&=\frac{1}{L}\sum_{i=1}^{L}\overline{\pi}_{N_i}(r_i|X_i)+\frac{1}{L}\sum_{i=1}^{L}\overline{\varphi}_r(U_i|X_i)+o_p(L^{-1/2})
\end{aligned}
\tag{4}
$$

Note that

$$
\begin{aligned}
E\big[\underline{\varphi}_r(U_i|X_i)\big|X_i\big]&=E\Big[E\Big[\underline{\psi}(r(X_j,N_j),U_i|X_j,N_j)\cdot\mathbb{1}\{X_j,N_j\in\mathcal{S}\}\;\Big|\;U_i\;,\;X_j=X_i\Big]\;\Big|\;X_i\Big]\cdot f_X(X_i)\\
&=E\Big[\underline{\psi}(r(X_i,N_j),U_i|X_i,N_j)\cdot\mathbb{1}\{X_i,N_j\in\mathcal{S}\}\;\Big|\;X_i\Big]\cdot f_X(X_i)\\
&=E\Big[E\big[\underline{\psi}(r(X_i,N_j),U_i|X_i,N_j)\big|X_i,N_i,N_j\big]\cdot\mathbb{1}\{X_i,N_j\in\mathcal{S}\}\;\Big|\;X_i\Big]\cdot f_X(X_i).
\end{aligned}
$$

But, by construction, if $\mathcal{S}$ satisfies the restrictions in Assumption 4′, having $X_i,N_j\in\mathcal{S}$ ensures that $E\big[\underline{\psi}(r(X_i,N_j),U_i|X_i,N_j)\big|X_i,N_i,N_j\big]$ is well-defined and equal to zero. Therefore, $E[\underline{\varphi}_r(U_i|X_i)]=0$. Similar arguments show that $E[\overline{\varphi}_r(U_i|X_i)]=0$. Define

$$
\begin{aligned}
\psi_r^{\ell}(U_i)&=\frac{\big[\underline{\pi}_{N_i}(r(X_i,N_i)|X_i)-\mathcal{T}_r^{\ell}\big]\cdot\mathbb{I}(U_i)+\underline{\varphi}_r(U_i|X_i)}{Pr(X,N\in\mathcal{S})},\\
\psi_r^{u}(U_i)&=\frac{\big[\overline{\pi}_{N_i}(r(X_i,N_i)|X_i)-\mathcal{T}_r^{u}\big]\cdot\mathbb{I}(U_i)+\overline{\varphi}_r(U_i|X_i)}{Pr(X,N\in\mathcal{S})}
\end{aligned}
\tag{5}
$$

Using (4), it is easy to show that our estimators in (1) have the following asymptotic linear representations,

$$\widehat{\mathcal{T}}_r^{\ell} = \mathcal{T}_r^{\ell} + \frac{1}{L}\sum_{i=1}^{L}\psi_r^{\ell}(U_i) + o_p(L^{-1/2}), \quad \widehat{\mathcal{T}}_r^{u} = \mathcal{T}_r^{u} + \frac{1}{L}\sum_{i=1}^{L}\psi_r^{u}(U_i) + o_p(L^{-1/2}). \tag{6}$$

In the policy experiments in Section 4.2, the status-quo was the reserve price policy $r_0(X,N) = v_0$ (i.e, reserve price equals appraisal value). In this case[2], $\pi_N^{\ell}(r_0(X,N)|X) = \pi_N^{u}(r_0(X,N)|X) = \pi_N(r_0(X,N)|X)$ and consequently $\widehat{\mathcal{T}}_r^{\ell} = \widehat{\mathcal{T}}_r^{u} = \widehat{\mathcal{T}}_r$. In this case, the influence functions in equation 9 satisfy $\psi_{r_0}^{\ell}(U_i) = \psi_{r_0}^{u}(U_i) \equiv \psi_{r_0}(U_i)$, and we have

$$\widehat{\mathcal{T}}_{r_0}^{\ell} = \widehat{\mathcal{T}}_{r_0}^{u} \equiv \widehat{\mathcal{T}}_{r_0} = \mathcal{T}_{r_0} + \frac{1}{L}\sum_{i=1}^{L}\psi_{r_0}(U_i) + o_p(L^{-1/2}). \tag{7}$$

Now we turn to the average measures we used to evaluate our policy exercises in Section 4.2. Two of the measures we estimate are

$$\underbrace{E_{X,N}\big[\pi_N(r_1(X,N)|X)\big|X,N\in\mathcal{S}\big]}_{\equiv\mathcal{T}_{r_1}}, \quad \underbrace{E_{X,N}\big[\pi_N(r_1(X,N)|X) - \pi_N(r_0(X,N)|X)\big|X,N\in\mathcal{S}\big]}_{\equiv\mathcal{T}_{r_1}-\mathcal{T}_{r_0}}.$$

Under the conditions of Theorem 1, we have $\mathcal{T}_{r_1} \in \big[\mathcal{T}_{r_1}^{\ell}\ ,\ \mathcal{T}_{r_1}^{u}\big]$ and $\mathcal{T}_{r_1} - \mathcal{T}_{r_0} \in \big[\mathcal{T}_{r_1}^{\ell} - \mathcal{T}_{r_0}\ ,\ \mathcal{T}_{r_1}^{u} - \mathcal{T}_{r_0}\big]$. Combining the results in (6) − (7), we have that under Assumption 4′,

$$\begin{pmatrix}\sqrt{L}(\widehat{\mathcal{T}}_{r_1}^{\ell} - \mathcal{T}_{r_1}^{\ell})\\ \sqrt{L}(\widehat{\mathcal{T}}_{r_1}^{u} - \mathcal{T}_{r_1}^{u})\end{pmatrix} \xrightarrow{d} \mathcal{N}\left(\begin{pmatrix}0\\0\end{pmatrix}\ ,\ \begin{pmatrix}\sigma_{\ell}^2(r_1) & \rho(r_1)\sigma_{\ell}(r_1)\sigma_u(r_1)\\ \rho(r_1)\sigma_{\ell}(r_1)\sigma_u(r_1) & \sigma_u^2(r_1)\end{pmatrix}\right),$$
$$\textit{where}\quad \sigma_{\ell}^2(r_1) = E\big[\psi_{r_1}^{\ell}(U_i)^2\big],\quad \sigma_u^2(r_1) = E\big[\psi_{r_1}^{u}(U_i)^2\big]\quad \textit{and}\quad \rho(r_1) = \frac{E[\psi_{r_1}^{\ell}(U_i)\cdot\psi_{r_1}^{u}(U_i)]}{\sigma_{\ell}(r_1)\sigma_u(r_1)} \tag{8A}$$

$$\begin{pmatrix}\sqrt{L}(\widehat{\mathcal{T}}_{r_1}^{\ell} - \widehat{\mathcal{T}}_{r_0} - (\mathcal{T}_{r_1}^{\ell} - \mathcal{T}_{r_0}))\\ \sqrt{L}(\widehat{\mathcal{T}}_{r_1}^{u} - \widehat{\mathcal{T}}_{r_0} - (\mathcal{T}_{r_1}^{\ell} - \mathcal{T}_{r_0}))\end{pmatrix}$$
$$\xrightarrow{d} \mathcal{N}\left(\begin{pmatrix}0\\0\end{pmatrix}\ ,\ \begin{pmatrix}\sigma_{\ell}^2(r_1,r_0) & \rho(r_1,r_0)\sigma_{\ell}(r_1,r_0)\sigma_u(r_1,r_0)\\ \rho(r_1,r_0)\sigma_{\ell}(r_1,r_0)\sigma_u(r_1,r_0) & \sigma_u^2(r_1,r_0)\end{pmatrix}\right),$$
$$\textit{where}\quad \sigma_{\ell}^2(r_1,r_0) = E\big[(\psi_{r_1}^{\ell}(U_i) - \psi_{r_0}(U_i))^2\big],\quad \sigma_u^2(r_1,r_0) = E\big[(\psi_{r_1}^{u}(U_i) - \psi_{r_0}(U_i))^2\big]$$
$$\textit{and}\quad \rho(r_1,r_0) = \frac{E[(\psi_{r_1}^{\ell}(U_i) - \psi_{r_0}(U_i))\cdot(\psi_{r_1}^{u}(U_i) - \psi_{r_0}(U_i))]}{\sigma_{\ell}(r_1,r_0)\sigma_u(r_1,r_0)} \tag{8B}$$

[2]By inspection, if $r = v_0$, the influence functions in equation 9 simply reduce to

$$\frac{(\max\{v_0\ ,\ B_i\} - T_{n-1:n}(v_0|x))}{q_{X,N}(x,n)}\mathbb{1}\{N_i = n\} \equiv \psi(r,U_i|x,n).$$

The resulting expression for $\psi_{r_0}^{\ell}(U_i) = \psi_{r_0}^{u}(U_i) \equiv \psi_{r_0}(U_i)$ follow immediately from here through the definitions leading to (5).

The elements in these variance-covariance matrices can be estimated nonparametrically. First, let

$$\widehat{\psi}^{\ell}_{r_1}(U_i) = \frac{\left[\widehat{\underline{\pi}}_{N_i}(r_1(X_i,N_i)|X_i) - \widehat{\mathcal{T}}^{\ell}_{r_1}\right]\cdot \mathbb{I}(U_i) + \widehat{\underline{\varphi}}_{r_1}(U_i|X_i)}{\widehat{Pr}(X,N\in\mathcal{S})},$$

$$\widehat{\psi}^{u}_{r_1}(U_i) = \frac{\left[\widehat{\overline{\pi}}_{N_i}(r_1(X_i,N_i)|X_i) - \widehat{\mathcal{T}}^{u}_{r_1}\right]\cdot \mathbb{I}(U_i) + \widehat{\overline{\varphi}}_{r_1}(U_i|X_i)}{\widehat{Pr}(X,N\in\mathcal{S})},$$

$$\widehat{\psi}_{r_0}(U_i) = \frac{\left[\widehat{\pi}_{N_i}(r_0(X_i,N_i)|X_i) - \widehat{\mathcal{T}}^{u}_{r_0}\right]\cdot \mathbb{I}(U_i) + \widehat{\varphi}_{r_0}(U_i|X_i)}{\widehat{Pr}(X,N\in\mathcal{S})},$$

where

$$\widehat{\underline{\varphi}}_{r_1}(U_i|X_i) = \frac{1}{(L-1)h^z_L}\sum_{j\neq i}\widehat{\underline{\psi}}\left(r_1(X_j,N_j),U_i|X_j,N_j\right)\cdot\mathbb{I}(U_j)\cdot\mathbb{1}\{X^d_j = X^d_i\}\cdot K\left(\frac{X^c_i - X^c_j}{h_L}\right),$$

$$\widehat{\overline{\varphi}}_{r_1}(U_i|X_i) = \frac{1}{(L-1)h^z_L}\sum_{j\neq i}\widehat{\overline{\psi}}\left(r_1(X_j,N_j),U_i|X_j,N_j\right)\cdot\mathbb{I}(U_j)\cdot\mathbb{1}\{X^d_j = X^d_i\}\cdot K\left(\frac{X^c_i - X^c_j}{h_L}\right),$$

$$\widehat{\varphi}_{r_0}(U_i|X_i) = \frac{1}{(L-1)h^z_L}\sum_{j\neq i}\widehat{\psi}\left(r_0(X_j,N_j),U_i|X_j,N_j\right)\cdot\mathbb{I}(U_j)\cdot\mathbb{1}\{X^d_j = X^d_i\}\cdot K\left(\frac{X^c_i - X^c_j}{h_L}\right),$$

where $\underline{\psi}(r,U_i|x,n)$ and $\overline{\psi}(r,U_i|x,n)$ are as described in equation 9 and $\psi(r,U_i|x,n)$ is given in footnote 2 (recall that $r_0(X_i,N_i) = X_{2,i}$ –appraisal value– throughout our policy experiments). The variance-covariance elements in (8A) can be consistently estimated by

$$\widehat{\sigma}^2_{\ell}(r_1) = \frac{1}{L}\sum_{i=1}^{L}\psi^{\ell}_{r_1}(U_i)^2, \quad \widehat{\sigma}^2_{u}(r_1) = \frac{1}{L}\sum_{i=1}^{L}\psi^{u}_{r_1}(U_i)^2 \quad \textit{and} \quad \widehat{\rho}(r_1) = \frac{\frac{1}{L}\sum_{i=1}^{L}\widehat{\psi}^{\ell}_{r_1}(U_i)\cdot\widehat{\psi}^{u}_{r_1}(U_i)}{\widehat{\sigma}_{\ell}(r_1)\widehat{\sigma}_{u}(r_1)}.$$

Similarly, those in (8B) can be estimated consistently by

$$\widehat{\sigma}^2_{\ell}(r_1,r_0) = \frac{1}{L}\sum_{i=1}^{L}(\widehat{\psi}^{\ell}_{r_1}(U_i) - \widehat{\psi}_{r_0}(U_i))^2, \quad \widehat{\sigma}^2_{u}(r_1,r_0) = \frac{1}{L}\sum_{i=1}^{L}(\widehat{\psi}^{u}_{r_1}(U_i) - \widehat{\psi}_{r_0}(U_i))^2,$$

$$\widehat{\rho}(r_1,r_0) = \frac{\frac{1}{L}\sum_{i=1}^{L}(\widehat{\psi}^{\ell}_{r_1}(U_i) - \widehat{\psi}_{r_0}(U_i))\cdot(\widehat{\psi}^{u}_{r_1}(U_i) - \widehat{\psi}_{r_0}(U_i))}{\widehat{\sigma}_{\ell}(r_1,r_0)\widehat{\sigma}_{u}(r_1,r_0)}$$

From the results in (8A)-(8B) and Theorem 1, we can construct confidence intervals for $\mathcal{T}_{r_1}$ and $\mathcal{T}_{r_1} - \mathcal{T}_{r_0}$ which achieve a prespecified asymptotic coverage probability $(1-\alpha)\%$ while allowing for $\mathcal{T}^u_{r_1} - \mathcal{T}^{\ell}_{r_1}$ to be arbitrarily close to zero. Before proceeding, note that our use of bias-reducing kernels implies that we may have $\widehat{\mathcal{T}}^{\ell}_{r_1} > \widehat{\mathcal{T}}^{u}_{r_1}$ with positive probability in finite samples. For this reason we will the type of shrinkage estimator for $\mathcal{T}^u_{r_1} - \mathcal{T}^{\ell}_{r_1}$ advocated in Stoye (2009). Let

$$\widetilde{\Delta}(r_1) = \begin{cases} \widehat{\mathcal{T}}^{u}_{r_1} - \widehat{\mathcal{T}}^{\ell}_{r_1} & \text{if} \quad \widehat{\mathcal{T}}^{u}_{r_1} - \widehat{\mathcal{T}}^{\ell}_{r_1} > b_L \\ 0 \quad \text{otherwise,} & \end{cases}$$

where b_L is a nonnegative sequence $b_L \to 0$ such that $b_L \sqrt{L} \to \infty$. The shrinkage estimator $\widetilde{\Delta}(r_1)$ will satisfy the type of "superefficiency" conditions required for the uniform validity of our confidence interval. Let c^1_ℓ and c^1_u minimize $c_\ell \cdot \widehat{\sigma}_\ell(r_1) + c_u \cdot \widehat{\sigma}_u(r_1)$ subject to the constraint

$$Pr\left(-c_\ell \leq Z_1 \ , \ \widehat{\rho}(r_1) \cdot Z_1 \leq c_u + \frac{\sqrt{L} \cdot \widetilde{\Delta}(r_1)}{\widehat{\sigma}_u(r_1)} + \sqrt{1 - \widehat{\rho}^2(r_1)} \cdot Z_2\right) \geq 1 - \alpha,$$

$$Pr\left(-c_\ell - \frac{\sqrt{L} \cdot \widetilde{\Delta}(r_1)}{\widehat{\sigma}_\ell(r_1)} - \sqrt{1 - \widehat{\rho}^2(r_1)} \cdot Z_2 \leq \widehat{\rho}(r_1) \cdot Z_1 \ , \ Z_1 \leq c_u\right) \geq 1 - \alpha,$$

where Z_1 and Z_2 are iid Standard Normal random variables. Our $(1-\alpha)\%$ CI for $\mathcal{T}_{r_1}$ is given by

$$CI_\alpha(\mathcal{T}_{r_1}) = \left[\widehat{\mathcal{T}}^\ell_{r_1} - c^1_\ell \cdot \frac{\widehat{\sigma}_\ell(r_1)}{\sqrt{L}} \ , \ \widehat{\mathcal{T}}^u_{r_1} + c^1_u \cdot \frac{\widehat{\sigma}_u(r_1)}{\sqrt{L}}\right] \tag{9}$$

Given our previous asymptotic results, if the conditions of Theorem 1 are satisfied, the validity of (9) follows from Proposition 3 in Stoye (2009). Conversely, a $(1-\alpha)\%$ CI for $\mathcal{T}_{r_1} - \mathcal{T}_{r_0}$ is given by

$$CI_\alpha(\mathcal{T}_{r_1} - \mathcal{T}_{r_0}) = \left[\widehat{\mathcal{T}}^\ell_{r_1} - \widehat{\mathcal{T}}_{r_0} - c^{1,0}_\ell \cdot \frac{\widehat{\sigma}_\ell(r_1, r_0)}{\sqrt{L}} \ , \ \widehat{\mathcal{T}}^u_{r_1} - \widehat{\mathcal{T}}_{r_0} + c^{1,0}_u \cdot \frac{\widehat{\sigma}_u(r_1, r_0)}{\sqrt{L}}\right], \tag{10}$$

where $c^{1,0}_\ell$ and $c^{1,0}_u$ minimize $c_\ell \cdot \widehat{\sigma}_\ell(r_1, r_0) + c_u \cdot \widehat{\sigma}_u(r_1, r_0)$ subject to the constraint

$$Pr\left(-c_\ell \leq Z_1 \ , \ \widehat{\rho}(r_1, r_0) \cdot Z_1 \leq c_u + \frac{\sqrt{L} \cdot \widetilde{\Delta}(r_1)}{\widehat{\sigma}_u(r_1, r_0)} + \sqrt{1 - \widehat{\rho}^2(r_1, r_0)} \cdot Z_2\right) \geq 1 - \alpha,$$

$$Pr\left(-c_\ell - \frac{\sqrt{L} \cdot \widetilde{\Delta}(r_1)}{\widehat{\sigma}_\ell(r_1, r_0)} - \sqrt{1 - \widehat{\rho}^2(r_1, r_0)} \cdot Z_2 \leq \widehat{\rho}(r_1, r_0) \cdot Z_1 \ , \ Z_1 \leq c_u\right) \geq 1 - \alpha.$$

The last object we studied in our counterfactual experiments was the average probability of no sale under the counterfactual policy r_1, that is,

$$E_{X,N}\left[F_{N:N}(r_1(X,N)|X)\big|X, N \in \mathcal{S}\right] \equiv \mathcal{P}_{r_1}.$$

Under the conditions of Theorem 1, $F_{N:N}(r_1(X,N)|X)$ is bounded between

$$\underline{F}_{N:N}(r_1(X,N)|X) = \sum_{m=N+1}^{\overline{n}} \frac{N}{(m-1)m} F_{m-1:m}(r_1(X,N)|X) + \frac{N}{\overline{n}}\left(\phi_{\overline{n}}(F_{\overline{n}-1:\overline{n}}(r_1(X,N)|X))\right)^{\overline{n}}, \quad and$$

$$\overline{F}_{N:N}(r_1(X,N)|X) \sum_{m=N+1}^{\overline{n}} \frac{N}{(m-1)m} F_{m-1:m}(r_1(X,N)|X) + \frac{N}{\overline{n}} F_{\overline{n}-1:\overline{n}}(r_1(X,N)|X)$$

and therefore $\mathcal{P}_{r_1} \in \left[\mathcal{P}^\ell_{r_1} \ , \ \mathcal{P}^u_{r_1}\right]$, where

$$\mathcal{P}^\ell_{r_1} = E_{X,N}\left[\underline{F}_{N:N}(r_1(X,N)|X)\big|X, N \in \mathcal{S}\right] \quad and \quad \mathcal{P}^u_{r_1} = E_{X,N}\left[\overline{F}_{N:N}(r_1(X,N)|X)\big|X, N \in \mathcal{S}\right].$$

These bounds are estimated nonparametrically by

$$\widehat{\mathcal{P}}^\ell_{r_1} = \frac{1}{L}\sum_{i=1}^{L} \frac{\widehat{\underline{F}}_{N_i:N_i}(r_1(X_i,N_i)|X_i)}{\widehat{Pr}(X, N \in \mathcal{S})} \cdot \mathbb{I}(U_i) \quad and \quad \widehat{\mathcal{P}}^u_{r_1} = \frac{1}{L}\sum_{i=1}^{L} \frac{\widehat{\overline{F}}_{N_i:N_i}(r_1(X_i,N_i)|X_i)}{\widehat{Pr}(X, N \in \mathcal{S})} \cdot \mathbb{I}(U_i). \tag{11}$$

Let

$$\underline{\psi}^{\mathcal{P}}(r,U_i|x,n) = \sum_{m=n+1}^{\overline{n}} \frac{n}{(m-1)m} \frac{\left(\mathbb{1}\{B_i \le r\} - F_{m-1:m}(r|x)\right)}{q_{X,N}(x,m)} \mathbb{1}\{N_i = m\}$$
$$+ n \cdot \left(\phi_{\overline{n}}(F_{\overline{n}-1:\overline{n}}(r|x))\right)^{\overline{n}-1} \cdot \phi'_{\overline{n}}(F_{\overline{n}-1:\overline{n}}(r|x)) \cdot \frac{\left(\mathbb{1}\{B_i \le r\} - F_{\overline{n}-1:\overline{n}}(r|x)\right)}{q_{X,N}(x,\overline{n})} \mathbb{1}\{N_i = \overline{n}\}$$
$$\overline{\psi}^{\mathcal{P}}(r,U_i|x,n) = \sum_{m=n+1}^{\overline{n}} \frac{n}{(m-1)m} \frac{\left(\mathbb{1}\{B_i \le r\} - F_{m-1:m}(r|x)\right)}{q_{X,N}(x,m)} \mathbb{1}\{N_i = m\} + \frac{n}{\overline{n}} \frac{\left(\mathbb{1}\{B_i \le r\} - F_{\overline{n}-1:\overline{n}}(r|x)\right)}{q_{X,N}(x,\overline{n})} \mathbb{1}\{N_i = \overline{n}\}$$

and

$$\underline{\varphi}_r^{\mathcal{P}}(U_i|x) = E\Big[\underline{\psi}^{\mathcal{P}}(r(X_j,N_j),U_i|X_j,N_j) \cdot \mathbb{1}\{X_j,N_j \in \mathcal{S}\} \;\Big|\; U_i \;,\; X_j = x\Big] \cdot f_X(x),$$
$$\overline{\varphi}_r^{\mathcal{P}}(U_i|x) = E\Big[\overline{\psi}^{\mathcal{P}}(r(X_j,N_j),U_i|X_j,N_j) \cdot \mathbb{1}\{X_j,N_j \in \mathcal{S}\} \;\Big|\; U_i \;,\; X_j = x\Big] \cdot f_X(x)$$

Define

$$\psi_{r_1}^{\mathcal{P}^\ell}(U_i) = \frac{\left[\underline{F}_{N_i:N_i}(r_1(X_i,N_i)|X_i) - \mathcal{P}_{r_1}^\ell\right] \cdot \mathbb{I}(U_i) + \underline{\varphi}_{r_1}^{\mathcal{P}}(U_i|X_i)}{Pr(X,N \in \mathcal{S})},$$
$$\psi_{r_1}^{\mathcal{P}^u}(U_i) = \frac{\left[\overline{F}_{N_i:N_i}(r_1(X_i,N_i)|X_i) - \mathcal{P}_{r_1}^u\right] \cdot \mathbb{I}(U_i) + \overline{\varphi}_{r_1}^{\mathcal{P}}(U_i|X_i)}{Pr(X,N \in \mathcal{S})}$$

Under Assumption $4'$ we have

$$\begin{pmatrix} \sqrt{L}\big(\widehat{\mathcal{P}}_{r_1}^\ell - \mathcal{P}_{r_1}^\ell\big) \\ \sqrt{L}\big(\widehat{\mathcal{P}}_{r_1}^u - \mathcal{P}_{r_1}^u\big) \end{pmatrix} \xrightarrow{d} \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} , \begin{pmatrix} \widetilde{\sigma}_\ell^2(r_1) & \widetilde{\rho}(r_1)\widetilde{\sigma}_\ell(r_1)\widetilde{\sigma}_u(r_1) \\ \widetilde{\rho}(r_1)\widetilde{\sigma}_\ell(r_1)\widetilde{\sigma}_u(r_1) & \widetilde{\sigma}_u^2(r_1) \end{pmatrix} \right),$$
$$where \quad \widetilde{\sigma}_\ell^2(r_1) = E\big[\psi_{r_1}^{\mathcal{P}^\ell}(U_i)^2\big], \quad \widetilde{\sigma}_u^2(r_1) = E\big[\psi_{r_1}^{\mathcal{P}^u}(U_i)^2\big] \quad and \quad \widetilde{\rho}(r_1) = \frac{E[\psi_{r_1}^{\mathcal{P}^\ell}(U_i) \cdot \psi_{r_1}^{\mathcal{P}^u}(U_i)]}{\widetilde{\sigma}_\ell(r_1)\widetilde{\sigma}_u(r_1)}$$

From here, a $(1-\alpha)\%$ CI for $\mathcal{P}_{r_1}$ is constructed as

$$CI_\alpha(\mathcal{P}_{r_1}) = \left[\widehat{\mathcal{P}}_{r_1}^\ell - \kappa_\ell^1 \cdot \frac{\widehat{\widetilde{\sigma}}_\ell(r_1)}{\sqrt{L}} \;,\; \widehat{\mathcal{P}}_{r_1}^u + \kappa_u^1 \cdot \frac{\widehat{\widetilde{\sigma}}_u(r_1)}{\sqrt{L}}\right], \tag{12}$$

where κ_ℓ^1, κ_ℓ^u are determined analogously to c_ℓ^1, c_ℓ^u in (9).

The IPV case

If IPV is satisfied and Assumption 2 in the paper holds, all the relevant objects in our counterfactual analysis are point-identified. In this case, the following is a consistent estimator of $\pi_n(r|x)$:

$$\widehat{\pi}_n^{IPV}(r|x) = \widehat{T}_{n-1:n}(r|x) - v_0 - \phi_n\left(\widehat{F}_{n-1:n}(r|x)\right)^n (r - v_0). \tag{13}$$

As a result, $\mathcal{T}_r$ can be consistently estimated by

$$\widehat{\mathcal{T}}_r^{IPV} = \frac{1}{L}\sum_{i=1}^{L} \frac{\widehat{\pi}_{N_i}^{IPV}(r_i|X_i)}{\widehat{Pr}(X,N \in \mathcal{S})} \cdot \mathbb{I}(U_i).$$

Let $\psi^{IPV}(r, U_i|x, n)$ be as defined in equation 12, and define

$$\varphi_r^{IPV}(U_i|x) = E\Big[\psi^{IPV}(r(X_j, N_j), U_i|X_j, N_j) \cdot \mathbb{1}\{X_j, N_j \in \mathcal{S}\} \;\Big|\; U_i \;,\; X_j = x\Big] \cdot f_X(x),$$
$$\psi_r^{IPV}(U_i) = \frac{\big[\pi_{N_i}(r(X_i, N_i)|X_i) - \mathcal{T}_r\big] \cdot \mathbb{I}(U_i) + \varphi_r^{IPV}(U_i|X_i)}{Pr(X, N \in \mathcal{S})}.$$

If IPV holds, we have

$$\sqrt{L}\left(\widehat{\mathcal{T}}_{r_1}^{IPV} - \mathcal{T}_{r_1}\right) \xrightarrow{d} \mathcal{N}\left(0, \sigma_{IPV}^2(r_1)\right), \quad where \quad \sigma_{IPV}^2(r_1) = E\left[\psi_{r_1}^{IPV}(U_i)^2\right],$$
$$\sqrt{L}\left(\widehat{\mathcal{T}}_{r_1}^{IPV} - \widehat{\mathcal{T}}_{r_0} - (\mathcal{T}_{r_1} - \mathcal{T}_{r_0})\right) \xrightarrow{d} \mathcal{N}\left(0, \sigma_{IPV}^2(r_1, r_0)\right),$$
$$where \quad \sigma_{IPV}^2(r_1, r_0) = E\left[(\psi_{r_1}^{IPV}(U_i) - \psi_{r_0}(U_i))^2\right].$$

From here, $(1-\alpha)\%$ CIs for $\mathcal{T}_{r_1}$ and $\mathcal{T}_{r_1} - \mathcal{T}_{r_0}$ can be constructed as

$$CI_\alpha^{IPV}(\mathcal{T}_{r_1}) = \left[\widehat{\mathcal{T}}_{r_1}^{IPV} - c_\alpha \cdot \frac{\widehat{\sigma}_{IPV}(r_1)}{\sqrt{L}} \;,\; \widehat{\mathcal{T}}_{r_1}^{IPV} + c_\alpha \cdot \frac{\widehat{\sigma}_{IPV}(r_1)}{\sqrt{L}}\right],$$
$$CI_\alpha^{IPV}(\mathcal{T}_{r_1} - \mathcal{T}_{r_0}) = \left[\widehat{\mathcal{T}}_{r_1}^{IPV} - \widehat{\mathcal{T}}_{r_0} - c_\alpha \cdot \frac{\widehat{\sigma}_{IPV}(r_1, r_0)}{\sqrt{L}} \;,\; \widehat{\mathcal{T}}_{r_1}^{IPV} - \widehat{\mathcal{T}}_{r_0} + c_\alpha \cdot \frac{\widehat{\sigma}_{IPV}(r_1, r_0)}{\sqrt{L}}\right], \tag{14}$$

where $\Phi(c_\alpha) - \Phi(-c_\alpha) = 1 - \alpha$.

Under IPV, we also have

$$\mathcal{P}_{r_1} = E_{X,N}\left[\phi_N(F_{N-1:N}(r_1(X,N)|X))^N \Big| X, N \in \mathcal{S}\right],$$

which can be estimated nonparametrically as

$$\widehat{\mathcal{P}}_{r_1}^{IPV} = \frac{1}{N}\sum_{i=1}^{N} \frac{\phi_{N_i}(\widehat{F}_{N_i-1:N_i}(r_1(X_i, N_i)|X_i))^{N_i}}{\widehat{Pr}(X, N \in \mathcal{S})} \cdot \mathbb{I}(U_i). \tag{15}$$

Let

$$\psi^{\mathcal{P}_{IPV}}(r, U_i|x, n) = n \cdot \left(\phi_n(F_{n-1:n}(r|x))\right)^{n-1} \cdot \phi_n'(F_{n-1:n}(r|x)) \cdot \frac{\left(\mathbb{1}\{B_i \le r\} - F_{n-1:n}(r|x)\right)}{q_{X,N}(x, n)} \mathbb{1}\{N_i = n\},$$
$$\varphi_r^{\mathcal{P}}(U_i|x) = E\Big[\psi^{\mathcal{P}_{IPV}}(r(X_j, N_j), U_i|X_j, N_j) \cdot \mathbb{1}\{X_j, N_j \in \mathcal{S}\} \;\Big|\; U_i \;,\; X_j = x\Big] \cdot f_X(x),$$
$$\psi_{r_1}^{\mathcal{P}_{IPV}}(U_i) = \frac{\big[\phi_{N_i}(F_{N_i-1:N_i}(r_1(X_i, N_i)|X_i))^{N_i} - \mathcal{P}_{r_1}\big] \cdot \mathbb{I}(U_i) + \varphi_{r_1}^{\mathcal{P}_{IPV}}(U_i|X_i)}{Pr(X, N \in \mathcal{S})}$$

If IPV holds, Assumption $4'$ yields

$$\sqrt{L}\big(\widehat{\mathcal{P}}_{r_1}^{IPV} - \mathcal{P}_{r_1}\big) \xrightarrow{d} \mathcal{N}\big(0, \widetilde{\sigma}_{IPV}^2(r_1)\big), \quad where \quad \widetilde{\sigma}_{IPV}^2(r_1) = E\big[\psi_{r_1}^{\mathcal{P}_{IPV}}(U_i)^2\big].$$

A $(1-\alpha)\%$ CI for $\mathcal{P}_{r_1}$ is given by

$$CI_\alpha^{IPV}(\mathcal{P}_{r_1}) = \left[\widehat{\mathcal{P}}_{r_1}^{IPV} - c_\alpha \cdot \frac{\widehat{\widetilde{\sigma}}_{IPV}(r_1)}{\sqrt{L}} \;,\; \widehat{\mathcal{P}}_{r_1}^{IPV} + c_\alpha \cdot \frac{\widehat{\widetilde{\sigma}}_{IPV}(r_1)}{\sqrt{L}}\right]. \tag{16}$$

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