Unobserved correlation in private-value ascending auctions[1]

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Abstract. In private-value ascending auctions, the winner's willingness to pay is not observed, leading to underidentification of many econometric models. I calculate tight bounds on expected revenue and optimal reserve price for the case of symmetric and affiliated private values.

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1 Introduction

A number of recent papers address the recovery of underlying economic primitives from auction data. (Guerre, Perrigne and Vuong (2000) discuss nonparametric estimation in first-price auctions; Athey and Haile (2002) give identification results for a wide range of auction rules and modeling specifications. See Athey and Haile (2007) for a thorough bibliography.)

In many auction formats – ascending auctions, button or clock auctions, and first-price auctions with proxy bidding (as on eBay), for example – the highest bidder's willingness to pay is not directly observed. In the independent private values paradigm, its distribution can be inferred from the distribution of the winning bid, and all the unobserved primitives are still identified (see Athey and Haile (2002)) or tightly bounded (see Haile and Tamer

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(2003)). However, under weaker assumptions than independence, its distribution is not identified.

The highest bidder's willingness to pay is important because it directly affects the seller's expected revenue in an auction with positive reserve price, as well as the optimal choice of reserve price. I calculate explicit upper and lower bounds on both these quantities in the case of symmetric affiliated private values; the upper bounds are equal to the values these quantities would achieve under independent private values.

2 Model and Results

A seller has one indivisible object to sell, which he values at v_0. There are a fixed number n of potential buyers, with private values $v_1, \ldots, v_n$. The joint distribution $f(v_1, v_2, \ldots, v_n)$ is symmetric and affiliated[3], with bounded support $[\underline{v}, \overline{v}]^n$. Let $v^1 \geq v^2 \geq \cdots \geq v^n$ be the order statistics of the values, and $F_i(\cdot)$ the cumulative distribution function[4] of v^i.

I abstract away from the precise details of the auction to be used, and assume only the following: the seller will specify a reserve price, and provided that at least one buyer's valuation exceeds this price, the object will be sold to the buyer with the highest valuation, at a price which is the greater of the reserve price and the second-highest valuation.[5] I similarly assume that past auctions exactly identify the distribution F_2 of the second-highest valuation, but give no further information about F_1. (Observations of other losing bids may tighten the upper bounds, but do not affect the lower bounds.) Note that I assume away variation in both the number of bidders and the reserve prices of previous auctions.[6]

Expected revenue in an ascending auction with reserve price r can be written as

$$\pi(r) = (r - v_0)(F_2(r) - F_1(r)) + \int_r^{\overline{v}} (v - v_0) dF_2(v) \tag{1}$$

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[3] See Milgrom and Weber (1982) for more on affiliation.

[4] In case of point masses, $F_i(r) \equiv \Pr(v^i < r)$, not $\Pr(v^i \leq r)$.

[5] This is exactly true in a second-price sealed-bid auction or a button or clock auction; it is true up to the minimum bid increment for an ascending auction with proxy bidding, and for any first-price ascending auction provided the bidders do not make "jump bids".

[6] Variation in the number of bidders in previous auctions would significantly alter my results. In most circumstances, it seems difficult to justify treating such variation as exogenous; treating it as endogenous requires a model of entry in auctions, which is outside the scope of this paper. Sufficient *exogenous* variation in past reserve prices would identify the distribution in question; if past reserve prices were correlated with bidder values, however, this would not suffice.

For a given distribution $F_2(\cdot)$, define a new distribution $H(\cdot)$ implicitly by

$$F_2(r) = n(n-1)\int_0^{H(r)} s^{n-2}(1-s)ds \tag{2}$$

so that the second-highest of n independent draws from the distribution H has distribution F_2.[7] Define

$$\overline{\pi}(r) \equiv (r - v_0)(F_2(r) - H^n(r)) + \int_r^{\overline{v}} (v - v_0)dF_2(v) \tag{3}$$

and note that $\overline{\pi}(r)$ would be the expected revenue of the auction with reserve price r if bidder values were independent draws from the distribution H. Define

$$\underline{\pi}(r) \equiv \int_r^{\overline{v}} (v - v_0)dF_2(v) \tag{4}$$

and note that for any $r > v_0$, $\underline{\pi}(r) < \int_{v_0}^{\overline{v}}(v - v_0)dF_2(v) = \pi(v_0)$.

Theorem 1. *Suppose bidders have private values which are symmetric and affiliated.*

1. *For any $r > v_0$, expected revenue $\pi(r)$ is bounded above by $\overline{\pi}(r)$ and below by $\underline{\pi}(r)$, and both bounds are tight*

2. *Suppose in addition[8] that $\overline{\pi}$ is continuous, differentiable, and strictly quasiconcave; let r^I be its maximizer. Then the optimal reserve price r^* is bounded above by r^I and below by v_0, and both bounds are tight*

In Quint (2008), I present an example where both expected revenue and optimal reserve price are strictly decreasing in the degree of unobserved correlation. I also show that the revenue bounds in Theorem 1 hold when values are conditionally independent but not affiliated; that the upper bound $\overline{\pi}$ can be tightened given data on other losing bids; and that similar bounds hold for auctions with entry fees.

3 Empirical Estimation

Now consider the problem of applying these results using an empirical estimate of the distribution F_2. For auctions in which v^2 is revealed by equilibrium bidding (such as button

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[7] Equation 2 is equivalent to $F_2(r) = nH(r)^{n-1} - (n-1)H(r)^n$; the latter is the cumulative distribution function of the second-highest of n independent draws on the distribution H.

[8] These additional requirements are similar to those made in Haile and Tamer. A sufficient (but not necessary) condition is that the derived distribution $H(\cdot)$ be continuous and differentiable with a nondecreasing hazard rate. When these additional conditions are not met, the inequality $\overline{\pi}(r^*) \geq \pi(r^*) \geq \pi(v_0)$ provides a weaker upper bound on r^*, since $\pi(v_0)$ is known exactly.

auctions), the distribution of winning bids yields point estimates $\hat{F}_2(r)$ and pointwise confidence intervals $[F_2^L(r), F_2^U(r)]$ for the true distribution. For general ascending auctions, observations of the highest two bids $b^1 > b^2$ defines upper and lower bounds on the empirical distribution of v^2, since under the weak assumptions on bidding behavior made in Haile and Tamer, $b^2 \leq v^2 \leq b^1 + \delta$ (where δ is the minimum bid increment); these bounds can similarly be expanded to pointwise confidence intervals $[F_2^L(r), F_2^U(r)]$ for $F_2(r)$.

For $r \geq v_0$, the revenue bounds $\underline{\pi}$ and $\overline{\pi}$ defined above are stochastically increasing[9] in the distribution of v^2, so calculating $\underline{\pi}(r)$ from F_2^U and $\overline{\pi}(r)$ from F_2^L yields appropriate bounds on expected revenue. Haile and Tamer give a technique for bounding optimal reserve price under independent values given bounds on π; a slight modification of their technique defines the upper bound

$$\overline{r}^I = \max\{r : \overline{\pi}_L(r) \geq \max_{r'} \overline{\pi}_U(r')\} \tag{5}$$

with $\overline{\pi}_L$ and $\overline{\pi}_U$ derived from F_2^L and F_2^U, respectively. The bounds $v_0 \leq r^* \leq \overline{r}^I$ then hold.

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Appendix – Proofs

Upper Bound on $\pi(r)$

$\overline{\pi}(r)$ was defined by replacing $F_1(r)$ in equation 1 by $F_1^I(r) \equiv H^n(r)$. I show below that $F_1(r) \geq F_1^I(r)$, implying $\pi(r) \leq \overline{\pi}(r)$ for $r > v_0$. Since independence is a special case of affiliation, the bound is tight.

Fix r. Choose $i \in \{0, 1, \ldots, n-2\}$. Let X and Y denote the following statements:

$$X = \text{“}v_1, \ldots, v_i \geq r, \;\; v_{i+1}, \ldots, v_{n-2} < r, \;\; v_{n-1} < r\text{''}$$

$$Y = \text{“}v_1, \ldots, v_i \geq r, \;\; v_{i+1}, \ldots, v_{n-2} < r, \;\; v_{n-1} \geq r\text{''}$$

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[9] That is, if F_2' first-order stochastically dominates F_2, then $\underline{\pi}$ and $\overline{\pi}$ are higher at every r when calculated from F_2'; see appendix for proof.

Under affiliation, $\Pr(v_n \geq r|X) \leq \Pr(v_n \geq r|Y)$ and $\Pr(v_n < r|X) \geq \Pr(v_n < r|Y)$ (since $\mathbf{1}_{v_n \geq r}$ is increasing in v_n, its expectation is increasing in v_{n-1}); so

$$\frac{\Pr(X)\Pr(v_n \geq r|X)}{\Pr(X)\Pr(v_n < r|X)} \leq \frac{\Pr(Y)\Pr(v_n \geq r|Y)}{\Pr(Y)\Pr(v_n < r|Y)} \quad \to \quad \frac{\Pr(X, v_n \geq r)}{\Pr(X, v_n < r)} \leq \frac{\Pr(Y, v_n \geq r)}{\Pr(Y, v_n < r)}$$

Let P_i be the (true) probability that exactly i bidders have values greater than or equal to r, and P_i^I be the probability under independently distributed values consistent with F_2. Recall that if X holds then i of the first $n-1$ values are above r, and if Y holds then $i+1$ are. By symmetry,

$$\begin{aligned} P_i &= {}_nC_i \quad \Pr(X, v_n < r) \\ P_{i+1} &= {}_nC_{i+1} \quad \Pr(X, v_n \geq r) \\ &= {}_nC_{i+1} \quad \Pr(Y, v_n < r) \\ P_{i+2} &= {}_nC_{i+2} \quad \Pr(Y, v_n \geq r) \end{aligned}$$

so the previous inequality becomes

$$\frac{\frac{1}{{}_nC_{i+1}}P_{i+1}}{\frac{1}{{}_nC_i}P_i} \leq \frac{\frac{1}{{}_nC_{i+2}}P_{i+2}}{\frac{1}{{}_nC_{i+1}}P_{i+1}}$$

If values are independent, $\Pr(v_n \geq r)$ does not depend on v_{n-1}, so the same inequalities hold with equality and

$$\frac{\frac{1}{{}_nC_{i+1}}P^I_{i+1}}{\frac{1}{{}_nC_i}P^I_i} = \frac{\frac{1}{{}_nC_{i+2}}P^I_{i+2}}{\frac{1}{{}_nC_{i+1}}P^I_{i+1}}$$

By definition, $F_1(r) = P_0$, $F_1^I(r) = P_0^I$, and $F_2(r) = P_0 + P_1 = P_0^I + P_1^I$. Suppose toward contradiction that $F_1(r) < F_1^I(r)$, or $P_0 < P_0^I$, implying $P_1 > P_1^I$. Then

$$\frac{\frac{1}{{}_nC_2}P_2}{\frac{1}{{}_nC_1}P_1} \geq \frac{\frac{1}{{}_nC_1}P_1}{\frac{1}{{}_nC_0}P_0} > \frac{\frac{1}{{}_nC_1}P^I_1}{\frac{1}{{}_nC_0}P^I_0} = \frac{\frac{1}{{}_nC_2}P^I_2}{\frac{1}{{}_nC_1}P^I_1}$$

so $\frac{P_2}{P_1} > \frac{P_2^I}{P_1^I}$; since $P_1 > P_1^I$, this means $P_2 > P_2^I$. It similarly follows that $\frac{P_3}{P_2} > \frac{P_3^I}{P_2^I}$, so $P_3 > P_3^I$, and similarly $P_4 > P_4^I$, etc. Since $P_0 + P_1 = P_0^I + P_1^I$, this means

$$1 \quad = \quad P_0 + P_1 + P_2 + \ldots + P_n \quad > \quad P_0^I + P_1^I + P_2^I + \ldots + P_n^I \quad = \quad 1$$

The contradiction proves that $F_1(r) \geq F_1^I(r)$.

Upper Bound on r^*

f continuous

By assumption, $\overline{\pi}$ is continuous, differentiable, and strictly quasiconcave. If the distribution f is continuous, then $\pi(\cdot)$ is continuous and differentiable as well, with f nondegenerate in the following sense: for any v, $\Pr(v_j = v | v_i = v) = 0$. Strict quasiconcavity implies $\overline{\pi}$ is strictly decreasing above r^I, so $\overline{\pi}'(r) < 0$ almost everywhere; I show below that $\overline{\pi}'(r) < 0 \rightarrow \pi'(r) < 0$, so π is strictly decreasing above r^I as well.

Pick $r > r^I$ with $\overline{\pi}'(r) < 0$. From equation 1,

$$\pi'(r) = F_2(r) - F_1(r) - (r - v_0) f_1(r)$$

and, letting f_1^I be the marginal density of the distribution F_1^I,

$$\overline{\pi}'(r) = F_2(r) - F_1^I(r) - (r - v_0) f_1^I(r)$$

Since $\overline{\pi}'(r) < 0$, $f_1^I(r) > 0$, and therefore $f_2(r) > 0$.[10]

I showed above that $F_1(r) \geq F_1^I(r)$, so if $f_1(r) \geq f_1^I(r)$ then $\pi'(r) \leq \overline{\pi}'(r) < 0$; assume therefore that $f_1(r) < f_1^I(r)$. In addition, $f_1(r) > 0$,[11] so let $f_1(r) = \alpha f_1^I(r)$ with $\alpha \in (0, 1)$.

$\Pr(v_n \geq r | v_1 = r, v_2, \ldots, v_{n-1} < r) \leq \Pr(v_n \geq r | v_1 > r, v_2, \ldots, v_{n-1} < r)$ by affiliation; letting $j(\cdot)$ be the marginal density function of v_1, this gives

$$\frac{j(r) \Pr(v_2, \ldots, v_{n-1} < r | v_1 = r) \Pr(v_n \geq r | v_1 = r, v_2, \ldots, v_{n-1} < r)}{j(r) \Pr(v_2, \ldots, v_{n-1} < r | v_1 = r) \Pr(v_n < r | v_1 = r, v_2, \ldots, v_{n-1} < r)} \leq$$

$$\frac{\Pr(v_1 \geq r, v_2, \ldots, v_{n-1} < r) \Pr(v_n \geq r | v_1 \geq r, v_2, \ldots, v_{n-1} < r)}{\Pr(v_1 \geq r, v_2, \ldots, v_{n-1} < r) \Pr(v_n < r | v_1 \geq r, v_2, \ldots, v_{n-1} < r)}$$

(We know $j(r) > 0$ since $f_2(r) > 0$.) Using the nondegeneracy of f, this is

$$\frac{\frac{1}{n(n-1)} f_2(r)}{\frac{1}{n} f_1(r)} \leq \frac{\frac{1}{{}_nC_2} P_2}{\frac{1}{{}_nC_1} P_1}$$

Let $\gamma = \left(\frac{1}{n(n-1)} f_2(r)\right) \Big/ \left(\frac{1}{n} f_1^I(r)\right)$, and recall that $f_1(r) = \alpha f_1^I(r)$, so

$$\frac{\gamma}{\alpha} = \frac{\frac{1}{n(n-1)} f_2(r)}{\frac{1}{n} f_1(r)} \leq \frac{\frac{1}{{}_nC_2} P_2}{\frac{1}{{}_nC_1} P_1} \leq \frac{\frac{1}{{}_nC_3} P_3}{\frac{1}{{}_nC_2} P_2} \leq \cdots \leq \frac{\frac{1}{{}_nC_n} P_n}{\frac{1}{{}_nC_{n-1}} P_{n-1}}$$

This gives

$$P_2 \geq \frac{P_1}{n} {}_nC_2 \left(\frac{\gamma}{\alpha}\right), \quad P_3 \geq \frac{P_1}{n} {}_nC_3 \left(\frac{\gamma}{\alpha}\right)^2, \quad \cdots, \quad P_n \geq \frac{P_1}{n} {}_nC_n \left(\frac{\gamma}{\alpha}\right)^{n-1}$$

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[10] $\frac{f_2(r)}{f_1^I(r)} = \frac{n(n-1)H^{n-2}(r)(1-H(r))h(r)}{nH^{n-1}(r)h(r)} = \frac{(n-1)(1-H(r))}{H(r)}$. If $H(r) = 1$, $F_2(r) = F_1^I(r) = 1$, so $\pi(r) \leq \overline{\pi}(r) = 0$.

[11] Let $j(\cdot)$ denote the marginal density of v_1; by symmetry,

$$\frac{f_1(r)}{f_2(r)} = \frac{nj(r)}{n(n-1)j(r)} \frac{\Pr(v_2,\ldots,v_{n-1}<r|v_1=r)}{\Pr(v_2,\ldots,v_{n-2}<r,v_{n-1}\geq r|v_1=r)} \frac{\Pr(v_n<r|v_1=r,v_2,\ldots,v_{n-2}<r,v_{n-1}<r)}{\Pr(v_n<r|v_1=r,v_2,\ldots,v_{n-2}<r,v_{n-1}\geq r)}$$

We can also write $f_2(r)$ using $\Pr(v_2, \ldots, v_{n-1} < r | v_1 = r)$, so the middle fraction is nonzero; the last fraction is greater than 1 by affiliation.

Summing these gives

$$1 - F_2(r) = \sum_{i=2,\dots,n} P_i \geq \frac{P_1}{n}\left({}_nC_2\left(\frac{\gamma}{\alpha}\right) + {}_nC_3\left(\frac{\gamma}{\alpha}\right)^2 + \cdots + {}_nC_n\left(\frac{\gamma}{\alpha}\right)^{n-1}\right)$$

Under independence, $\alpha = 1$ and all these inequalities hold with equality, so

$$1 - F_2(r) = \frac{P_1^I}{n}\left({}_nC_2\gamma + {}_nC_3\gamma^2 + \cdots + {}_nC_n\gamma^{n-1}\right)$$

so when $f_1(r) = \alpha f_1^I(r)$,

$$P_1 \leq \frac{{}_nC_2\gamma + {}_nC_3\gamma^2 + \cdots + {}_nC_n\gamma^{n-1}}{{}_nC_2\left(\frac{\gamma}{\alpha}\right) + {}_nC_3\left(\frac{\gamma}{\alpha}\right)^2 + \ldots + {}_nC_n\left(\frac{\gamma}{\alpha}\right)^{n-1}} P_1^I < \alpha P_1^I$$

But $P_1 = F_2(r) - F_1(r)$ and $P_1^I = F_2(r) - F_1^I(r)$, so

$$\pi'(r) = P_1 - (r - v_0)f_1(r) < \alpha P_1^I - (r - v_0)\alpha f_1^I(r) = \alpha\overline{\pi}'(r) < 0$$

f not continuous

For $r' > r^I$ and f continuous, $\pi(r') - \pi(r^I)$ is not only negative, but bounded away from zero.[12] If f is not continuous, take any series of symmetric, affiliated, continuous distributions $f^1, f^2, \dots$ which converge to f; pointwise convergence of f^k to f suffices to show $\pi^k(r') - \pi^k(r^I)$ converges to $\pi(r') - \pi(r^I)$, which is therefore negative.

Lower Bounds on $\pi(r)$ and r^*

$v^1 \geq v^2$, so $F_1(r) \leq F_2(r)$, so $\pi(r) \geq \int_r^{\overline{v}}(v - v_0)dF_2(v)$. Consider the case where bidder values are perfectly correlated and together take the observed distribution of the second-highest value; this joint distribution is symmetric, affiliated, and conditionally independent, and since $F_1(r) = F_2(r)$, the bound is achieved. Since $\underline{\pi}$ is strictly decreasing above v_0, this distribution also achieves the lower bound on r^*.

$\underline{\pi}$ and $\overline{\pi}$ Stochastically Increasing

Write $\underline{\pi}(r)$ as $E_{v^2}\left[(v^2 - v_0)\mathbf{1}_{v^2 \geq r}\right]$. For $r \geq v_0$, $(v^2 - v_0)\mathbf{1}_{v^2 \geq r}$ is increasing in v^2, so its expectation increases with a first-order stochastic increase in the distribution of v^2.

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[12] Pick r_1, r_2 s.t. $r^I < r_1 < r_2 < r'$, and let $\alpha^* \equiv \inf_{r \in [r_2, r']} \int_{r_1}^{r}(v - v_0)dF_2(v)/[(r - v_0)(F_2(r) - F_1^I(r))] > 0$. If $\frac{f_1(r_3)}{f_1^I(r_3)} \leq \alpha^*$ for any $r_3 \in [r_2, r']$, then $\pi(r') \leq \pi(r_3) \leq \underline{\pi}(r_1)$. If not, then $\pi(r') - \pi(r_2) \leq \alpha^*(\overline{\pi}(r') - \overline{\pi}(r_2))$. So for any continuous f, $\pi(r') - \pi(r^I) \leq \max\{\underline{\pi}(r_1) - \underline{\pi}(r^I), \alpha^*(\overline{\pi}(r') - \overline{\pi}(r_2))\} < 0$.

Similarly, write $\overline{\pi}(r)$ as $E_{v^2}\left[\mathbf{1}_{v^2<r}\psi(r)(r-v_0)+\mathbf{1}_{v^2\geq r}(v^2-v_0)\right]$ where

$$\psi(r)=\frac{F_2(r)-H^n(r)}{F_2(r)}=1-\frac{H^n(r)}{nH^{n-1}(r)-(n-1)H^n(r)}=1-\frac{1}{\frac{n}{H(r)}-(n-1)}$$

is decreasing in $H(r)$. A stochastic increase in v^2 decreases $F_2(r)$ and therefore $H(r)$, increasing $\psi(r)$; and with $\psi(r)$ fixed and $r\geq v_0$, the expression in square brackets is increasing in v^2; so a stochastic increase in v^2 increases $\overline{\pi}(r)$.

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