Unobserved Correlation in Ascending Auctions: Example And Extensions

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1 Introduction

In private-value ascending auctions, the winning bidder's willingness to pay is not observed. Under assumptions weaker than independent private values, the joint distribution of bidder valuations is not identified (see Athey and Haile (2002)), so the expected revenue at a positive reserve price, and the reserve price that would maximize expected revenue, are not uniquely pinned down. In a separate paper, Quint (2008), I calculate tight upper and lower bounds on these two measures for the symmetric affiliated private values case; the upper bounds coincide with the values achieved under the special case of independent private values. Here, I give an illustrative example and several extensions.

2 Model

A seller has one indivisible object to sell, and values it at v_0. There are n potential buyers, with private values $v_1, \ldots, v_n$ whose joint distribution f is symmetric and has bounded support $[\underline{v}, \overline{v}]^n$. Let $v^1 \geq v^2 \geq \cdots \geq v^n$ be the order statistics of the values, and $F_i(\cdot)$ the cumulative distribution function[2] of v^i.

I consider a stylized version of an ascending auction: the seller announces a reserve price, and as long as at least one buyer's valuation exceeds this price, the object is sold to the buyer with the highest valuation, at a price which is the greater of the reserve price and the second-highest valuation. I assume that the distribution of this second-highest valuation (F_2) is known exactly, but that no further information is available about F_1.[3]

Note that expected seller profit can be written as

$$\pi(r) \quad = \quad (r - v_0)(F_2(r) - F_1(r)) + \int_r^{\overline{v}} (v - v_0) dF_2(v) \tag{1}$$

Given the distribution $F_2(\cdot)$, define $H(\cdot)$ implicitly by

$$F_2(r) \quad = \quad n(n-1) \int_0^{H(r)} s^{n-2}(1-s) ds \tag{2}$$

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[2] Cumulative distribution functions in this paper exclude any mass at the point being considered, that is, $F_i(r) \equiv \Pr(v^i < r)$, not $\Pr(v^i \leq r)$.

[3] The revenue assumption, and precise knowledge of F_2, would hold exactly for second-price sealed-bid auctions and button auctions, and up to a bid increment for first-price auctions with proxy bidding and any ascending auction without jump bids.

or, equivalently, $F_2(r) = nH^{n-1}(r) - (n-1)H^n(r)$, and define

$$\begin{aligned}
\underline{\pi}(r) &\equiv \int_r^{\overline{v}} (v - v_0) dF_2(v) \\
\overline{\pi}(r) &\equiv (r - v_0)(F_2(r) - H^n(r)) + \int_r^{\overline{v}} (v - v_0) dF_2(v)
\end{aligned} \tag{3}$$

The main result from Quint (2008):

Theorem 1. *Suppose bidders have private values which are symmetric and affiliated.*

1. *For any $r > v_0$, expected revenue $\pi(r)$ is bounded above by $\overline{\pi}(r)$ and below by $\underline{\pi}(r)$, and both bounds are tight*

2. *Suppose in addition that $\overline{\pi}$ is continuous, differentiable, and strictly quasiconcave; let r^I be its maximizer. Then the optimal reserve price r^* is bounded above by r^I and below by v_0, and both bounds are tight*

3 An Example With A Parameter For Correlation

Let $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ be *i.i.d.* draws from the uniform distribution on $[0, 1]$, and let $\epsilon^1 \geq \epsilon^2 \geq \ldots \geq \epsilon^n$ be their order statistics. Let bidders i's private value be

$$v_i = \rho\epsilon^2 + (1 - \rho)\epsilon_i \tag{4}$$

Since $v^2 = \epsilon^2$, the observed distribution F_2 does not depend on ρ; thus, this example allows us to parameterize the correlation between bidder values while holding fixed the data that would be observed. ($\rho = 0$ corresponds to the case of independent private values, while $\rho = 1$ would be perfectly correlated values.) For simplicity, let $v_0 = 0$.

Result 1. *For $\rho < 1$, expected revenue is*

$$\pi(r) = \begin{cases} r^n - \frac{n-1}{n+1} r^{n+1} - \frac{r^{n+1}}{1-\rho} + \frac{n-1}{n+1} & for \quad r \leq 1 - \rho \\ \\ r^n - \frac{n-1}{n+1} r^{n+1} - \frac{r^{n+1}}{1-\rho} + \frac{n-1}{n+1} + \frac{r(r+\rho-1)^n}{\rho^{n-1}(1-\rho)} & for \quad r > 1 - \rho \end{cases} \tag{5}$$

and the revenue-maximizing reserve price is

$$r^* = \frac{n}{n - 1 + \frac{n+1}{1-\rho}} \tag{6}$$

both of which are strictly decreasing in ρ.

Since $\rho = 0$ corresponds to independent private values, both $\pi(r)$ and r^* are bounded above by their value under IPV, and both are decreasing in the degree of unobserved correlation.

4 Relaxing Affiliation

Theorem 2. *Suppose that $v_1, v_2, \ldots, v_n$ are conditionally independent[4] but not necessarily affiliated.*

1. *The same revenue bounds hold: $\underline{\pi}(r) \leq \pi(r) \leq \overline{\pi}(r)$, with both bounds being tight.*
2. *The lower bound on r^* is still v_0, and still tight.*
3. *It is not necessarily true that $r^* \leq r^I = \arg\max_r \overline{\pi}(r)$. An upper bound (not tight) on r^* is provided by*
$$(r^* - v_0)H^n(r^*) \leq \int_{v_0}^{r^*} F_2(v)dv \tag{7}$$

Thus, the first part of Theorem 1 extends to conditionally independent values. (In fact, a sufficient condition for the revenue bounds is that for any $v \in [v_0, \overline{v}]$, $\Pr(v_i < v)$ is increasing in the number of other bidders with values $v_j < v$.) However, the second part of Theorem 1 does not fully extend to conditionally independent values: in the appendix, we give an example where $r^* > r^I$.

Equation 7 is still a nontrivial upper bound on r^*, since as v approaches $\overline{v}$, $(v - v_0)H^n(v)$ approaches $v - v_0$ and $\int_{v_0}^{v} F_2(v)$ does not.

5 What If Losing Bids Are Observed

Above, I assumed that the distribution of v^2 was known exactly, but that no other information was available about the joint distribution f of values. Here, I consider the inferences that can be made from other losing bids. Let b_i denote bidder i's bid, and b^i the i^{th} highest bid.

As in Haile and Tamer (2003), I do not interpret a losing bids as an exact indication of that bidder's willingness to pay, only as a lower bound on it. Thus, no observations will be able to falsify perfect correlation of bidder values, which is used to prove the lower bounds on both $\pi(r)$ and r^*. These lower bounds, therefore, are unchanged if losing bids are observed.

On the other hand, if losing bids are sufficiently high (close enough to v_2), they may falsify the assumption of independence, in which case a tighter upper bound on $\pi(r)$ will follow, which may in turn lead to a tighter upper bound on r^*. As a demonstration, consider the case of symmetric, affiliated private values when the distribution of the third-highest bid b^3 is observed along with F_2. Similar results will hold for other losing bids.

Let $G_3(\cdot)$ be the observed distribution of b^3, and note that by assumption, $v^3 \geq b^3$, and therefore $F_3(r) \leq G_3(r)$. Then under symmetry and affiliation,

$$\frac{F_1(r)}{\frac{1}{n}(F_2(r) - F_1(r))} \geq \frac{\frac{1}{n}(F_2(r) - F_1(r))}{\frac{1}{{}_nC_2}(F_3(r) - F_2(r))} \geq \frac{\frac{1}{n}(F_2(r) - F_1(r))}{\frac{1}{{}_nC_2}(G_3(r) - F_2(r))} \tag{8}$$

(The first inequality is from the proof of Theorem 1 in Quint (2008).) Simplifying gives

$$\frac{F_1(r)}{(F_2(r) - F_1(r))^2} \geq \frac{n-1}{2n} \frac{1}{G_3(r) - F_2(r)} \tag{9}$$

[4] Conditionally independent values satisfy $f(v_1, v_2, \ldots, v_n) = E_\theta \{f(v_1|\theta)f(v_2|\theta)\cdots f(v_n|\theta)\}$ for some family of distributions $f(\cdot|\theta)$.

Since the left-hand side is strictly increasing in $F_1(r)$, Equation 9 gives a lower bound on $F_1(r)$, which gives an upper bound on $\pi(r)$. As we saw above, an upper bound $\pi(r) \leq \overline{\pi}(r)$ imposes an upper bound $\overline{\pi}(r^*) \geq \underline{\pi}(v_0)$ on r^*; if the losing bids are high enough, this bound may be lower than r^I.

6 Auctions With Entry Fees

Results for auctions with entry fees are similar to the results for auctions without. First, consider auctions where potential bidders must pay an entry fee e before learning their private values and participating in the auction. (That is, players learn e and r but not v_i, decide (simultaneously) whether to pay e and participate, learn v_i, and then the auction is held.) I refer to this as an *early* entry fee. It is easy to show that in such an auction with symmetric bidders, the seller maximizes expected revenue by setting $r = v_0$ and using the entry fee to extract all expected surplus from the sellers by setting

$$e = e^* \equiv \frac{1}{n}\left(\int_{v_0}^{\overline{v}} (v - v_0) dF_1(v) - \pi(v_0)\right) \tag{10}$$

Let e^I denote the value of e^* when bidder values are independent (that is, substituting $H^n(v)$ for $F_1(v)$ in Equation 10).

Theorem 3. *Suppose bidder values are symmetric and affiliated or conditionally independent. In an auction with an early entry fee, the optimal reserve price is v_0; the optimal entry fee is bounded below by 0 and above by e^I, with both bounds being tight.*

Finally, consider the harder problem of auctions with an entry fee which is paid *after* bidders learn their valuations. That is, bidders learn e, r, and v_i, decide (simultaneously) whether to participate, and then the auction is held among those who enter. The results are not as complete, but I do offer the following bounds on the revenue-maximizing parameters:

Theorem 4. *Suppose bidder values are symmetric and affiliated or conditionally independent. In an auction with a late entry fee, the optimal reserve price and entry fee (r^*, e^*) are not bounded away from $(v_0, 0)$; an upper bound on $r^* + e^*$ is given by*

$$\int_{r^*+e^*}^{\overline{v}} (v - v_0) d(H^n(v)) \geq \pi(v_0, 0) \tag{11}$$

References

1. Athey, S. and P. Haile (2002), "Identification of Standard Auction Models", Econometrica 70.6

2. Haile, P. and E. Tamer (2003), "Inference with an Incomplete Model of English Auctions", JPE 111.1

3. Quint, D. (2008), "Unobserved Correlation in Private-Value Ascending Auctions", Economics Letters 100.3

Appendix – Proof of Result 1

We begin by calculating $F_1(r)$, the cumulative density function of v^1:

$$F_1(r) = \Pr(v^1 < r) = \Pr(\rho\epsilon^2 + (1-\rho)\epsilon^1 < r)$$

Since we know (by construction) the distribution of ϵ^2, F_2, we can rewrite this as

$$\begin{aligned} F_1(r) &= \int \Pr(\rho\epsilon^2 + (1-\rho)\epsilon^1 < r | \epsilon^2 = x) dF_2(x) \\ &= \int_0^r \Pr(\epsilon^1 < \tfrac{r-\rho\epsilon^2}{1-\rho} \,|\epsilon^2 = x) dF_2(x) \end{aligned}$$

Now, the distribution of ϵ^1, conditioned on a given value of ϵ^2, is simply the distribution of ϵ conditional on being above that value. That is, knowing that $\epsilon^2 = x$ makes the conditional distribution of ϵ^1 the uniform distribution on $[x, 1]$. So

$$\Pr\left(\epsilon^1 < \frac{r-\rho\epsilon^2}{1-\rho}\middle| \epsilon^2 = x\right) = \begin{cases} 1 & \text{if } x < \frac{r+\rho-1}{\rho} \\ (\frac{r-\rho x}{1-\rho} - x)/(1-x) & \text{if } x \in [\frac{r+\rho-1}{\rho}, r] \\ 0 & \text{if } x > r \end{cases}$$

Plugging this into the integrals above gives

$$F_1(r) = \begin{cases} \int_0^r \frac{r-x}{(1-\rho)(1-x)} dF_2(x) & \text{if } r \leq 1-\rho \\ \\ F_2(\frac{r+\rho-1}{\rho}) + \int_{(r+\rho-1)/\rho}^r \frac{r-x}{(1-\rho)(1-x)} dF_2(x) & \text{if } r > 1-\rho \end{cases}$$

Since $v^2 = \epsilon^2$, $F_2(x) = nx^{n-1} - (n-1)x^n$; plugging in, integrating, and simplifying then gives

$$F_1(r) = \begin{cases} \frac{r^n}{1-\rho} & \text{if } r \leq 1-\rho \\ \\ \frac{r^n}{1-\rho} - \frac{(r+\rho-1)^n}{\rho^{n-1}(1-\rho)} & \text{if } r > 1-\rho \end{cases}$$

Case 1 : $r > 1-\rho$

As noted in equation 1, when $v_0 = 0$, expected revenue is

$$\pi(r) = r(F_2(r) - F_1(r)) + \int_r^1 x dF_2(x)$$

When $r > 1-\rho$, this is

$$\begin{aligned} \pi(r) &= r(F_2(r) - F_1(r)) + \int_r^1 x dF_2(x) \\ &= r\left(nr^{n-1} - (n-1)r^n - \tfrac{r^n}{1-\rho} + \tfrac{(r+\rho-1)^n}{\rho^{n-1}(1-\rho)}\right) + \int_r^1 x\left(n(n-1)x^{n-2}(1-x)\right)dx \\ &= nr^n - (n-1)r^{n+1} - \tfrac{r^{n+1}}{1-\rho} + \tfrac{r(r+\rho-1)^n}{\rho^{n-1}(1-\rho)} + n(n-1)\int_r^1 \left(x^{n-1} - x^n\right)dx \\ &= nr^n - (n-1)r^{n+1} - \tfrac{r^{n+1}}{1-\rho} + \tfrac{r(r+\rho-1)^n}{\rho^{n-1}(1-\rho)} + (n-1)\; x^n\big|_{x=r}^1 - n(n-1)\; \tfrac{x^{n+1}}{n+1}\Big|_{x=r}^1 \\ &= nr^n - (n-1)r^{n+1} - \tfrac{r^{n+1}}{1-\rho} + \tfrac{r(r+\rho-1)^n}{\rho^{n-1}(1-\rho)} + (n-1) - (n-1)r^n - \tfrac{n(n-1)}{n+1} + \tfrac{n(n-1)}{n+1}r^{n+1} \\ &= nr^n - (n-1)r^n - (n-1)r^{n+1} + \tfrac{n(n-1)}{n+1}r^{n+1} - \tfrac{r^{n+1}}{1-\rho} + \tfrac{r(r+\rho-1)^n}{\rho^{n-1}(1-\rho)} + (n-1) - \tfrac{n(n-1)}{n+1} \\ &= r^n - \tfrac{n-1}{n+1}r^{n+1} - \tfrac{r^{n+1}}{1-\rho} + \tfrac{r(r+\rho-1)^n}{\rho^{n-1}(1-\rho)} + \tfrac{n-1}{n+1} \end{aligned}$$

and so

$$\begin{aligned}
\pi'(r) &= nr^{n-1} - (n-1)r^n - \tfrac{n+1}{1-\rho}r^n + \tfrac{(r+\rho-1)^n}{\rho^{n-1}(1-\rho)} + \tfrac{nr(r+\rho-1)^{n-1}}{\rho^{n-1}(1-\rho)} \\
&= n(1-\tfrac{r}{1-\rho})r^{n-1} - (n-1)r^n - \tfrac{r^n}{1-\rho} + (\tfrac{r+\rho-1}{\rho})^{n-1}\tfrac{1}{1-\rho}(r+\rho-1+nr) \\
&= \tfrac{1}{1-\rho}\big(-n(r+\rho-1)r^{n-1} - (n-1)(1-\rho)r^n - r^n + \\
&\quad (\tfrac{r+\rho-1}{\rho})^{n-1}(r+\rho-1) + nr(\tfrac{r+\rho-1}{\rho})^{n-1})\big) \\
&= \tfrac{1}{1-\rho}\Big(-(n-1)(r+\rho-1)r^{n-1} - (n-1)(1-\rho)r^n + (n-1)r(\tfrac{r+\rho-1}{\rho})^{n-1} \\
&\quad -(r+\rho-1)r^{n-1} + (r+\rho-1)(\tfrac{r+\rho-1}{\rho})^{n-1} - r^n + r(\tfrac{r+\rho-1}{\rho})^{n-1}\Big) \\
&< \tfrac{n-1}{1-\rho}\Big(-(r+\rho-1)r^{n-1} - (1-\rho)r^n + r(\tfrac{r+\rho-1}{\rho})^{n-1}\Big)
\end{aligned}$$

(Since $(1-r)(1-\rho) = 1-r-\rho+r\rho > 0$, $r > \frac{r+\rho-1}{\rho}$; the inequality then follows, since $r^n > r(\frac{r+\rho-1}{\rho})^{n-1}$ and $(r+\rho-1)r^{n-1} > (r+\rho-1)(\frac{r+\rho-1}{\rho})^{n-1}$.) Then

$$\begin{aligned}
\pi'(r) &< \tfrac{n-1}{1-\rho}\Big(-r\rho(\tfrac{r+\rho-1}{\rho})r^{n-2} - r(1-\rho)r^{n-1} + r(\tfrac{r+\rho-1}{\rho})^{n-1}\Big) \\
&< \tfrac{n-1}{1-\rho}\Big(-r\rho(\tfrac{r+\rho-1}{\rho})r^{n-2} - r(1-\rho)(\tfrac{r+\rho-1}{\rho})r^{n-2} + r(\tfrac{r+\rho-1}{\rho})^{n-1}\Big) \\
&< \tfrac{n-1}{1-\rho}\Big(-r(\tfrac{r+\rho-1}{\rho})r^{n-2} + r(\tfrac{r+\rho-1}{\rho})^{n-1}\Big) \\
&< 0
\end{aligned}$$

Since $\pi'(r) < 0$ for $r > 1-\rho$, we know that $r^* \in [0, 1-\rho]$.

To show that $\pi(r)$ is decreasing in ρ, rewrite expected revenue as

$$\pi(r) = r\left(F_2(r) - \int_0^r \min\left\{1, \frac{r-x}{(1-\rho)(1-x)}\right\} dF_2(x)\right) + \int_r^1 x dF_2(x)$$

which is strictly decreasing in ρ by inspection. (This simply requires that $\min\left\{1, \frac{r-x}{(1-\rho)(1-x)}\right\} \neq 1$ on a positive-measure (w.r.t. F_2) subset of $[0, r]$; this is the case on $[\frac{r+\rho-1}{\rho}, r]$.)

Case 2 : $r \leq 1-\rho$

For $r \leq 1-\rho$, $F_1(r) = \frac{r^n}{1-\rho}$, and so

$$\pi(r) = r^n - \frac{n-1}{n+1}r^{n+1} - \frac{r^{n+1}}{1-\rho} + \frac{n-1}{n+1}$$

which is decreasing in ρ by inspection. Differentiating,

$$\pi'(r) = nr^{n-1} - (n-1)r^n - \frac{n+1}{1-\rho}r^n$$

Note that π' has the same sign as $n - (n-1+\frac{n+1}{1-\rho})r$, so π is strictly quasiconcave. Thus, the first-order condition gives us the maximizer, which is

$$r^* = \frac{n}{n-1+\frac{n+1}{1-\rho}}$$

Proof of Theorem 2

As in the proof of Theorem 1 in Quint (2008), we show $F_1(r) \geq H^n(r)$; then $\pi(r) \leq \overline{\pi}(r)$ for $r \geq v_0$.

Define $\psi_1, \psi_2 : [0,1] \to [0,1]$ by $\psi_1(x) = x^n$ and $\psi_2(x) = nx^{n-1} - (n-1)x^n$. For a given distribution $H(\cdot)$ of one variable, then, $\psi_1(H(\cdot))$ and $\psi_2(H(\cdot))$ are the distributions of the highest and second highest, respectively, of n independent draws on H.

If values are conditionally independent, let $\{H_\theta\}$ be the set of distributions from which values may be independently drawn. It is easy to show that

$$F_2(r) = E_\theta \psi_2(H_\theta(r)) \quad \text{and} \quad F_1(r) = E_\theta \psi_1(H_\theta(r))$$

Next, we show that $\psi_1 \circ \psi_2^{-1}$ is convex. This is because

$$\left(\psi_1 \circ \psi_2^{-1}\right)'(s) = \frac{\psi_1'(\psi_2^{-1}(s))}{\psi_2'(\psi_2^{-1}(s))} = \frac{nt^{n-1}}{n(n-1)t^{n-2}(1-t)} = \frac{t}{(n-1)(1-t)}$$

where $t = \psi_2^{-1}(s)$; since this is increasing in t, and therefore s, $\left(\psi_1 \circ \psi_2^{-1}\right)'$ is increasing so $\psi_1 \circ \psi_2^{-1}$ is convex. Recall also that $H(r)$ was defined by $F_2(r) = \psi_2(H(r))$. Applying Jensen's inequality,

$$\begin{aligned}
F_1(r) &= E_\theta \psi_1(H_\theta(r)) \\
&= E_\theta \psi_1\left(\psi_2^{-1} \circ \psi_2(H_\theta(r))\right) \\
&= E_\theta \left(\psi_1 \circ \psi_2^{-1}\right)(\psi_2(H_\theta(r))) \\
&= E_\theta \left(\psi_1 \circ \psi_2^{-1}\right)(\psi_2(H_\theta(r))) \\
&\geq \left(\psi_1 \circ \psi_2^{-1}\right)(E_\theta \psi_2(H_\theta(r))) \\
&= \psi_1\left(\psi_2^{-1}(F_2(r))\right) \\
&= H^n(r)
\end{aligned}$$

From Equations 1 and 3, then,

$$\pi(r) - \overline{\pi}(r) \quad = \quad (r - v_0)(H^n(r) - F_1(r)) \quad \leq \quad 0$$

and the bound is tight because IPV is a special case of conditionally independent private values. The lower bound on π, as well as on r^*, is proved the same way as in Quint (2008).

To show that r^* is not necessarily lower than $\arg\max_r \overline{\pi}(r)$, we offer a counterexample. Let $n = 3$, $v_0 = 0$, and suppose that θ takes the values $0, 1$ with equal probabilities and

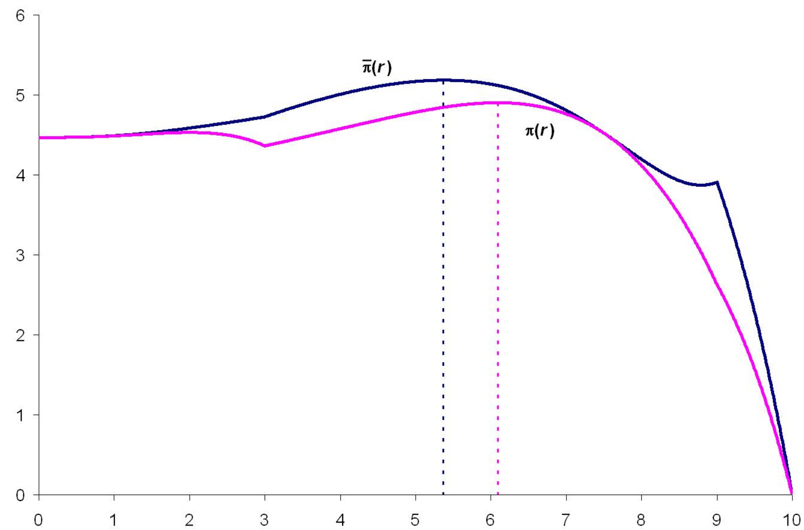
- when $\theta = 0$, bidder valuations are $i.i.d. \sim U[3,9]$
- when $\theta = 1$, bidder valuations are $i.i.d. \sim U\left[[0,3] \cup [9,10]\right]$.

For $i \in \{1,2\}$,

$$F_i(x) = \begin{cases} \frac{1}{2}\psi_i\left(\frac{x}{4}\right) & for \quad x \leq 3 \\ \frac{1}{2}\psi_i\left(\frac{3}{4}\right) + \frac{1}{2}\psi_i\left(\frac{x-3}{6}\right) & for \quad x \in (3,9) \\ \frac{1}{2}\psi_i\left(\frac{x-6}{4}\right) + \frac{1}{2} & for \quad x \geq 9 \end{cases}$$

This allows us to calculate a closed-form (if messy) expression for $\pi(r)$. While we don't have a closed-form expression for $H(r) = \psi_2^{-1}(F_2(r))$, we can calculate it, and therefore $\overline{\pi}(r)$, numerically.

Figure 1: An example with CIPV where $\pi(r) \leq \overline{\pi}(r)$ but $\arg\max_r \pi(r) > \arg\max_r \overline{\pi}(r)$.



It turns out that while $\pi(r) \leq \overline{\pi}(r)$ everywhere (as required by Theorem 2), $\pi(r)$ is maximized at $r^* = 6.09$, and $\overline{\pi}(r)$ at $r^I = 5.37$, as shown in Figure 1.

(In Figure 1, $\overline{\pi}$ is not quasi-concave, so this example is not an exact contradiction of Theorem 1 in Quint (2008) when affiliation is relaxed. However, we can eliminate the "lip" in $\overline{\pi}$ near $r = 9$ without changing the result. If rather than the uniform distribution on $[3, 9]$, the CDF of each bidder's value when $\theta = 0$ is $1 - \sqrt{1 - \left(\frac{x-3}{6}\right)}$ for x between 3 and 9, then $\overline{\pi}$ is strictly quasi-concave, and $\arg\max_r \pi(r)$ is still strictly greater than $\arg\max_r \overline{\pi}(r)$.)

As for the new upper bound on r^*,

$$\overline{\pi}(r^*) \geq \pi(r^*) \geq \pi(v_0) \geq \underline{\pi}(v_0)$$

(the middle inequality is the optimality of r^*, the first and third are simply the bounds on π). $\overline{\pi}(r^*) \geq \underline{\pi}(v_0)$ can be written as

$$(r^* - v_0)(F_2(r^*) - H^n(r^*)) \geq \int_{v_0}^{r^*} (v - v_0) dF_2(v)$$

Integrating the right-hand side by parts and simplifying gives

$$(r^* - v_0)H^n(r^*) \leq \int_{v_0}^{r^*} F_2(v) dv$$

Proof of Theorem 3

An auction with a reserve price $r^* = v_0$ and entry fee

$$e^* = \frac{1}{n} E\left(\max\{v^1, v_0\} - \max\{v^2, v_0\}\right) \tag{12}$$

achieves efficiency and extracts all bidder surplus; thus, it must be optimal. (It is not hard to show that in nondegenerate cases, this auction is uniquely optimal.) Since $\max\{v^1, v_0\}$ is an increasing function of v^1, its expectation, and therefore e^*, are increasing functions in the distribution F_1 (with respect to first-order stochastic dominance).

We argued above that $F_1(r) \leq F_2(r)$ everywhere, with equality being attained for the perfectly-correlated joint distribution. Thus,

$$e^* \geq \frac{1}{n} E\left(\max\{v^2, v_0\} - \max\{v^2, v_0\}\right) = 0$$

forms a tight lower bound. Similarly, we showed that $F_1(r) \geq F_1^I(r)$ everywhere, and since independent values are a special case of symmetric affiliated values and conditionally independent values, equality is attainable, so

$$e^* \leq \frac{1}{n}\left(\int_{\underline{v}}^{\overline{v}} \max\{v, v_0\} dF_1^I(v) - E \max\{v^2, v_0\}\right) = e^I$$

forms a tight upper bound.

Proof of Theorem 4

Lower Bound

We again use the perfectly-correlated values example consistent with the observed distribution F_2, and claim that for any $(r, e) \neq (v_0, 0)$, $\pi(r, e) < \pi(v_0, 0)$.

If $e = 0$ and $r \neq v_0$, all players enter, and the expected revenue is

$$\int_r^{\overline{v}} (v - v_0) dF_2(v) < \int_{v_0}^{\overline{v}} (v - v_0) dF_2(v) = \pi(v_0, 0)$$

Suppose, therefore, that $e > 0$ for the rest of the proof. For a given value of v, let $\pi_v(r, e)$ be the expected revenue (including entry fees) from the auction with reserve price r and entry fee e when all bidders have the private value v for the good, so that

$$\pi(r, e) = E_v \pi_v(r, e) = \int_{\underline{v}}^{\overline{v}} \pi_v(r, e) dF_2(v)$$

We assume that each player has an independent entry strategy $\tau_i : [\underline{v}, \overline{v}] \to [0, 1]$ giving their probability of entering for each realization of their private value v. Note that no player will ever enter when $v < e + r$, so $\pi_v = 0$ for $v < e + r$.

For a given v, we consider two cases: when only one player considers entering ($\tau_i(v) = 0$ for all i but at most one), and when more than one consider entering. In the first case, letting x be the player who may enter, $\pi_v(r, e) = \tau_x(v)(e + r - v_0)$; since τ_x is zero when $v < e + r$, we know that

$$\pi_v(r, e) \leq \max\{0, (e + r - v_0) \times \mathbf{1}_{v \geq e+r}\}$$

In the second case, note that the revenue from the auction, *excluding* the entry fees, is 0 when nobody enters, $r - v_0$ when one player enters, and $v - v_0$ when at least two enter; thus, we can express total expected revenue as

$$\begin{aligned}\pi_v(r,e) &= \left(\sum_i \tau_i(v)\right) e + \left(\sum_i \left(\tau_i(v)\prod_{j\neq i}(1-\tau_j(v))\right)\right)(r-v_0) \\ &\quad + \left(1-\prod_i(1-\tau_i(v)) - \sum_i \left(\tau_i(v)\prod_{j\neq i}(1-\tau_j(v))\right)\right)(v-v_0)\end{aligned}$$

Now, entering bidders get no surplus from an auction if any other bidders enter, since the price paid is equal to their private value; so equilibrium play requires that for each i, either $\tau_i(v)=0$, or $-e+(\prod_{j\neq i}(1-\tau_j(v)))(v-r)\geq 0$. In either case,

$$\tau_i(v)e \leq \tau_i(v)\prod_{j\neq i}(1-\tau_j(v))(v-r)$$

Plugging this into the expression for $\pi_v(r,e)$ and simplifying gives

$$\pi_v(r,e) \leq \left(1-\prod_i(1-\tau_i(v))\right)(v-v_0)$$

Now, if more than one player considers entering, let y be the player with the second-highest value of $\tau_i(v)$. By assumption, $\tau_y(v)>0$, so equilibrium play requires

$$e \leq \prod_{j\neq y}(1-\tau_j(v))(v-r)$$

or $\prod_{j\neq y}(1-\tau_j(v)) \geq \frac{e}{v-r} \geq \frac{e}{\overline{v}-r}$. Letting x again be the player with the highest value of $\tau_i(v)$, we know that $1-\tau_y(v)) \geq 1-\tau_x(v) \geq \frac{e}{\overline{v}-r}$, so $\prod_i(1-\tau_i(v)) \geq \left(\frac{e}{\overline{v}-r}\right)^2$; thus, when more than two bidders consider entering,

$$\pi_v(r,e) \leq \left(1-\left(\frac{e}{\overline{v}-r}\right)^2\right)(v-v_0)$$

Thus, we have now shown that given equilibrium play by the bidders,

$$\pi_v(r,e) \leq \mathbf{1}_{v\geq e+r}\max\left\{0, e+r-v_0, \left(1-\left(\frac{e}{\overline{v}-r}\right)^2\right)(v-v_0)\right\}$$

This expression is everywhere weakly less than $\max\{0, v-v_0\}$, and strictly less than $v-v_0$ on $(v_0,\overline{v}]-\{e+r\}$. Thus,

$$\pi(r,e) = \int_{\underline{v}}^{\overline{v}} \pi_v(r,e) < \int_{v_0}^{\overline{v}}(v-v_0)dF_2(v) = \pi(v_0,0)$$

Since this argument holds for any $(r,e)\neq(v_0,0)$, it follows that $(r^*,e^*)=(v_0,0)$ must be optimal.

Upper Bound

For the upper bound on r^*+e^*, note that for any (r,e), the maximum possible surplus (to both the seller and the buyers) is $\int_{r+e}^{\overline{v}}(v-v_0)dF_1(v)$ since nobody will enter when $v^1<r^*+e^*$. Since in equilibrium, bidders must have nonnegative expected payoff,

$$\pi(r,e) \leq \int_{r+e}^{\overline{v}}(v-v_0)dF_1(v) \leq \int_{r+e}^{\overline{v}}(v-v_0)dF_1^I(v)$$

(The last inequality is because $(v-v_0)\mathbf{1}_{v>r+e}$ is a nondecreasing function of v, so its expectation is increasing with respect to first-order stochastic dominance, and we showed in the proof of Theorem 1 that $F_1(\cdot)\geq F_1^I(\cdot)$ everywhere.) Optimality of (r,e) implies $\pi(r,e)\geq\pi(v_0,0)=\pi(v_0)$; combining the inequalities gives $\int_{r+e}^{\overline{v}}(v-v_0)dF_1^I(v)\geq\pi(v_0)$, completing the proof.