Supplement to: "A Simple Test for Moment Inequality Models with an Application to English Auctions"

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Abstract

This document includes supplementary material for the paper "A Simple Test for Moment Inequality Models with an Application to English Auctions". It includes a step-by-step proof of Theorem 1, an alternative (computationally simpler) way to construct our test-statistic, detailed descriptions for the constructions of our test-statistics for the auction models analyzed, additional Monte Carlo experiment results, and the proofs of the auction model results contained in Section 4 of the paper.

Labeling Conventions

Every result, equation, assumption, table, etc., introduced in this supplement will be labeled starting with an 'S'. Specifically, equations will be labeled (S.1), (S.2), etc. Every equation referenced here that is not of that type refers to an equation in the main paper. Sections in this supplement will be labeled S-A, S-B, and so on. Sections referenced here that are not of that format refer to sections in the main paper. Similarly, all claims, propositions, theorems and results introduced here will be labeled S1, S2, S3, etc. Any other labeling refers to results in the main paper. The same is true for tables and assumptions.

S-A Econometric Results

Here we present the step-by-step proof of Theorem 1, an alternative (computationally simpler) way to construct our test-statistic and a detailed descriptions for the constructions of our test-statistics for the auction models analyzed.

S-A.1 Proof of Theorem 1

S-A.1.1 A useful probability inequality

Let $\mathcal{W} \equiv \mathcal{X} \times \mathcal{N} \times \mathcal{Z}$ denote our overall testing range and recall that $\mathcal{X}$ is chosen such that $(x^c, x^d) \equiv x \in \mathcal{X}$ implies $x^c \in \text{int}\,(\text{Supp}(X^c))$. Also recall that

$$f_{X,N}(x,n) \geq \underline{f} > 0 \quad \forall\ (x,n) \in \mathcal{W}.$$

In this section we will describe conditions that yield a exponential bound for the probability

$$Pr\left(\underset{(x,z,n,n')\in\mathcal{W}}{sup} \left| \widehat{R}^q(x,z;n,n') - R^q(x,z;n,n') \right| \geq b_L \right),$$

where $b_L \longrightarrow 0$ is the bandwidth sequence used in our construction (4). The bound we obtain is given in (S.12) and its usefulness will become evident in sections to follow. Invoking Lemma 22 in Nolan and Pollard (1987) and Lemmas 2.4 and 2.14 in Pakes and Pollard (1989) (see also Example 10 there), having a kernel of bounded variation implies that the class of functions

$$\mathscr{G}_K = \left\{ g : g(x) = \mathcal{H}\left(x - v; h\right) \quad \text{for some } v \in \mathbb{R}^{dim(X)} \text{ and some } h > 0 \right\}$$

is *Euclidean*[1] with respect to the constant envelope $\overline{K}$. For a given (x,z,n) and for each of the $\ell = 1, \ldots, d_s$ elements in the vector-valued function $S(Y_i, x, z, n)$ denote

$$\begin{aligned}
\widetilde{q}_i^{\ell}(x,z,n;h) &= S^{\ell}(Y_i,x,z,n) \cdot \mathcal{H}(X_i - x;h) \cdot \mathbb{1}\left\{N_i = n\right\}, \\
\widetilde{v}_i^{\ell}(x,z,n;h) &= \left(\frac{S^{\ell}(Y_i,x,z,n) - s^{\ell}(x,z,n)}{f_{X,N}(x,n)} \right) \cdot \mathcal{H}(X_i - x;h) \cdot \mathbb{1}\left\{N_i = n\right\}, \\
\widehat{\nu}^{\ell}(x,z,n) &= \frac{1}{L \cdot h_L^r} \sum_{i=1}^{L} \widetilde{v}_i^{\ell}\left(x,z,n;h_L\right), \\
\widehat{Q}^{\ell}(x,z,n) &= \frac{1}{L \cdot h_L^r} \sum_{i=1}^{L} \widetilde{q}_i^{\ell}\left(x,z,n;h_L\right), \\
Q^{\ell}(x,z,n) &= s^{\ell}(x,z,n) \cdot f_{X,N}(x,n).
\end{aligned}$$

Using an M^{th} order approximation, the smoothness conditions in Assumption 3.1 imply the existence of a finite constant $\overline{M}$ such that,

$$\begin{aligned}
&\underset{(x,n)\in\mathcal{W}}{sup} \left| E\left[\widehat{f}_{X,N}(x,n)\right] - f_{X,N}(x,n) \right| \leq \overline{M} \cdot h_L^M, \\
&\underset{(x,z,n)\in\mathcal{W}}{sup} \left| E\left[\widehat{\nu}^{\ell}(x,z,n)\right] \right| \leq \overline{M} \cdot h_L^M, \\
&\underset{(x,z,n)\in\mathcal{W}}{sup} \left| E\left[\widehat{Q}^{\ell}(x,z,n)\right] - Q^{\ell}(x,z,n) \right| \leq \overline{M} \cdot h_L^M.
\end{aligned} \tag{S.1}$$

[1] See Definition 2.7 in Pakes and Pollard (1989).

If the Euclidean properties in Assumption 3.4 hold, Lemma 2.14 in Pakes and Pollard (1989) implies that the processes

$$\left\{\widetilde{q}_i^{\ell}(x,z,n;h)\colon\ (x,z,n)\in\mathcal{W},\ h>0,\ 1\le i\le L\right\}$$

$$\left\{\widetilde{v}_i^{\ell}(x,z,n;h)\colon\ (x,z,n)\in\mathcal{W},\ h>0,\ 1\le i\le L\right\}$$

are *manageable* (as described in Definition 7.9 of Pollard (1990)) with respect to the envelopes[2] $\overline{K}\cdot\overline{S}(\cdot)$ and $\left(\overline{K}/\underline{f}\right)\cdot\left(\overline{S}(\cdot)+\max\{|\overline{s}|,|\underline{s}|\}\right)$ respectively. These envelopes possess a moment generating function by Assumption 3.4. The Euclidean property of the class of functions $\mathscr{G}_K$ described above also implies that the process

$$\{\mathcal{H}(X_i-x;h)\cdot\mathbb{1}\{N_i=n\}\colon\ (x,n)\in\mathcal{W},\ h>0,\ 1\le i\le L\}$$

is manageable with respect to the constant envelope $\overline{K}$. Using the maximal inequality results in Chapter 7 of Pollard (1990) combined with the bias conditions in S.1 imply that there exist positive constants A_1, A_2 and A_3 such that for each $\ell=1,\ldots,d_s$ and any $\delta>0$,

$$\begin{aligned}
&Pr\left(\sup_{(x,n)\in\mathcal{W}}\left|\widehat{f}_{X,N}(x,n)-f_{X,N}(x,n)\right|\ge\delta\right)\le A_1\cdot\exp\left\{-\sqrt{L}\cdot h_L^r\left(A_2\cdot\delta-A_3\cdot h_L^M\right)\right\},\\
&Pr\left(\sup_{(x,z,n)\in\mathcal{W}}\left|\widehat{\nu}^{\ell}(x,z,n)\right|\ge\delta\right)\le A_1\cdot\exp\left\{-\sqrt{L}\cdot h_L^r\left(A_2\cdot\delta-A_3\cdot h_L^M\right)\right\},\\
&Pr\left(\sup_{(x,z,n)\in\mathcal{W}}\left|\widehat{Q}^{\ell}(x,z,n)-Q^{\ell}(x,z,n)\right|\ge\delta\right)\le A_1\cdot\exp\left\{-\sqrt{L}\cdot h_L^r\left(A_2\cdot\delta-A_3\cdot h_L^M\right)\right\}.
\end{aligned}\tag{S.2}$$

Group

$$\widehat{\mu}^{\ell}(x,z,n)=\left(\left[\widehat{Q}^{\ell}(x,z,n)-Q^{\ell}(x,z,n)\right]\quad\left[\widehat{f}_{X,N}(x,n)-f_{X,N}(x,n)\right]\right)'.$$

Using (S.2), for any $\delta>0$ we have

$$Pr\left(\sup_{(x,z,n)\in\mathcal{W}}\left\|\widehat{\mu}^{\ell}(x,z,n)\right\|\ge\delta\right)\le 2\cdot A_1\cdot\exp\left\{-\sqrt{L}\cdot h_L^r\left(A_2\cdot\frac{\delta}{\sqrt{2}}-A_3\cdot h_L^M\right)\right\}.\tag{S.2$'$}$$

For any $(x,z,n)\in\mathcal{W}$, a second order approximation yields (recall that $f_{X,N}(x,n)\ge\underline{f}>0$ for all $(x,n)\in\mathcal{W}$)

$$\begin{aligned}
s^{\ell}(x,z,n)-s^{\ell}(x,z,n)&=\frac{1}{f_{X,N}(x,n)}\cdot\left[\widehat{Q}^{\ell}(x,z,n)-Q^{\ell}(x,z,n)\right]-\frac{s^{\ell}(x,z,n)}{f_{X,N}(x,n)}\cdot\left[\widehat{f}_{X,N}(x,n)-f_{X,N}(x,n)\right]\\
&\quad+\frac{1}{2}\cdot\widehat{\mu}^{\ell}(x,z,n)'\begin{pmatrix}0 & -\frac{1}{\widetilde{f}_{X,N}^2(x,n)}\\ -\frac{1}{\widetilde{f}_{X,N}^2(x,n)} & \frac{2\widetilde{Q}^{\ell}(x,z,n)}{\widetilde{f}_{X,N}^3(x,n)}\end{pmatrix}\cdot\widehat{\mu}^{\ell}(x,z,n)\\
&=\widehat{\nu}^{\ell}(x,z,n)+\frac{1}{2}\cdot\widehat{\mu}^{\ell}(x,z,n)'\begin{pmatrix}0 & -\frac{1}{\widetilde{f}_{X,N}^2(x,n)}\\ -\frac{1}{\widetilde{f}_{X,N}^2(x,n)} & \frac{2\widetilde{Q}^{\ell}(x,z,n)}{\widetilde{f}_{X,N}^3(x,n)}\end{pmatrix}\cdot\widehat{\mu}^{\ell}(x,z,n)
\end{aligned}$$

[2] If a class is Euclidean it is also necessarily manageable. See page 1033 in Pakes and Pollard (1989).

where $\left(\widetilde{Q}^{\ell}(x,z,n),\ \widetilde{f}_{X,N}(x,n)\right)$ belong in the line segment that connects $\left(\widehat{Q}^{\ell}(x,z,n),\ \widehat{f}_{X,N}(x,n)\right)$ with $\left(Q^{\ell}(x,z,n),\ f_{X,N}(x,n)\right)$. Denote

$$\xi^{\ell}_{L}(x,z,n)=\frac{1}{2}\cdot\widehat{\mu}^{\ell}(x,z,n)'\begin{pmatrix}0 & -\frac{1}{\widetilde{f}^{2}_{X,N}(x,n)}\\ -\frac{1}{\widetilde{f}^{2}_{X,N}(x,n)} & \frac{2\widetilde{Q}^{\ell}(x,z,n)}{\widetilde{f}^{3}_{X,N}(x,n)}\end{pmatrix}\cdot\widehat{\mu}^{\ell}_{L}(x,z,n).$$

Then we can express

$$\widehat{s}^{\ell}(x,z,n)-s^{\ell}(x,z,n)=\widehat{\nu}^{\ell}(x,z,n)+\xi^{\ell}_{L}(x,z,n). \tag{S.3}$$

By Assumption 3.1 there exists a constant $\overline{Q}<\infty$ such that

$$\underset{(x,z,n)\in\mathcal{W}}{sup}\left|Q^{\ell}(x,z,n)\right|\leq\overline{Q}.$$

Define

$$J=\left\|\begin{matrix}0 & -\frac{1}{\left(\underline{f}/2\right)^{2}}\\ -\frac{1}{\left(\underline{f}/2\right)^{2}} & \frac{3\overline{Q}}{\left(\underline{f}/2\right)^{3}}\end{matrix}\right\|. \tag{S.4}$$

et J be as described in (S.4). Combining (S.2)-($S.2'$), for any $\delta>0$ we have

$$\begin{aligned}&Pr\left(\underset{(x,z,n)\in\mathcal{W}}{sup}\left|\xi^{\ell}_{L}(x,z,n)\right|\geq\delta\right)\leq Pr\left(\underset{(x,z,n)\in\mathcal{W}}{sup}\left|\widehat{Q}^{\ell}(x,z,n)-Q^{\ell}(x,z,n)\right|\geq\overline{Q}\right)\\&+Pr\left(\underset{(x,n)\in\mathcal{W}}{sup}\left|\widehat{f}_{X,N}(x,n)-f_{X,N}(x,n)\right|\geq\underline{f}/2\right)+Pr\left(\underset{(x,n)\in\mathcal{W}}{sup}\left|\widehat{\mu}^{\ell}(x,z,n)\right|\geq\sqrt{\frac{2\delta}{J}}\right)\\&\leq 4\cdot A_{1}\cdot\exp\left\{-\sqrt{L}\cdot h^{r}_{L}\left(A_{2}\cdot\min\left\{\sqrt{\frac{\delta}{J}},\,\overline{Q},\,\underline{f}/2\right\}-A_{3}\cdot h^{M}_{L}\right)\right\}.\end{aligned} \tag{S.5}$$

Combining (S.2), (S.3) and (S.5), for any $\delta>0$ we have

$$\begin{aligned}&Pr\left(\underset{(x,z,n)\in\mathcal{W}}{sup}\left|\widehat{s}^{\ell}(x,z,n)-s^{\ell}(x,z,n)\right|\geq\delta\right)\\&\leq Pr\left(\underset{(x,z,n)\in\mathcal{W}}{sup}\left|\widehat{\nu}^{\ell}(x,z,n)\right|\geq\frac{\delta}{2}\right)+Pr\left(\underset{(x,z,n)\in\mathcal{W}}{sup}\left|\xi^{\ell}(x,z,n)\right|\geq\frac{\delta}{2}\right)\\&\leq 5\cdot A_{1}\cdot\exp\left\{-\sqrt{L}\cdot h^{r}_{L}\left(A_{2}\cdot\min\left\{\frac{\delta}{2},\,\sqrt{\frac{\delta}{J}},\,\overline{Q},\,\underline{f}/2\right\}-A_{3}\cdot h^{M}_{L}\right)\right\}.\end{aligned}$$

Our estimator $\widehat{s}(x,z,n)$ is

$$\widehat{s}(x,z,n)=\left(\widehat{s}^{1}(x,z,n),\ldots,\widehat{s}^{d_s}(x,z,n)\right)'.$$

Let b_L be the vanishing sequence used in our construction. For reasons that will become clear below, we are interested in a bound for $Pr\left(\underset{(x,z,n)\in\mathcal{W}}{sup}\|\widehat{s}(x,z,n)-s(x,z,n)\|\geq b_L\right)$. A Bonferroni inequality implies

$$Pr\left(\underset{(x,z,n)\in\mathcal{W}}{sup}\|\widehat{s}(x,z,n)-s(x,z,n)\|\geq b_L\right)\leq\sum_{\ell=1}^{d_s}Pr\left(\underset{(x,z,n)\in\mathcal{W}}{sup}\left|\widehat{s}^{\ell}(x,z,n)-s^{\ell}(x,z,n)\right|\geq\frac{b_L}{\sqrt{d_s}}\right).$$

For large enough L we will have $\min\left\{\frac{b_L}{2},\, \sqrt{\frac{b_L}{J}},\, \overline{Q},\, \underline{f}/2\right\} = \frac{b_L}{2}$ and consequently

$$Pr\left(\sup_{(x,z,n)\in\mathcal{W}} \|\widehat{s}(x,z,n) - s(x,z,n)\| \geq b_L\right) \leq B_1 \cdot \exp\left\{-\sqrt{L}\cdot h_L^r \left(B_2 \cdot b_L - A_3 \cdot h_L^M\right)\right\}, \quad \text{(S.6)}$$

where $B_1 \equiv 5 \cdot d_s \cdot A_1$ and $B_2 \equiv \frac{A_2}{2\sqrt{d_s}}$.

We move on to studying the properties of $\widehat{R}^q(x,z;n,n') - R^q(x,z;n,n')$. We begin with the following condition. For a given (x,z,n) group

$$\widetilde{v}_i(x,z,n;h) = \left(\widetilde{v}_i^\ell(x,z,n;h)\right)_{\ell=1}^{d_s} \quad \text{and} \quad \xi_L(x,z,n) = \left(\xi_L^\ell(x,z,n)\right)_{\ell=1}^{d_s}$$

and for a pair n, n' let

$$v_i(x,z,n,n';h) = \left(\widetilde{v}_i(x,z,n;h)',\, \widetilde{v}_i(x,z,n';h)'\right)' \quad \text{and} \quad \xi_L(x,z,n,n') = \left(\xi_L(x,z,n)',\, \xi_L(x,z,n')'\right)'.$$

Fix (x,z,n,n') and let $\nabla_s m^q(x,z;n,n')$ be as defined in (3.1). Denote

$$v_i^q(x,z,n,n';h) = \nabla_s m^q(x,z;n,n')'\, v_i(x,z,n,n';h),$$
$$\widehat{\nu}^q(x,z,n,n') = \frac{1}{L\cdot h_L^r}\sum_{i=1}^{L} v_i^q(x,z,n,n';h_L).$$

Using an M^{th} order approximation, the smoothness conditions in Assumption 3.1 imply the existence of a finite constant $\overline{M}'$ such that,

$$\sup_{(x,z,n,n')\in\mathcal{W}} \left|E\left[\widehat{\nu}^q(x,z,n,n')\right]\right| \leq \overline{M}' \cdot h_L^M. \quad (S.1')$$

Denote

$$\widehat{\mu}(x,z,n,n') = \left(\left(\widehat{s}(x,z,n) - s(x,z,n)\right)' \quad \left(\widehat{s}(x,z,n') - s(x,z,n')\right)'\right)'.$$

Let $\nabla_s m^q(x,z;n,n')$ and $\frac{\partial^2 m^q(s_1,s_2;n,n')}{\partial s \partial s'}$ be as described in Assumption 3.1. By the smoothness conditions in Assumption 3.1, for any $(x,z,n,n') \in \mathcal{W}$, a second order approximation yields

$$\widehat{R}^q(x,z;n,n') - R^q(x,z;n,n') = \widehat{\nu}^q(x,z,n,n') + \xi_L^{q,1}(x,z,n,n') + \xi_L^{q,2}(x,z,n,n'), \quad \text{(S.7)}$$

where

$$\xi_L^{q,1}(x,z,n,n') = \nabla_s m^q(x,z;n,n')'\, \xi_L(x,z,n,n'),$$
$$\xi_L^{q,2}(x,z,n,n') = \widehat{\mu}(x,z,n,n')'\widetilde{\nabla}_{ss'} m^q(x,z;n,n')\, \widehat{\mu}(x,z,n,n'),$$

with

$$\widetilde{\nabla}_{ss'} m^q(x,z;n,n') = \frac{\partial^2 m^q(\widetilde{s}(x,z,n), \widetilde{s}(x,z,n'); n, n')}{\partial s \partial s'}.$$

and $(\widetilde{s}(x,z,n), \widetilde{s}(x,z,n'))$ lie in the line segment connecting $(\widehat{s}(x,z,n), \widehat{s}(x,z,n'))$ and $(s(x,z,n), s(x,z,n'))$. Let D be as described in Assumption 3.1. The smoothness conditions described there along with the Euclidean properties in Assumption 3.4 imply that the process

$$\{v_i^q(x,z,n,n';h)\colon\ (x,z,n,n') \in \mathcal{W},\ h > 0,\ 1 \leq i \leq L\}$$

is manageable with respect to the envelope $\left(D/\underline{f}\right)\cdot\left(\overline{S}(\cdot)+\max\{|\underline{s}|,|\overline{s}|\}\right)$, which has a moment generating function by Assumption 3.4. As before, this allows us to use the maximal inequality results in Chapter 7 of Pollard (1990) which, combined with the bias conditions in S.1', imply that there exist positive constants B_1', B_2' and B_3' such that, for any $\delta>0$,

$$Pr\left(\underset{(x,z,n,n')\in\mathcal{W}}{sup}|\widehat{\nu}^q(x,z,n,n')|\geq\delta\right)\leq B_1'\cdot\exp\left\{-\sqrt{L}\cdot h_L^r\left(B_2'\cdot\delta-B_3'\cdot h_L^M\right)\right\}. \tag{S.8}$$

Our ultimate goal here is to bound $Pr\left(\underset{(x,z,n,n')\in\mathcal{W}}{sup}\left|\widehat{R}^q(x,z;n,n')-R^q(x,z;n,n')\right|\geq b_L\right)$. By (S.7),

$$\begin{aligned}&Pr\left(\underset{(x,z,n,n')\in\mathcal{W}}{sup}\left|\widehat{R}^q(x,z;n,n')-R^q(x,z;n,n')\right|\geq b_L\right)\leq Pr\left(\underset{(x,z,n,n')\in\mathcal{W}}{sup}|\widehat{\nu}^q(x,z,n,n')|\geq\frac{b_L}{3}\right)\\&+Pr\left(\underset{(x,z,n,n')\in\mathcal{W}}{sup}\left|\xi_L^{q,1}(x,z,n,n')\right|\geq\frac{b_L}{3}\right)+Pr\left(\underset{(x,z,n,n')\in\mathcal{W}}{sup}\left|\xi_L^{q,2}(x,z,n,n')\right|\geq\frac{b_L}{3}\right)\end{aligned} \tag{S.9}$$

Let D and η be as described in Assumption 3.1. Then

$$\begin{aligned}&Pr\left(\underset{(x,z,n,n')\in\mathcal{W}}{sup}\left|\xi_L^{q,1}(x,z,n,n')\right|\geq\frac{b_L}{3}\right)\\&\leq Pr\left(\underset{(x,z,n)\in\mathcal{W}}{sup}\|\widehat{s}(x,z,n)-s(x,z,n)\|\geq\eta\right)+Pr\left(\underset{(x,z,n,n')\in\mathcal{W}}{sup}\|\xi_L(x,z,n,n')\|\geq\frac{b_L}{3\cdot D}\right)\\&\leq Pr\left(\underset{(x,z,n)\in\mathcal{W}}{sup}\|\widehat{s}(x,z,n)-s(x,z,n)\|\geq\eta\right)+2\cdot Pr\left(\underset{(x,z,n)\in\mathcal{W}}{sup}\|\xi_L(x,z,n)\|\geq\frac{b_L}{\sqrt{18\cdot D^2}}\right)\end{aligned}$$

where the last line follows from a Bonferroni inequality and the definition of $\xi_L(x,z,n,n')$. We also have

$$\begin{aligned}&Pr\left(\underset{(x,z,n,n')\in\mathcal{W}}{sup}\left|\xi_L^{q,2}(x,z,n,n')\right|\geq\frac{b_L}{3}\right)\\&\leq Pr\left(\underset{(x,z,n)\in\mathcal{W}}{sup}\|\widehat{s}(x,z,n)-s(x,z,n)\|\geq\eta\right)+Pr\left(\underset{(x,z,n,n')\in\mathcal{W}}{sup}\|\widehat{\mu}(x,z,n,n')\|\geq\sqrt{\frac{b_L}{3\cdot D}}\right)\\&\leq Pr\left(\underset{(x,z,n)\in\mathcal{W}}{sup}\|\widehat{s}(x,z,n)-s(x,z,n)\|\geq\eta\right)+2\cdot Pr\left(\underset{(x,z,n)\in\mathcal{W}}{sup}\|\widehat{s}(x,z,n)-s(x,z,n)\|\geq\sqrt{\frac{b_L}{6\cdot D}}\right)\end{aligned}$$

Combining these we have

$$\begin{aligned}&Pr\left(\underset{(x,z,n,n')\in\mathcal{W}}{sup}\left|\xi_L^{q,1}(x,z,n,n')\right|\geq\frac{b_L}{3}\right)+Pr\left(\underset{(x,z,n,n')\in\mathcal{W}}{sup}\left|\xi_L^{q,2}(x,z,n,n')\right|\geq\frac{b_L}{3}\right)\\&\leq 2\cdot Pr\left(\underset{(x,z,n)\in\mathcal{W}}{sup}\|\widehat{s}(x,z,n)-s(x,z,n)\|\geq\min\left\{\eta,\sqrt{\frac{b_L}{6\cdot D}}\right\}\right)\\&+2\cdot Pr\left(\underset{(x,z,n)\in\mathcal{W}}{sup}\|\xi_L(x,z,n)\|\geq\frac{b_L}{\sqrt{18\cdot D^2}}\right)\end{aligned}$$

Let J be as described in (S.4). Using (S.5) and a Bonferroni inequality,

$$Pr\left(\sup_{(x,z,n)\in\mathcal{W}} \|\xi_L(x,z,n)\| \geq \frac{b_L}{\sqrt{18\cdot D^2}}\right) \leq \sum_{\ell=1}^{d_s} Pr\left(\sup_{(x,z,n)\in\mathcal{W}} \left|\xi_L^\ell(x,z,n)\right| \geq \frac{b_L}{\sqrt{18\cdot D^2\cdot d_s}}\right)$$
$$\leq 4\cdot A_1\cdot d_s\cdot \exp\left\{-\sqrt{L}\cdot h_L^r\left(A_2\cdot \min\left\{\frac{b_L^{1/2}}{(J^2\cdot 18\cdot D^2\cdot d_s)^{1/4}},\, \overline{Q},\, \underline{f}/2\right\} - A_3\cdot h_L^M\right)\right\}. \tag{S.10}$$

For large enough L we will have $\left(\frac{B_2}{\sqrt{6\cdot D}} + \frac{A_2}{(J^2\cdot 18\cdot D^2\cdot d_s)^{1/4}}\right)\times b_L^{1/2} \leq \min\{\overline{Q},\, \underline{f}/2\}$. Using (S.6) and (S.10) we obtain

$$Pr\left(\sup_{(x,z,n,n')\in\mathcal{W}} \left|\xi_L^{q,1}(x,z,n,n')\right| \geq \frac{b_L}{3}\right) + Pr\left(\sup_{(x,z,n,n')\in\mathcal{W}} \left|\xi_L^{q,2}(x,z,n,n')\right| \geq \frac{b_L}{3}\right)$$
$$\leq B_1''\cdot \exp\left\{-\sqrt{L}\cdot h_L^r\left(B_2''\cdot b_L^{1/2} - A_3\cdot h_L^M\right)\right\}, \tag{S.11}$$

where $B_1'' \equiv 4\cdot A_1\cdot d_s + B_1$ and $B_2'' \equiv \min\left\{\frac{B_2}{\sqrt{6\cdot D}},\, \frac{A_2}{(J^2\cdot 18\cdot D^2\cdot d_s)^{1/4}}\right\}$. Combined with (S.7) and (S.9), we obtain

$$Pr\left(\sup_{(x,z,n,n')\in\mathcal{W}} \left|\widehat{R}^q(x,z;n,n') - R^q(x,z;n,n')\right| \geq b_L\right)$$
$$\leq B_1'\cdot \exp\left\{-\sqrt{L}\cdot h_L^r\left(B_2'\cdot \frac{b_L}{3} - B_3'\cdot h_L^M\right)\right\} + B_1''\cdot \exp\left\{-\sqrt{L}\cdot h_L^r\left(B_2''\cdot b_L^{1/2} - A_3\cdot h_L^M\right)\right\}$$

For L large enough we have $\left(\frac{B_2'}{3}\right)\cdot b_L \leq B_2''\cdot b_L^{1/2}$ and therefore the above bound becomes

$$Pr\left(\sup_{(x,z,n,n')\in\mathcal{W}} \left|\widehat{R}^q(x,z;n,n') - R^q(x,z;n,n')\right| \geq b_L\right) \quad \leq \overline{K}_1\cdot \exp\left\{-\sqrt{L}\cdot h_L^r\left(\overline{K}_2\cdot b_L - \overline{K}_3\cdot h_L^M\right)\right\}, \tag{S.12}$$

where $\overline{K}_1 \equiv \max\{B_1',\, B_1''\}$, $\overline{K}_2 \equiv \min\left\{\frac{B_2'}{3},\, B_2''\right\}$ and $\overline{K}_3 \equiv \max\{A_3,\, B_3'\}$. Going back to (S.7), our results also imply

$$\widehat{R}^q(x,z;n,n') - R^q(x,z;n,n') = \frac{1}{L\cdot h_L^r}\sum_{i=1}^{L} v_i^q(x,z,n,n';h_L) + \xi_L^q(x,z,n,n'),$$
$$\text{where} \quad \sup_{(x,z,n,n')\in\mathcal{W}} \left\|\xi_L^q(x,z,n,n')\right\| = O_p\left(\frac{\log(L)^2}{L\cdot h_L^r}\right). \tag{S.13}$$

S-A.1.2 Proving Theorem 1

Using the main results from our previous section (equations (S.12) and (S.13)), we show that under our assumptions there is a linear representation for $\widehat{\mathcal{T}}^q_{n,n'}$. First, fix $z\in\mathcal{Z}$ and recall from (4) that

$$\widehat{\mathcal{T}}^q_{n,n'}(z) = \frac{1}{L}\sum_{i=1}^{L} \widehat{R}^q(X_i,z;n,n')\cdot \mathbb{1}\left\{\widehat{R}^q(X_i,z;n,n') \geq -b_L\right\}\cdot \mathbb{I}_{\mathcal{X}}(X_i).$$

We have

$$\widehat{\mathcal{T}}^q_{n,n'}(z) = \frac{1}{L}\sum_{i=1}^{L} \widehat{R}^q(X_i, z; n, n') \cdot \mathbb{1}\left\{R^q(X_i, z; n, n') \geq 0\right\} \cdot \mathbb{I}_{\mathcal{X}}(X_i) + \zeta^q_L(z, n, n'),$$

where

$$|\zeta^q_L(z,n,n')| \leq \underbrace{\frac{1}{L}\sum_{i=1}^{L}\left|\widehat{R}^q(X_i,z;n,n')\right| \cdot \mathbb{1}\left\{-2\cdot b_L \leq R^q(X_i,z;n,n') < 0\right\} \cdot \mathbb{I}_{\mathcal{X}}(X_i)}_{=\left|\zeta^{q,1}_L(z,n,n')\right|}$$

$$+ \underbrace{\frac{2}{L}\sum_{i=1}^{L}\left|\widehat{R}^q(X_i,z;n,n')\right| \cdot \mathbb{1}\left\{\left|\widehat{R}^q(X_i,z;n,n') - R^q(X_i,z;n,n')\right| \geq b_L\right\} \cdot \mathbb{I}_{\mathcal{X}}(X_i)}_{=\left|\zeta^{q,2}_L(z,n,n')\right|}.$$

We have

$$\begin{aligned}
&\left|\zeta^{q,1}_L(z,n,n')\right| \\
&\leq \frac{1}{L}\sum_{i=1}^{L}\left|R^q(X_i,z;n,n')\right| \cdot \mathbb{1}\left\{-2\cdot b_L \leq R^q(X_i,z;n,n') < 0\right\} \cdot \mathbb{I}_{\mathcal{X}}(X_i) \\
&+ \frac{1}{L}\sum_{i=1}^{L}\left|\widehat{R}^q(X_i,z;n,n') - R^q(X_i,z;n,n')\right| \cdot \mathbb{1}\left\{-2\cdot b_L \leq R^q(X_i,z;n,n') < 0\right\} \cdot \mathbb{I}_{\mathcal{X}}(X_i) \\
&\leq 2\cdot b_L \times \frac{1}{L}\sum_{i=1}^{L}\mathbb{1}\left\{-2\cdot b_L \leq R^q(X_i,z;n,n') < 0\right\} \cdot \mathbb{I}_{\mathcal{X}}(X_i) \\
&+ \underset{(x,z,n,n')\in\mathcal{W}}{sup}\left|\widehat{R}^q(x,z;n,n') - R^q(x,z;n,n')\right| \times \frac{1}{L}\sum_{i=1}^{L}\mathbb{1}\left\{-2\cdot b_L \leq R^q(X_i,z;n,n') < 0\right\} \cdot \mathbb{I}_{\mathcal{X}}(X_i) \\
&= \left(2\cdot b_L + O_p\left(\frac{\log(L)}{\sqrt{L\cdot h^r_L}}\right)\right) \times \frac{1}{L}\sum_{i=1}^{L}\mathbb{1}\left\{-2\cdot b_L \leq R^q(X_i,z;n,n') < 0\right\} \cdot \mathbb{I}_{\mathcal{X}}(X_i),
\end{aligned} \tag{S.14}$$

where the term $O_p\left(\frac{\log(L)}{\sqrt{L\cdot h^r_L}}\right)$ in the last line follows from (S.13). For $b > 0$ define

$$g^{q,1}_i(z,b;n,n') = \mathbb{1}\left\{-2\cdot b \leq R^q(X_i,z;n,n') < 0\right\} \cdot \mathbb{I}_{\mathcal{X}}(X_i).$$

And let

$$\widetilde{g}^{q,1}_i(z,b;n,n') = g^{q,1}_i(z,b;n,n') - E\left[g^{q,1}_i(z,b;n,n')\right], \quad \widehat{\nu}^{q,1}_L(z,n,n') = \frac{1}{L}\sum_{i=1}^{L}\widetilde{g}^{q,1}_i\left(z,b_L;n,n'\right).$$

Lemmas 2.4 and 2.14 in Pakes and Pollard (1989) imply that the process

$$\left\{\widetilde{g}^{q,1}_i(z,b;n,n'):\ b > 0,\ (z,n,n') \in \mathcal{W},\ 1 \leq i \leq L\right\}$$

is manageable with respect to the envelope 1. Let $\bar{b}$ and $\overline{A}$ be as described in Assumption 3.2. For large enough L we have $2 \cdot b_L \leq \bar{b}$, and therefore the regularity condition described in Assumption 3.2 and the aforementioned manageability property yield

$$\underset{(z,n,n')\in\mathcal{W}}{sup} \left|\widehat{\nu}_L^{q,1}(z,n,n')\right| = O_p\left(\sqrt{\frac{b_L}{L}}\right).$$

When L is large enough that $2 \cdot b_L \leq \bar{b}$, the regularity condition in Assumption 3.2 implies

$$\frac{1}{L}\sum_{i=1}^{L}\mathbb{1}\left\{-2\cdot b_L \leq R^q(X_i,z;n,n') < 0\right\}\cdot \mathbb{I}_{\mathcal{X}}(X_i) = \widehat{\nu}_L^{q,1}(z,n,n') + \varsigma_L^{q,1}(z,n,n'),$$
$$\text{where} \quad \underset{(z,n,n')\in\mathcal{W}}{sup}\left|\varsigma_L^{q,1}(z,n,n')\right| \leq 2\cdot\overline{A}\cdot b_L.$$

Therefore,

$$\begin{aligned}\underset{(z,n,n')\in\mathcal{W}}{sup}\left|\frac{1}{L}\sum_{i=1}^{L}\mathbb{1}\left\{-2\cdot b_L \leq R^q(X_i,z;n,n') < 0\right\}\cdot \mathbb{I}_{\mathcal{X}}(X_i)\right| &\leq O_p\left(\sqrt{\frac{b_L}{L}}\right) + 2\cdot\overline{A}\cdot b_L \\ &= O_p\left(b_L\right),\end{aligned}$$

where the last equality follows from the bandwidth convergence conditions in Assumption 3.3. Going back to (S.14), this yields

$$\underset{(z,n,n')\in\mathcal{W}}{sup}\left|\zeta_L^{q,1}(z,n,n')\right| \leq O_p\left(b_L^2\right) + O_p\left(\frac{\log(L)\cdot b_L}{\sqrt{L\cdot h_L^r}}\right) = O_p\left(L^{-1/2-\epsilon}\right) \quad \text{for some } \epsilon > 0,$$

where the last equality follows from the bandwidth convergence properties in Assumption 3.3. Using (S.12) and (S.13),

$$\begin{aligned}\underset{(z,n,n')\in\mathcal{W}}{sup}\left|\zeta_L^{q,2}(z,n,n')\right| &\leq \underset{(x,z,n,n')\in\mathcal{W}}{sup}\left|\widehat{R}^q(x,z;n,n')\right| \times \mathbb{1}\left\{\underset{(x,z,n,n')\in\mathcal{W}}{sup}\left|\widehat{R}^q(x,z;n,n') - R^q(x,z;n,n')\right| \geq b_L\right\} \\ &= O_p(1)\times O_p\left(\overline{K}_1^{1/2}\cdot\exp\left\{-\frac{1}{2}\sqrt{L}\cdot h_L^r\left(\overline{K}_2\cdot b_L - \overline{K}_3\cdot h_L^M\right)\right\}\right) \\ &= O_p\left(L^{-1/2-\epsilon}\right) \quad \text{for some } \epsilon > 0.\end{aligned}$$

From here we conclude that

$$\begin{aligned}\widehat{\mathcal{T}}_{n,n'}^q(z) = \frac{1}{L}\sum_{i=1}^{L}\widehat{R}^q(X_i,z;n,n')\cdot\mathbb{1}\left\{R^q(X_i,z;n,n')\geq 0\right\}\cdot\mathbb{I}_{\mathcal{X}}(X_i) + \zeta_L^q(z,n,n'), \\ \text{where} \quad \underset{(z,n,n')\in\mathcal{W}}{sup}\left|\zeta_L^q(z,n,n')\right| = O_p\left(L^{-1/2-\epsilon}\right) \quad \text{for some } \epsilon > 0.\end{aligned} \tag{S.15}$$

This can be re-expressed as

$$\begin{aligned}\widehat{\mathcal{T}}_{n,n'}^q(z) = &\frac{1}{L}\sum_{i=1}^{L}\max\left\{R^q(X_i,z;n,n'),\,0\right\}\cdot\mathbb{I}_{\mathcal{X}}(X_i) \\ &+\frac{1}{L}\sum_{i=1}^{L}\left(\widehat{R}^q(X_i,z;n,n') - R^q(X_i,z;n,n')\right)\cdot\mathbb{1}\left\{R^q(X_i,z;n,n')\geq 0\right\}\cdot\mathbb{I}_{\mathcal{X}}(X_i) + \zeta_L^q(z,n,n'),\end{aligned} \tag{S.16}$$

We will begin by studying the second term. Denote $\varepsilon(y,x,z,n) = S(y,x,z,n) - s(x,z,n)$. Fix $z \in \mathcal{Z}$, $(n,n') \in \mathcal{N}^2$, $x_1 \in \mathcal{X}$ and $u_2 \equiv (y_2, x_2, n_2) \in \text{Supp}(Y) \times \mathcal{X} \times \mathcal{N}$ and define

$$\widetilde{\varepsilon}(u_2, x_1, z; n, n') = \left(\frac{\varepsilon(y_2, x_1, z, n)' \cdot \mathbb{1}\{n_2 = n\}}{f_{X,N}(x_1, n)}, \frac{\varepsilon(y_2, x_1, z, n')' \cdot \mathbb{1}\{n_2 = n'\}}{f_{X,N}(x_1, n')} \right)'.$$

Denote

$$\begin{aligned} \phi^q(u_2, x_1, z; n, n') &= \left[\nabla_s m^q(x_1, z; n, n')' \widetilde{\varepsilon}(u_2, x_1, z; n, n') \right] \cdot \mathbb{1}\{R^q(x_1, z; n, n') \geq 0\}, \\ f^q(x_1, u_2, z, n, n'; h) &= \phi^q(u_2, x_1, z; n, n') \cdot \mathbb{I}_{\mathcal{X}}(x_1) \cdot \frac{1}{h^r} \mathcal{H}(x_2 - x_1; h). \end{aligned}$$

Using (S.13), we have

$$\begin{aligned} &\frac{1}{L} \sum_{i=1}^{L} \left(\widehat{R}^q(X_i, z; n, n') - R^q(X_i, z; n, n') \right) \cdot \mathbb{1}\{R^q(X_i, z; n, n') \geq 0\} \cdot \mathbb{I}_{\mathcal{X}}(X_i) \\ =& \frac{1}{L^2} \sum_{i=1}^{L} \sum_{j=1}^{L} f^q(X_i, U_j, z, n, n'; h_L) + \varrho_L^q(z, n, n'), \end{aligned} \tag{S.17}$$

where

$$\sup_{(z,n,n') \in \mathcal{W}} |\varrho_L^q(z,n,n')| = O_p\left(\frac{\log(L)^2}{L \cdot h_L^r} \right) = O_p\left(L^{-1/2-\epsilon} \right) \quad \text{for some } \epsilon > 0.$$

Let us analyze the properties of the first term in the right hand side of (S.17) which is a second-order U process (Serfling (1980), Sherman (1994)). Denote

$$\mu_L^q(z,n,n') = E\left[f^q(X_i, U_j, z, n, n'; h_L) \right].$$

Given the smoothness conditions in Assumption 3.1 there exists a constant $\overline{D}$ such that for any $i \neq j$,

$$\sup_{(z,n,n') \in \mathcal{W}} \left| E\left[f^q(X_i, U_j, z, n, n'; h_L) \,\middle|\, X_i \right] \right| \leq \overline{D} \cdot h_L^M. \tag{S.18}$$

And a dominated convergence argument and iterated expectations imply

$$\sup_{(z,n,n') \in \mathcal{W}} \left| \mu_L^q(z,n,n') \right| \leq \overline{D} \cdot h_L^M.$$

Let

$$\begin{aligned} \widetilde{f}^q(X_i, U_j, z, n, n'; h_L) &= f^q(X_i, U_j, z, n, n'; h_L) - \mu_L^q(z,n,n'), \\ \widetilde{g}^q(U_i, U_j, z, n, n'; h_L) &= \frac{\widetilde{f}^q(X_i, U_j, z, n, n'; h_L) + \widetilde{f}^q(X_j, U_i, z, n, n'; h_L)}{2}, \\ V_L^q(z,n,n') &= \binom{L}{2}^{-1} \sum_{i<j} \widetilde{g}^q(U_i, U_j, z, n, n'; h_L). \end{aligned}$$

Note first that under our previous assumptions,

$$\sup_{(z,n,n') \in \mathcal{W}} \left| \frac{1}{L^2} \sum_{i=1}^{L} f^q(X_i, U_i, z, n, n'; h_L) \right| = O_p\left(\frac{1}{L \cdot h_L^r} \right) = o_p\left(L^{-1/2-\epsilon} \right) \quad \text{for some } \epsilon > 0.$$

Combined with the vanishing properties of $\mu_L^q(z,n,n')$, this means that we can express

$$\frac{1}{L^2}\sum_{i=1}^{L}\sum_{j=1}^{L} f^q\left(X_i,U_j,z,n,n';h_L\right)=\left(\frac{L-1}{L}\right)\cdot V_L^q(z,n,n')+\vartheta_L(z,n,n'),$$
$$\text{where}\quad \sup_{(z,n,n')\in\mathcal{W}}|\vartheta_L(z,n,n')|=O\left(h_L^M\right)+O_p\left(\frac{1}{L\cdot h_L^r}\right)=o_p\left(L^{-1/2-\epsilon}\right)\quad\text{for some }\epsilon>0.$$

Fix u. Symmetry of $\widetilde{g}^q$ implies $E_U\left[\widetilde{g}^q\left(u,U,z,n,n';h_L\right)\right]=E_U\left[\widetilde{g}^q\left(U,u,z,n,n';h_L\right)\right]$. We will denote

$$\theta_L^q\left(u,z,n,n'\right)=E_U\left[\widetilde{g}^q\left(u,U,z,n,n';h_L\right)\right].$$

Note that $E\left[\theta_L^q\left(U,z,n,n'\right)\right]=0$. Let

$$t^q\left(U_i,U_j,z,n,n';h_L\right)=\widetilde{g}^q\left(U_i,U_j,z,n,n';h_L\right)-\theta_L^q\left(U_i,z,n,n'\right)-\theta_L^q\left(U_j,z,n,n'\right),$$
$$V_L^{q,2}(z,n,n')=\binom{L}{2}^{-1}\sum_{i<j}t^q\left(U_i,U_j,z,n,n';h_L\right).$$

The properties of $\mu_L^q(z,n,n')$ and the Hoeffding decomposition of V_L^q ((Serfling (1980)) imply that

$$V_L^q(z,n,n')=\frac{2}{L}\sum_{i=1}^{L}\theta_L^q(U_i,z,n,n')+V_L^{q,2}(z,n,n')+\widetilde{\tau}_L^q(z,n,n'),\quad\text{where}\quad\sup_{(z,n,n')\in\mathcal{W}}|\widetilde{\tau}_L^q(z,n,n')|=O\left(h_L^M\right).$$

$V_L^{q,2}(z,n,n')$ is a *degenerate* U-statistic of order 2. Given our assumptions and previous results, it satisfies $\sup_{(z,n,n')\in\mathcal{W}} V_L^{q,2}(z,n,n')=O_p\left(\frac{1}{L\cdot h_L^r}\right)$ (see Serfling (1980), Sherman (1994)). Let

$$\Delta_L^q\left(u,z,n,n'\right)=E_X\left[f^q\left(X,u,z,n,n';h_L\right)\right]. \tag{S.19}$$

Using (S.18), our smoothness conditions imply that the last result can be re-expressed as

$$V_L^q(z,n,n')=\frac{1}{L}\sum_{i=1}^{L}\left(\Delta_L^q\left(U_i,z,n,n'\right)-E\left[\Delta_L^q\left(U_i,z,n,n'\right)\right]\right)+\tau_L^q(z,n,n'),$$
$$\text{where}\quad\sup_{(z,n,n')\in\mathcal{W}}\left|\tau_L^q(z,n,n')\right|=O_p\left(\frac{1}{L\cdot h_L^r}\right)+O\left(h_L^M\right)=O_p\left(L^{-1/2-\epsilon}\right)\quad\text{for some }\epsilon>0. \tag{S.20}$$

Recall from (3) that we defined $\mathcal{T}_{n,n'}^q(z)=E_X\left[\max\left\{R^q(X,z;n,n'),0\right\}\cdot\cdot\mathbb{I}_{\mathcal{X}}(X)\right]$. Let

$$\lambda_L^q\left(U_i,z;n,n'\right)=\Big(\max\left\{R^q(X_i,z;n,n'),0\right\}\cdot\mathbb{I}_{\mathcal{X}}(X_i)-\mathcal{T}_{n,n'}^q(z)\Big)$$
$$+\Big(\Delta_L^q\left(U_i,z,n,n'\right)-E_U\left[\Delta_L^q\left(U,z,n,n'\right)\right]\Big). \tag{S.21}$$

Combining (S.16) and (S.20) we obtain

$$\widehat{\mathcal{T}}_{n,n'}^q(z)=\mathcal{T}_{n,n'}^q(z)+\frac{1}{L}\sum_{i=1}^{L}\lambda_L^q\left(U_i,z;n,n'\right)+\xi_L^q(z,n,n'),$$
$$\text{where}\quad|\xi_L^q(n,n')|=O_p\left(L^{-1/2-\epsilon}\right)\text{ for some }\epsilon>0. \tag{S.22}$$

Examining the structure of $\lambda_L^q\left(U_i,z;n,n'\right)$ it is easy to see that for each $z\in\mathcal{Z}$,

(i) $E\left[\lambda_L^q\left(U_i, z; n, n'\right)\right] = 0$.

(ii) If $P_X\left(R^q(X, z; n, n') < 0 \middle| X \in \mathcal{X}\right) = 1$, then $\lambda_L^q\left(U_i, z; n, n'\right) = 0$ w.p.1. That is, if the contact set for z has measure zero then $\lambda_L^q\left(U_i, z; n, n'\right) = 0$ almost surely.

Using the previous results we obtain the asymptotic properties of $\widehat{\mathcal{T}}^q_{n,n'}$ described in Theorem 1. Recall that $\widehat{\mathcal{T}}^q_{n,n'} = \int \widehat{\mathcal{T}}^q_{n,n'}(z) d\mathcal{P}(z)$ and $\mathcal{T}^q_{n,n'} = \int \mathcal{T}^q_{n,n'}(z) d\mathcal{P}(z)$. Let

$$
\begin{aligned}
&\varphi^q\left(u_2, x_1; n, n'\right) = \int\limits_{z \in \mathcal{Z}} \left[\nabla_s m^q\big(x_1, z; n, n'\big)' \widetilde{\varepsilon}(u_2, x_1, z; n, n')\right] \cdot \mathbb{1}\left\{R^q(x_1, z; n, n') \geq 0\right\} d\mathcal{P}(z), \\
&f^q(x_1, u_2, n, n'; h) = \varphi^q\left(u_2, x_1; n, n'\right) \cdot \mathbb{I}_{\mathcal{X}}(x_1) \cdot \frac{1}{h^r} \mathcal{H}(x_2 - x_1; h), \\
&\Delta^q(u_2, n, n'; h) = E_X\left[f^q(X, u_2, n, n'; h)\right], \\
&\Delta^q(u_2, n, n'; h_L) \equiv \Delta^q_L(u_2, n, n'),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{D}^q_{n,n'}(X) &= \int_{z \in \mathcal{Z}} \max\left\{R^q(X, z; n, n'),\, 0\right\} d\mathcal{P}(z), \\
\lambda^q_L(U_i; n, n') &= \Big(\mathcal{D}^q_{n,n'}(X_i) \cdot \mathbb{I}_{\mathcal{X}}(X_i) - \underbrace{E_X\left[\mathcal{D}^q_{n,n'}(X) \cdot \mathbb{I}_{\mathcal{X}}(X)\right]}_{=\mathcal{T}^q_{n,n'}}\Big) + \Big(\Delta^q_L(U_i, n, n') - E_U\left[\Delta^q_L(U, n, n')\right]\Big).
\end{aligned}
\tag{S.23}
$$

From (S.22), we have

$$
\widehat{\mathcal{T}}^q_{n,n'} = \mathcal{T}^q_{n,n'} + \frac{1}{L}\sum_{i=1}^{L} \lambda^q_L\left(U_i; n, n'\right) + \xi^q_L(n, n'),
$$
$$
\text{where } |\xi^q_L(n, n')| = O_p\left(L^{-1/2-\epsilon}\right) \text{ for some } \epsilon > 0.
$$

Note that

(i) $E\left[\lambda_L^q\left(U_i; n, n'\right)\right] = 0$.

(ii) If $P_X\left(R^q(X, z; n, n') < 0 \middle| X \in \mathcal{X}\right) = 1$ for a.e $z \in \mathcal{Z}$ (i.e, if the contact set has measure zero), then $\lambda_L^q\left(U_i; n, n'\right) = 0$ w.p.1.

This proves Theorem 1. □

Let

$$
\lambda_L\left(U_i\right) = \sum_{n,n' \in \mathcal{N}} \sum_{q=1}^{Q_{n,n'}} \lambda^q_L\left(U_i; n, n'\right).
$$

By our previous results,

(i) $E\left[\lambda_L\left(U_i\right)\right] = 0$.

(ii) If $P_X\left(R^q(X,z;n,n')<0\big|X\in\mathcal{X}\right) = 1$ for a.e $z\in\mathcal{Z}$ and each (n,n') and q (i.e, if every contact set has measure zero), then $\lambda_L(U_i)=0$ w.p.1.

And

$$\widehat{\mathcal{T}} = \mathcal{T} + \frac{1}{L}\sum_{i=1}^{L}\lambda_L(U_i) + \xi_L, \quad \text{where} \quad \xi_L = O_p\left(L^{-1/2-\epsilon}\right) \text{ for some } \epsilon>0.$$

Note that the asymptotic properties of $\widehat{\mathcal{T}}$ adapt to the contact sets. This is captured by

$$\sigma_L^2 = \text{Var}\left(\lambda_L(U_i)\right).$$

If the inequalities are satisfied but the contact sets have measure zero we will have $\sigma_L^2=0$; otherwise it will be positive. σ_L^2 is the relevant measure for the *slackness* in (1).

S-A.2 An alternative way to construct the test-statistic

There is a way to circumvent the numerical integration (over z) that is involved in the construction of our test-statistic. Recall that, from the onset we have stated that $\mathcal{P}$ can be assumed to be a pre-specified distribution with Lebesgue density and support $\mathcal{Z}$. Suppose that, independent of all other observable covariates, we generate an iid sample[3] $\{Z_i\}_{i=1}^L$ with $Z_i\sim\mathcal{P}$. By iterated expectations,

$$\begin{aligned}
\mathcal{T}^q_{n,n'} &= \int_{z\in\mathcal{Z}} \mathcal{T}^q_{n,n'}(z)d\mathcal{P}(z)\\
&= \int_{z\in\mathcal{Z}} \Big(E_X\left[\max\left\{R^q(X,z;n,n'),\,0\right\}\cdot\mathbb{I}_{\mathcal{X}}(X)\right]\Big)d\mathcal{P}(z)\\
&= E_{X,Z}\big[\max\left\{R^q(X,z;n,n'),\,0\right\}\cdot\mathbb{I}_{\mathcal{X}}(X)\big].
\end{aligned}$$

Given this, instead of estimating $\mathcal{T}^q_{n,n'}$ with $\int_{z\in\mathcal{Z}}\widehat{\mathcal{T}}^q_{n,n'}(z)d\mathcal{P}(z)$ we can use

$$\widetilde{\mathcal{T}}^q_{n,n'} = \frac{1}{L}\sum_{i=1}^{L}\widehat{R}^q(X_i,Z_i;n,n')\cdot\mathbb{1}\left\{\widehat{R}^q(X_i,Z_i;n,n')\geq -b_L\right\}\cdot\mathbb{I}_{\mathcal{X}}(X_i). \tag{S.24}$$

The steps used to prove Theorem 1 also provide the asymptotic properties of $\widetilde{\mathcal{T}}^q_{n,n'}$. Let $\Delta^q_L(u,z,n,n')$ and $\Delta^q_L(u,n,n')$ be as defined in (S.19) and (S.23), respectively. The counterpart to the influence function $\lambda^q_L(U;n,n')$ in (S.23) would now be

$$\begin{aligned}
\lambda^q_L(U,Z;n,n') = &\Big(\max\left\{R^q(X,Z;n,n'),\,0\right\}\cdot\mathbb{I}_{\mathcal{X}}(X) - \underbrace{E_{X,Z}\big[\max\left\{R^q(X,Z;n,n'),\,0\right\}\cdot\mathbb{I}_{\mathcal{X}}(X)\big]}_{=\mathcal{T}^q_{n,n'}}\Big)\\
&+\Big(\Delta^q_L(U,n,n') - E_U\left[\Delta^q_L(U,n,n')\right]\Big).
\end{aligned} \tag{S.25}$$

[3]The sample generated for Z_i does not necessarily have to be of size L, but this is the computationally simplest case.

Note that, under H_0, $\lambda_L^q(U,Z;n,n') = \Delta_L^q(U,n,n') - E_U\left[\Delta_L^q(U,n,n')\right] = \lambda_L^q(U;n,n')$ (the original influence function in (S.23)). Accordingly, let

$$\lambda_L\left(U_i, Z_i\right) = \sum_{n,n'\in\mathcal{N}} \sum_{q=1}^{Q_{n,n'}} \lambda_L^q\left(U_i, Z_i; n, n'\right). \tag{S.26}$$

Let $\widetilde{\mathcal{T}} = \sum_{n,n'\in\mathcal{N}} \sum_{q=1}^{Q_{n,n'}} \widetilde{\mathcal{T}}_{n,n'}^q$. The steps leading to the proof of Theorem 1 now yield the following,

$$\widetilde{\mathcal{T}} = \mathcal{T} + \frac{1}{L}\sum_{i=1}^{L} \lambda_L\left(U_i, Z_i\right) + \vartheta_L, \quad \text{where} \quad \vartheta_L = O_p\left(L^{-1/2-\epsilon}\right) \text{ for some } \epsilon > 0.$$

The resulting test-statistic and the corresponding rejection rule would have the same asymptotic properties as the one described in the paper.

S-A.2.1 Constructing an estimator for $\sigma_L^2 = Var\left(\lambda_L\left(U_i, Z_i\right)\right)$

Perhaps the biggest computational gains from using this alternative construction can be found in the computation of $\widehat{\sigma}_L^2$, the estimator of $Var\left(\lambda_L\left(U_i, Z_i\right)\right)$. We can proceed first by estimating the influence function $\lambda_L\left(U_i, Z_i\right)$. As we defined above, for $i \neq j$ let

$$\widehat{v}^{\ell}\left(U_i, U_j; Z_i, n, h\right) = \left(\frac{S^{\ell}(Y_j, X_i, Z_i, n) - \widehat{s}^{\ell}(X_i, Z_i, n)}{\widehat{f}_{X,N}(X_i, n)}\right) \cdot \mathcal{H}\left(X_j - X_i; h\right) \cdot \mathbb{1}\left\{N_j = n\right\},$$

$$\widehat{v}(U_i, U_j; Z_i, n, h) = \left(\widehat{v}^{1}\left(U_i, U_j; Z_i, n, h\right), \ldots, \widehat{v}^{d_s}\left(U_i, U_j; Z_i, n, h\right)\right)',$$

$$\widehat{v}\left(U_i, U_j, Z_i, n, n'; h\right) = \left(\widehat{v}(U_i, U_j; Z_i, n, h)', \widehat{v}(U_i, U_j; Z_i, n', h)'\right)',$$

and

$$\widehat{f}^q\left(U_i, U_j, Z_i, n, n'; h\right) =$$
$$\frac{1}{h^r} \cdot \nabla_s m^q\left(X_i, Z_i; n, n'\right)' \widehat{v}\left(U_i, U_j, Z_i, n, n'; h\right) \cdot \mathbb{1}\left\{\widehat{R}^q(X_i, Z_i; n, n') \geq -b_L\right\} \cdot \mathbb{I}_{\mathcal{X}}(X_i)$$

We can estimate $\Delta_L^q(U_i, n, n')$ as

$$\widehat{\Delta}_L^q(U_i, n, n') = \frac{1}{L-1}\sum_{j\neq i} \widehat{f}^q\left(U_j, U_i, Z_j, n, n'; h_L\right),$$

(note the order of the subscripts on the right hand side). And from here our estimators for $\lambda_L^q(U_i, Z_i; n, n')$ and $\lambda_L(U_i, Z_i)$ are

$$\widehat{\lambda}_L^q(U_i, Z_i; n, n') = \left(\widehat{R}^q(X_i, Z_i; n, n') \cdot \mathbb{1}\left\{\widehat{R}^q(X_i, Z_i; n, n') \geq -b_L\right\} \cdot \mathbb{I}_{\mathcal{X}}(X_i) - \widehat{\mathcal{T}}_{n,n'}^q\right)$$
$$+ \left(\widehat{\Delta}_L^q(U_i, n, n') - \widehat{E}\left[\widehat{\Delta}_L^q(U_i, n, n')\right]\right),$$

$$\widehat{\lambda}_L(U_i, Z_i) = \sum_{q=1}^{Q_{n,n'}} \sum_{n,n'\in\mathcal{N}} \widehat{\lambda}_L^q(U_i, Z_i; n, n').$$

From here we can estimate σ_L^2 as

$$\widehat{\sigma}_L^2 = \frac{1}{L}\sum_{i=1}^{L} \widehat{\lambda}_L^2(U_i, Z_i).$$

Under Assumptions 3.1-3.4 (the conditions leading to Theorem 1), we will have $\left|\widehat{\sigma}_L^2 - \sigma_L^2\right| = o_p(1)$.

S-A.3 Construction of the test statistics in our auction models

For computational simplicity the test-statistics used in our auction models (both in the Monte Carlo simulations and in the timber data application) were constructed as described in Section S-A.2. In all cases the auxiliary sample $\{Z_i\}_{i=1}^{L}$ for the index z was generated as $Z_i \sim Unif\left[B_{(0.01)},\, B_{(0.99)}\right]$, where $B_{(p)}$ denotes the (estimated) p^{th} quantile of transaction price in the data. Thus, we used $\mathcal{P} = Unif_{\left[B_{(0.01)},\, B_{(0.99)}\right]}$. In what follows, $\lambda_L^q\left(U_i, Z_i; n, n'\right)$ and $\lambda_L\left(U_i, Z_i\right)$ are as defined in (S.25) and (S.26), respectively. The estimator for $\sigma_L^2 = \text{Var}\left(\lambda_L\left(U_i, Z_i\right)\right)$ was obtained following the generic construction described in Section S-A.2.1, above.

S-A.3.1 Detailed expressions for the auction models test statistics

In all auction model tests our statistics are of the form

$$\widehat{t}_L = \frac{\sqrt{L}\cdot\widetilde{\mathcal{T}}}{\max\left\{\widehat{\sigma}_L, \kappa_L\right\}},$$

where $\widetilde{\mathcal{T}}$ is as described in Section S-A.2, with $Z_i \sim Unif\left[B_{(0.01)},\, B_{(0.99)}\right]$, where $B_{(p)}$ denotes the (estimated) p^{th} quantile of transaction price in the data. Here we describe the precise expressions for $\widehat{\mathcal{T}}$ and $\widehat{\sigma}_L$ for each of the auction models examples. We let $\mathcal{N}$ be the entire range of values of N. As we just described, the range used for the index z was the interval $\mathcal{Z} = \left[B_{(0.01)},\, B_{(0.99)}\right]$, where $B_{(p)}$ denotes the p^{th} quantile of transaction price observed in the data. The testing range used for x was

$$\mathcal{X} = \left\{x\!:\ \widehat{f}_X(x) \geq \widehat{f}_X^{(.005)} \text{ and } 10^{-4} \leq \widehat{G}_{k:n}(z|x) \leq 1 - 10^{-4}\ \forall (n,z) \in \mathcal{N}\times\mathcal{Z} \text{ and each } 2 \leq k \leq n\right\}$$

where $\widehat{f}_X^{(.005)}$ denotes the $.005^{th}$ quantile of $\widehat{f}_X(\cdot)$.

For $s \in (0,1)$ and $1 \leq k \leq n$ denote

$$\nabla\psi_{k:n}^{-1}(s) = \frac{(n-k)!\cdot(k-1)!}{n!\cdot\left[\psi_{k:n}^{-1}(s)\right]^{k-1}\cdot\left(1-\left[\psi_{k:n}^{-1}(s)\right]\right)^{n-k}}$$

IPV with fixed N

For equation (12) we have $Q_n = n-2$ and for each $q = 1, \ldots, n-2$,

$$\widehat{R}^q(X_j, Z_j; n) = \psi_{n-1:n}^{-1}\left(\widehat{G}_{n:n}^{\Delta}(Z_j|X_j)\right) - \psi_{q:n}^{-1}\left(\widehat{G}_{q:n}(Z_j|X_j)\right),$$

$$\widehat{f}^q\left(U_j, U_i, Z_j, n; h_L\right) =$$

$$\Bigg\{\frac{1}{h_L^r\cdot\widehat{f}_{X,N}(X_j,n)}\cdot\Bigg[\nabla\psi_{n-1:n}^{-1}\left(\widehat{G}_{n:n}^{\Delta}(Z_j|X_j)\right)\cdot\left(\mathbb{1}\left\{B_{N:N,i}+\Delta \leq Z_j\right\} - \widehat{G}_{n:n}^{\Delta}(Z_j|X_j)\right)$$

$$-\nabla\psi_{q:n}^{-1}\left(\widehat{G}_{q:n}(Z_j|X_j)\right)\cdot\left(\mathbb{1}\left\{B_{q:N,i} \leq Z_j\right\} - \widehat{G}_{q:n}(Z_j|X_j)\right)\Bigg]\cdot\mathbb{1}\left\{N_i = n\right\}\cdot\mathcal{H}\left(X_i - X_j; h_L\right)$$

$$\times\,\mathbb{1}\left\{\widehat{R}^q(X_j, Z_j; n) \geq -b_L\right\}\cdot\mathbb{I}_{\mathcal{X}}(X_j)$$

From here,

$$\widetilde{\mathcal{T}}_n^q = \frac{1}{L}\sum_{i=1}^{L} \widehat{R}^q(X_i, Z_i; n) \cdot \mathbb{1}\left\{\widehat{R}^q(X_i, Z_i; n) \geq -b_L\right\} \cdot \mathbb{I}_{\mathcal{X}}(X_i), \quad \widetilde{\mathcal{T}} = \sum_{q=1}^{n-2}\sum_{n\in\mathcal{N}} \widetilde{\mathcal{T}}_n^q,$$

$$\widehat{\Delta}_L^q(U_i; n) = \frac{1}{L-1}\sum_{j\neq i} \widehat{f}^q\left(U_j, U_i, Z_j, n; h_L\right),$$

$$\widehat{\lambda}_L^q(U_i; n) = \left(\widehat{R}^q(X_i, Z_i; n) \cdot \mathbb{1}\left\{\widehat{R}^q(X_i, Z_i; n) \geq -b_L\right\} \cdot \mathbb{I}_{\mathcal{X}}(X_i) - \widehat{\mathcal{T}}_n^q\right)$$

$$+ \left(\widehat{\Delta}_L^q(U_i; n) - \widehat{E}\left[\widehat{\Delta}_L^q(U_i; n)\right]\right), \quad \widehat{\lambda}_L(U_i) = \sum_{q=1}^{n-2}\sum_{n\in\mathcal{N}} \widehat{\lambda}_L^q(U_i; n), \quad \widehat{\sigma}_L^2 = \frac{1}{L}\sum_{i=1}^{L} \widehat{\lambda}_L^2(U_i).$$

Nonnegatively correlated values and $V \perp N$

For equation (14) we have $Q_{n,n'} = 1$ and

$$\widehat{R}(X_j, Z_j; n, n') = \left(\psi_{n'-1:n'}^{-1}\left(\widehat{G}_{n':n'}(Z_j|X_j)\right) - \psi_{n-1:n}^{-1}\left(\widehat{G}_{n:n}(Z_j|X_j)\right)\right) \cdot \mathbb{1}\left\{n > n'\right\},$$

$$\widehat{f}\left(U_j, U_i, Z_j, n, n'; h_L\right) =$$

$$\left\{\frac{1}{h_L^r} \cdot \left[\nabla\psi_{n'-1:n'}^{-1}\left(\widehat{G}_{n':n'}(Z_j|X_j)\right) \cdot \frac{\left(\mathbb{1}\left\{B_{N:N,i} \leq Z_j\right\} - \widehat{G}_{n':n'}(Z_j|X_j)\right)}{\widehat{f}_{X,N}(X_j, n')} \cdot \mathbb{1}\left\{N_i = n'\right\}\right.\right.$$

$$\left. - \nabla\psi_{n-1:n}^{-1}\left(\widehat{G}_{n:n}(Z_j|X_j)\right) \cdot \frac{\left(\mathbb{1}\left\{B_{N:N,i} \leq Z_j\right\} - \widehat{G}_{n:n}(Z_j|X_j)\right)}{\widehat{f}_{X,N}(X_j, n)} \cdot \mathbb{1}\left\{N_i = n\right\}\right]$$

$$\left. \times \mathcal{H}\left(X_i - X_j; h_L\right) \cdot \mathbb{1}\left\{\widehat{R}(X_j, Z_j; n, n') \geq -b_L\right\} \cdot \mathbb{I}_{\mathcal{X}}(X_j)\right\} \cdot \mathbb{1}\left\{n > n'\right\}.$$

From here,

$$\begin{aligned}
&\widehat{\mathcal{T}}_{n,n'} = \frac{1}{L}\sum_{i=1}^{L} \widehat{R}(X_i, Z_i; n, n') \cdot \mathbb{1}\left\{\widehat{R}(X_i, Z_i; n, n') \geq -b_L\right\} \cdot \mathbb{I}_{\mathcal{X}}(X_i), \quad \widehat{\mathcal{T}} = \sum_{n\in\mathcal{N}} \widehat{\mathcal{T}}_{n,n'},\\
&\widehat{\Delta}_L(U_i; n, n') = \frac{1}{L-1}\sum_{j\neq i} \widehat{f}\left(U_j, U_i, Z_j, n, n'; h_L\right),\\
&\widehat{\lambda}_L(U_i; n, n') = \left(\widehat{R}(X_i, Z_i; n, n') \cdot \mathbb{1}\left\{\widehat{R}(X_i, Z_i; n, n') \geq -b_L\right\} \cdot \mathbb{I}_{\mathcal{X}}(X_i) - \widehat{\mathcal{T}}_{n,n'}\right) \qquad \text{(S.27)}\\
&+ \left(\widehat{\Delta}_L(U_i; n, n') - \widehat{E}\left[\widehat{\Delta}_L(U_i; n, n')\right]\right), \quad \widehat{\lambda}_L(U_i) = \sum_{n,n'\in\mathcal{N}} \widehat{\lambda}_L(U_i; n, n'),\\
&\widehat{\sigma}_L^2 = \frac{1}{L}\sum_{i=1}^{L} \widehat{\lambda}_L^2(U_i).
\end{aligned}$$

IPV without independence between V and N

For equation (15) we have $Q_{n,n'} = 1$ and

$$\widehat{R}(X_j, Z_j; n, n') = \left(\psi^{-1}_{n-1:n}\left(\widehat{G}_{n:n}(Z_j|X_j)\right) - \psi^{-1}_{n'-1:n'}\left(\widehat{G}_{n':n'}(Z_j|X_j)\right)\right) \cdot \mathbb{1}\left\{n > n'\right\},$$

$$\widehat{f}\left(U_j, U_i, Z_j, n, n'; h_L\right) =$$

$$\left\{\frac{1}{h_L^r} \cdot \left[\nabla\psi^{-1}_{n-1:n}\left(\widehat{G}_{n:n}(Z_j|X_j)\right) \cdot \frac{\left(\mathbb{1}\left\{B_{N:N,i} \leq Z_j\right\} - \widehat{G}_{n:n}(Z_j|X_j)\right)}{\widehat{f}_{X,N}(X_j, n)} \cdot \mathbb{1}\left\{N_i = n\right\}\right.\right.$$

$$\left. - \nabla\psi^{-1}_{n'-1:n'}\left(\widehat{G}_{n':n'}(Z_j|X_j)\right) \cdot \frac{\left(\mathbb{1}\left\{B_{N:N,i} \leq Z_j\right\} - \widehat{G}_{n':n'}(Z_j|X_j)\right)}{\widehat{f}_{X,N}(X_j, n')} \cdot \mathbb{1}\left\{N_i = n'\right\}\right]$$

$$\left. \times \mathcal{H}\left(X_i - X_j; h_L\right) \cdot \mathbb{1}\left\{\widehat{R}(X_j, Z_j; n, n') \geq -b_L\right\} \cdot \mathbb{I}_{\mathcal{X}}(X_j)\right\} \cdot \mathbb{1}\left\{n > n'\right\}$$

We then construct $\widehat{\mathcal{T}}$ and $\widehat{\sigma}_L$ using the generic expression (S.27).

S-B Additional Monte Carlo Experiment Results

Here we supplement the results in Appendix B of the main paper by describing the full set of results of our test for *each* one of the individual combinations of constants (c_b, c_h) used to construct the bandwidths.

Table S1: Supplement of Table B.3: complete results of our test for "$H_0 : E[Y - \theta | X = x] \leq 0 \ \forall \ x \in \mathcal{X}$" for $\theta = \theta_1 \equiv \max\{f(x) \colon x \in \mathcal{X}\}$.

Rejection rates. Nominal level $\alpha = 0.05$.															
		$c_b = 0.01$			$c_b = 0.10$			$c_b = 0.50$			$c_b = 1.0$			Jackknifed bandwidths	
		$c_h =$			$c_h =$			$c_h =$			$c_h =$			uniform	weights
DGP	L	0.2	0.4	0.6	0.2	0.4	0.6	0.2	0.4	0.6	0.2	0.4	0.6	weights	as (16)
$f = f_{AS1}$	100	.000	.000	.002	.002	.000	.006	.000	.000	.002	.000	.000	.000	.000	.000
$D = 1$	250	.020	.004	.002	.018	.012	.010	.004	.002	.008	.004	.004	.000	.000	.000
	500	.038	.022	.034	.036	.010	.006	.034	.014	.020	.008	.006	.006	.000	.000
	1000	.066	.020	.020	.072	.024	.032	.036	.022	.016	.024	.014	.014	.000	.000
$f = f_{AS1}$	100	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
$D = 5$	250	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
	500	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
	1000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
$f = f_{AS2}$	100	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
$D = 1$	250	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
	500	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
	1000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
$f = f_{AS2}$	100	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
$D = 5$	250	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
	500	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
	1000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000

Table S2: Supplement of Table B.4: complete results of our test for "$H_0 : E[Y - \theta | X = x] \leq 0 \; \forall \; x \in \mathcal{X}$" for $\theta = \theta_2 \equiv \max\{f(x) \colon x \in \mathcal{X}\} - 0.02$.

Rejection rates. Nominal level $\alpha = 0.05$.														
		$c_b = 0.01$			$c_b = 0.10$			$c_b = 0.50$			$c_b = 1.0$			Jackknifed bandwidths	
		$c_h =$			$c_h =$			$c_h =$			$c_h =$			uniform	weights
DGP	L	0.2	0.4	0.6	0.2	0.4	0.6	0.2	0.4	0.6	0.2	0.4	0.6	weights	as (16)
$f = f_{AS1}$	100	.648	.318	.142	.656	.300	.132	.636	.284	.148	.584	.238	.138	.086	.280
$D = 1$	250	.752	.376	.190	.750	.356	.162	.716	.342	.176	.664	.320	.184	.258	.556
	500	.790	.438	.270	.790	.392	.262	.766	.416	.242	.740	.322	.210	.448	.746
	1000	.896	.516	.338	.868	.516	.338	.828	.462	.312	.820	.472	.276	.746	.930
$f = f_{AS1}$	100	.732	.476	.284	.728	.392	.284	.726	.398	.324	.628	.364	.284	.400	.712
$D = 5$	250	.894	.610	.552	.866	.652	.542	.876	.590	.530	.818	.570	.458	.938	.988
	500	.972	.860	.746	.950	.838	.778	.958	.790	.708	.932	.786	.702	1.00	1.00
	1000	.998	.960	.950	.996	.966	.944	.992	.950	.936	.982	.928	.934	1.00	1.00
$f = f_{AS2}$	100	.694	.374	.212	.708	.334	.214	.676	.342	.202	.608	.326	.182	.212	.538
$D = 1$	250	.854	.490	.344	.846	.488	.296	.808	.444	.304	.790	.402	.220	.722	.898
	500	.938	.660	.448	.936	.610	.388	.898	.572	.338	.880	.490	.364	.974	.992
	1000	.968	.706	.526	.978	.704	.524	.954	.692	.430	.924	.592	.388	.992	1.00
$f = f_{AS2}$	100	.576	.352	.350	.596	.348	.354	.592	.346	.326	.584	.332	.288	.306	.558
$D = 5$	250	.862	.722	.726	.852	.738	.664	.858	.672	.632	.818	.646	.618	.986	1.00
	500	.956	.888	.894	.958	.894	.912	.952	.856	.856	.932	.818	.866	1.00	1.00
	1000	.994	.976	.986	.994	.976	.982	.996	.984	.968	.986	.938	.970	1.00	1.00

S-C Proofs of the Auctions Models Results in Section 4

S-C.1 Proof of Proposition 1

Under IPV, $F_{k:n}(v|x) = \psi_{k:n}(F_V(v|x))$, so

$$\psi_{k:n}^{-1}(F_{k:n}(v|x)) = F_V(v|x) = \psi_{n-1:n}^{-1}(F_{n-1:n}(v|x))$$

For the second part, fix n, x, and v, and let

$$\Pr(v|m) = \Pr(V_i < v | X = x, N = n, \|\{j \neq i : V_j < v\}\| = m)$$

By assumption, this is weakly increasing in m, and not constant across all values of m.

Suppress the dependence of value distributions on x. Let P_i denote the probability that exactly i valuations are greater than or equal to v, so $P_0 = F_{n:n}(v)$, $P_n = 1 - F_{1:n}(v)$, and $P_i = F_{n-i:n}(v) - F_{n-i+1:n}(v)$ for $1 \leq i < n$. Let $\Pr(m)$ be the probability that $V_1, \ldots, V_m \geq v$ and $V_{m+1}, \ldots, V_{n-1} < v$. By symmetry, $P_{i+1} = {}_nC_{i+1} \Pr(i)(1 - \Pr(v|n-1-i))$ and $P_i = {}_nC_i \Pr(i) \Pr(v|n-1-i)$; so

$$\frac{\frac{1}{{}_nC_{i+1}} P_{i+1}}{\frac{1}{{}_nC_i} P_i} = \frac{\Pr(i)(1 - \Pr(v|n-1-i))}{\Pr(i)\Pr(v|n-1-i)} = \frac{1 - \Pr(v|n-1-i)}{\Pr(v|n-1-i)}$$

By assumption, this is weakly increasing in i, and strictly increasing for some i.

Let $p = \psi_{n-1:n}^{-1}(F_{n-1:n}(v))$, and let $P_i^I = {}_nC_i p^{n-i}(1-p)^i$, and $F_{k:n}^I(v) = \psi_{k:n}(p) = \sum_{i=0}^{n-k} P_i^I$, so P_i^I and $F_{k:n}^I$ are what P_i and $F_{k:n}$ would be if valuations were independent draws from the distribution $F_V(\cdot) = \psi_{n-1:n}^{-1}(F_{n-1:n}(\cdot))$. By construction,

$$\frac{\frac{1}{{}_nC_{i+1}} P_{i+1}^I}{\frac{1}{{}_nC_i} P_i^I} = \frac{1-p}{p}$$

and therefore does not vary with i. Note that $P_0 + P_1 = F_{n-1:n}(v) = F_{n-1:n}^I(v) = P_0^I + P_1^I$.

Claim S1. $P_0 > P_0^I$.

Proof is by contradiction. Since $P_0 + P_1 = P_0^I + P_1^I$, if $P_0 \leq P_0^I$, then $P_1 \geq P_1^I$. Then

$$\frac{\frac{1}{{}_nC_2} P_2}{\frac{1}{{}_nC_1} P_1} \geq \frac{\frac{1}{{}_nC_1} P_1}{\frac{1}{{}_nC_0} P_0} \geq \frac{\frac{1}{{}_nC_1} P_1^I}{\frac{1}{{}_nC_0} P_0^I} = \frac{1-p}{p} = \frac{\frac{1}{{}_nC_2} P_2^I}{\frac{1}{{}_nC_1} P_1^I}$$

and so since $P_1 \geq P_1^I$ and $\frac{P_2}{P_1} \geq \frac{P_2^I}{P_1^I}$, then $P_2 \geq P_2^I$. Similarly, since $\frac{P_3}{P_2} \geq \frac{P_2}{P_1} \geq \frac{P_2^I}{P_1^I} = \frac{P_3^I}{P_2^I}$, $P_3 \geq P_3^I$; and likewise, $P_i \geq P_i^I$ for every $i > 3$, with at least one strict inequality due to the requirement that Assumption 4.2 holds strictly. This leads to $\sum_{i=0}^n P_i > \sum_{i=0}^n P_i^I$, which is a contradiction since both must be equal to 1.

Claim S2. $P_2 < P_2^I$.

Since $P_1 < P_1^I$, if $P_2 \geq P_2^I$, then $\frac{P_2}{P_1} > \frac{P_2^I}{P_1^I}$, and so $\frac{P_3}{P_2} > \frac{P_3^I}{P_2^I}$ giving $P_3 > P_3^I$ and so on; this would give $P_0 + P_1 = P_0^I + P_1^I$, $P_2 \geq P_2^I$, and $P_i > P_i^I$ for $i \geq 3$, yielding a contradiction.

(Note that if $\Pr(v|m)$ is only weakly increasing in m, everything up to here applies as weak inequalities and $P_2 \leq P_2^I$, which will be used below in the proof of Proposition 2.)

Claim S3. If $P_k > P_k^I$, then $P_{k'} > P_{k'}^I$ for all $k' > k$.

We know that $P_1 < P_1^I$. Let j denote the smallest $i > 0$ such that $P_i > P_i^I$. This means $P_j > P_j^I$ but $P_{j-1} \leq P_{j-1}^I$, and therefore

$$\frac{\frac{1}{{}_nC_j} P_j}{\frac{1}{{}_nC_{j-1}} P_{j-1}} > \frac{\frac{1}{{}_nC_j} P_j^I}{\frac{1}{{}_nC_{j-1}} P_{j-1}^I} = \frac{1-p}{p}$$

Which means that

$$\frac{\frac{1}{{}_nC_{j+1}} P_{j+1}}{\frac{1}{{}_nC_j} P_j} \geq \frac{\frac{1}{{}_nC_j} P_j}{\frac{1}{{}_nC_{j-1}} P_{j-1}} > \frac{1-p}{p} = \frac{\frac{1}{{}_nC_{j+1}} P_{j+1}^I}{\frac{1}{{}_nC_j} P_j^I}$$

and so $P_j > P_j^I$ and $\frac{P_{j+1}}{P_j} > \frac{P_{j+1}^I}{P_j^I}$, meaning $P_{j+1} > P_{j+1}^I$. Likewise,

$$\frac{\frac{1}{{}_nC_{j+2}} P_{j+2}}{\frac{1}{{}_nC_{j+1}} P_{j+1}} \geq \frac{\frac{1}{{}_nC_{j+1}} P_{j+1}}{\frac{1}{{}_nC_j} P_j} > \frac{1-p}{p} = \frac{\frac{1}{{}_nC_{j+2}} P_{j+2}^I}{\frac{1}{{}_nC_{j+1}} P_{j+1}^I}$$

and so $P_{j+2} > P_{j+2}^I$, and so on, proving the claim.

Claim S4. For $k > 1$, if $F_{n-k:n}(v) \geq F_{n-k:n}^I(v)$ then $P_k > P_k^I$.

By construction, $F_{n-k:n}(v) = \sum_{i=0}^k P_i$ and $F_{n-k:n}^I(v) = \sum_{i=0}^k P_i^I$. We know that $P_0 + P_1 = P_0^I + P_1^I$, and $P_2 < P_2^I$; so if $\sum_{i=0}^k P_k \geq \sum_{i=0}^k P_k^I$, there must be some j $(2 < j \leq k)$ such that $P_j > P_j^I$. But then by the previous claim, $P_k > P_k^I$.

Claim S5. For $1 < k < n$, $F_{n-k:n}(v) < F_{n-k:n}^I(v)$.

If $F_{n-k:n}(v) \geq F_{n-k:n}^I(v)$, then by the last claim, $P_k > P_k^I$. But then by the previous claim, $P_{k'} > P_{k'}^I$ for all $k' > k$. So

$$1 = F_{n-k:n}(v) + \sum_{k'>k} P_k > F_{n-k:n}^I(v) + \sum_{k'>k} P_k^I = 1$$

a contradiction. So it must be that $F_{n-k:n}(v) < F_{n-k:n}^I(v)$. But

$$F_{n-k:n}^I(v) = \psi_{n-k:n}(p) = \psi_{n-k:n}\left(\psi_{n-1:n}^{-1}(F_{n-1:n}(v))\right)$$

so the last claim is that $F_{n-k:n}(v) < \psi_{n-k:n}\left(\psi_{n-1:n}^{-1}(F_{n-1:n}(v))\right)$, proving the proposition.

S-C.2 Proof of Propositions 2 and 3

Proposition 2 part a. Under IPV and the exclusion restriction,

$$\psi^{-1}_{n-1:n}\left(F_{n-1:n}(v|x)\right) \quad = \quad F_V(v|x,n) \quad = \quad F_V(v|x,n') \quad = \quad \psi^{-1}_{n'-1:n'}\left(F_{n'-1:n'}(v|x)\right)$$

$B_{n:n} = V_{n-1:n}$ implies $G_{n:n}(v|x) = F_{n-1:n}(v|x)$ and $G_{n':n'}(v|x) = F_{n'-1:n'}(v|x)$, giving (13).

Proposition 2 part b. Fix n, x, and v. As above, let $p = \psi^{-1}_{n-1:n}(F_{n-1:n}(v|x))$, and let P_i be the probability that exactly i (of n) valuations are at least v. Under Assumption 4.2, as noted above in the proof of Proposition 12, $P_2 \leq {}_nC_2 p^{n-2}(1-p)^2$. If valuations are independent of N, plugging $r = n-2$ into equation 9 of Athey and Haile (2002) and rearranging gives

$$\begin{aligned}
F_{n-2:n-1}(v|x) &= F_{n-1:n}(v|x) + \tfrac{2}{n}\left[F_{n-2:n}(v|x) - F_{n-1:n}(v|x)\right] \\
&= np^{n-1} - (n-1)p^n + \tfrac{2}{n}P_2 \\
&\leq np^{n-1} - (n-1)p^n + \tfrac{2}{n}\tfrac{n(n-1)}{2}p^{n-2}(1-p)^2 \\
&= (n-1)p^{n-2} - (n-2)p^{n-1} \\
&= \psi_{n-2:n-1}(p) \\
&= \psi_{n-2:n-1}\left(\psi^{-1}_{n-1:n}\left(F_{n-1:n}(v|x)\right)\right)
\end{aligned}$$

or $\psi^{-1}_{n-2:n-1}(F_{n-2:n-1}(v|x)) \leq \psi^{-1}_{n-1:n}(F_{n-1:n}(v|x))$; if Assumption 4.2 holds strictly at (v,x,n), then (from the proof of Proposition 1 above) $P_2 < {}_nC_2 p^{n-2}(1-p)^2$ and the inequality is strict. From there, transitivity gives $\psi^{-1}_{n'-1:n'}(F_{n'-1:n'}(v|x)) \leq \psi^{-1}_{n-1:n}(F_{n-1:n}(v|x))$ for any $n' < n$; $G_{n:n} = F_{n-1:n}$ and $G_{n':n'} = F_{n'-1:n'}$ then imply (14).

Proposition 3. Let $n > n'$; under Assumption 4.3, $F_V(\,\cdot\,|x,n) \succsim_{FOSD} F_V(\,\cdot\,|x,n')$, so

$$\psi^{-1}_{n-1:n}\left(F_{n-1:n}(v|x)\right) \quad = \quad F_V(v|x,n) \quad \leq \quad F_V(v|x,n') \quad = \quad \psi^{-1}_{n'-1:n'}\left(F_{n'-1:n'}(v|x)\right)$$

Since $G_{n:n} = F_{n-1:n}$ and $G_{n':n'} = F_{n'-1:n'}$, (15) follows.

S-C.3 Violations of (14) Under Two Entry Models

Here, we show how dependence of valuations on N generated by two standard models of endogenous participation in auctions would lead to rejection of the exclusion restriction due to a violation of (14).

Consider a model of independent private values with unobserved heterogeneity. There is a one-dimensional variable $\theta \in \Re$ which is observed by bidders but not the analyst. Valuations are *i.i.d.* $\sim F_V(\cdot \mid \theta)$, and $\theta > \theta'$ implies $F(\,\cdot\,|\theta) \succsim_{FOSD} F(\,\cdot\,|\theta')$. For each θ, assume $F_V(\cdot|\theta)$ is twice differentiable and has bounded support $[\underline{v}, \overline{v}]$. Let $f_V(\cdot \mid \theta)$ denote the density function.

Now, we apply two standard models of endogeous entry to this environment. In the first model, that of Levin and Smith (1994), there are $\overline{n}$ potential bidders, who each observe θ but not their own valuations before deciding whether to enter (in which case they incur a cost c and participate in the auction) or not (earning a payoff of 0). Bidders play a different symmetric mixed strategy for each realization of θ, leading to a stochastic N with a different distribution for each θ.

In the second model, that of Samuelson (1985), bidders observe both θ and their own valuation before making their entry decision, and play a different pure-strategy symmetric equilibrium in cutoff strategies for each θ.

Proposition 1. *In the Levin-Smith entry game, if $f_V(\overline{v}|\theta)$ and the equilibrium entry probability are both strictly increasing in θ, then the valuations generated would violate Equation* (14) *over some range of v.*

In the Sameulson entry game, if valuations and θ are related via the Strict Monotone Likelihood Ratio Property, then the valuations generated would violate (14) *over some range of v.*

Proof. The Taylor expansion of $\psi^{-1}_{n-1:n}(F_{n-1:n}(v))$ around $v = \overline{v}$, after a lot of algebra, gives

$$\psi^{-1}_{n-1:n}(F_{n-1:n}(v)) = 1 - (\overline{v} - v)\sqrt{E_{\theta|n}\left(f_V(\overline{v}|\theta)\right)^2} + O\left((\overline{v} - v)^2\right) \tag{S.28}$$

Let $n > n'$. If $\theta|N = n \succ_{FOSD} \theta|N = n'$ and $f_V(\overline{v}|\theta)$ is increasing in θ, then $E_{\theta|n}\left(f_V(\overline{v}|\theta)\right)^2 > E_{\theta|n'}\left(f_V(\overline{v}|\theta)\right)^2$, in which case $\psi^{-1}_{n-1:n}(F_{n-1:n}(v)) < \psi^{-1}_{n'-1:n'}(F_{n'-1:n'}(v))$ for v sufficiently close to $\overline{v}$, violating (14).

For the Levin-Smith result, this is all we need. We showed in Aradillas-López, Gandhi, and Quint (2013) that if the entry probability is increasing in θ, then $n > n'$ implies $\theta|N = n \succsim_{FOSD} \theta|N = n'$; the same argument shows this is strict when the entry probability is strictly increasing.

In the Samuelson game, the entry cutoff $v^*(\theta)$ is the solution to $vF_V^{\bar{n}-1}(v|\theta) = c$. Under the strict MLRP, $F_V(v|\theta)$ is strictly decreasing in θ on $(\underline{v}, \overline{v})$, so $v^*(\theta)$ and $1 - F_V(v^*(\theta)|\theta)$ are both strictly increasing in θ; so $\theta|N = n \succ_{FOSD} \theta|N = n'$ when $n > n'$, as above. But now we require the density of valuations at $\overline{v}$ *conditional on entry* to be strictly increasing in θ. This density can be written as

$$\frac{f_V(\overline{v}|\theta)}{1 - F_V(v^*(\theta)|\theta)} \quad = \quad \frac{f_V(\overline{v}|\theta)}{\int_{v^*(\theta)}^{\overline{v}} f_V(v|\theta)dv} \quad = \quad \left(\int_{v^*(\theta)}^{\overline{v}} \frac{f_V(v|\theta)}{f_V(\overline{v}|\theta)}dv\right)^{-1}$$

Since $v^*(\theta)$ is increasing in θ, an increase in θ shrinks the interval $[v^*(\theta), \overline{v}]$ over which the integral is taken; and if v and θ are related by the strict MLRP, since $v < \overline{v}$, $\frac{f_V(v|\theta)}{f_V(\overline{v}|\theta)}$ is strictly decreasing in θ. So $\int_{v^*(\theta)}^{\overline{v}} \frac{f_V(v|\theta)}{f_V(\overline{v}|\theta)}dv$ is strictly decreasing in θ, meaning $\frac{f_V(\overline{v}|\theta)}{1-F_V(v^*(\theta)|\theta)}$ is strictly increasing in θ; so (S.28) gives $\psi^{-1}_{n-1:n}(F_{n-1:n}(v)) < \psi^{-1}_{n'-1:n'}(F_{n'-1:n'}(v))$ for v close to $\overline{v}$.

S-C.4 Generalizing (13), (14), and (15) to Haile-and-Tamer Bidding

The proofs of Propositions 2 and 3 establish that under IPV and the exclusion restriction,

$$\psi^{-1}_{n-1:n}\left(F_{n-1:n}(v|x)\right) \quad = \quad \psi^{-1}_{n'-1:n'}\left(F_{n'-1:n'}(v|x)\right)$$

for any n and n'; under Assumption 4.2 and the exclusion restriction,

$$\psi_{n-1:n}^{-1}\left(F_{n-1:n}(v|x)\right) \quad \geq \quad \psi_{n'-1:n'}^{-1}\left(F_{n'-1:n'}(v|x)\right)$$

for $n > n'$; and under IPV and Assumption 4.3,

$$\psi_{n-1:n}^{-1}\left(F_{n-1:n}(v|x)\right) \quad \leq \quad \psi_{n'-1:n'}^{-1}\left(F_{n'-1:n'}(v|x)\right)$$

for $n > n'$. As discussed in the text, the Haile-and-Tamer bidding assumptions imply that $B_{n-1:n} \leq V_{n-1:n} \leq B_{n:n} + \Delta$, and therefore $G_{n-1:n}(v|x) \geq F_{n-1:n}(v|x) \geq G^{\Delta}_{n:n}(v|x)$. Combining these with the relationships above gives the following results, which are the analogs of the tests (13), (14), and (15) under Haile and Tamer bidding:

Proposition 2. *Assume bidding behavior satisfies the Haile-and-Tamer assumptions and valuations are independent of N.*

(a) Under IPV, if valuations are independent of N then for any (x, n, n', v),

$$\psi_{n-1:n}^{-1}\left(G_{n-1:n}(v|x)\right) \quad \geq \quad \psi_{n'-1:n'}^{-1}\left(G^{\Delta}_{n':n'}(v|x)\right) \tag{S.29}$$

(b) Under Assumption 4.2, if valuations are independent of N, then for any (x, n, n', v),

$$n > n' \quad \longrightarrow \quad \psi_{n-1:n}^{-1}\left(G_{n-1:n}(v|x)\right) \quad \geq \quad \psi_{n'-1:n'}^{-1}\left(G^{\Delta}_{n':n'}(v|x)\right) \tag{S.30}$$

(c) Under IPV and Assumption 4.3, for any (x, n, n', v),

$$n > n' \quad \longrightarrow \quad \psi_{n-1:n}^{-1}\left(G^{\Delta}_{n:n}(v|x)\right) \quad \leq \quad \psi_{n'-1:n'}^{-1}\left(G_{n'-1:n'}(v|x)\right) \tag{S.31}$$

Structural functions S and s and the resulting functions m can be derived for (S.29), (S.30), and (S.31) in exactly the analogous way as for (13), (14), and (15) in the text.

References

Aradillas-López, A., A. Gandhi, and D. Quint (2013). Identification and inference in ascending auctions with correlated private values. *Econometrica 81*(2), 489–534.

Athey, S. and P. Haile (2002). Identification of standard auction models. *Econometrica 70*(6), 2107–2140.

Levin, D. and J. Smith (1994). Equilibrium in auctions with entry. *The American Economic Review 84*(3), 585–599.

Nolan, D. and D. Pollard (1987). U-processes: Rates of convergence. *Annals of Statistics 15*, 780–799.

Pakes, A. and D. Pollard (1989). Simulation and the asymptotics of optimization estimators. *Econometrica 57*(5), 1027–1057.

Pollard, D. (1990). *Empirical Processes: Theory and Applications*. Institute of Mathematical Statistics.

Samuelson, W. (1985). Competitive bidding with entry costs. *Economics Letters 17*(1-2), 53–57.

Serfling, R. (1980). *Approximation Theorems of Mathematical Statistics*. Wiley. New York, NY.

Sherman, R. (1994). Maximal inequalities for degenerate u-processes with applications to optimization estimators. *Annals of Statistics 22*, 439–459.