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Pooling with Essential and Nonessential Patents†

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Several recent technological standards were accompanied by patent pools—arrangements to license relevant intellectual property as a package. A key distinction made by regulators—between patents essential to a standard and patents with substitutes—has not been addressed in the theoretical literature. I show that pools of essential patents are always welfare increasing, while pools which include nonessential patents can be welfare reducing—even pools limited to complementary patents and stable under compulsory individual licensing. If pools gain commitment power and price as Stackelberg leaders, this reduces, and can reverse, the gains from welfare-increasing pools. (*JEL* D43, D45, K21, L13, L24, O34)

When firms with market power sell complementary goods, their combined price will typically be higher than if both were sold by a single monopolist. This effect, first understood by Cournot (1838) and later termed double marginalization, can be particularly severe in the context of intellectual property. In high-tech fields where innovation is rapid and cumulative, a large number of patents may touch on the same new technology; double marginalization can make the technology expensive to commercialize, harming downstream producers and consumers as well as the innovators the patent system was designed to reward.

One tool to address this problem is a *patent pool*—an agreement by multiple patent holders to share a group of patents among themselves or to license them as a package to third parties. Patent pools in the United States go back to the 1850s, the first one involving sewing machine patents. Lerner and Tirole (2007) claim that "in the early days of the twentieth century … many (if not most) important manufacturing industries had a patent pooling arrangement." In 1917, with airplanes needed for World War I, Franklin Roosevelt (then Assistant Secretary of the Navy) pushed US aircraft manufacturers into a patent pool because ongoing litigation between the Wright Company and Curtiss Company had choked off aircraft production. In the last decade, several patent pools have formed in conjunction with technological

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standards, beginning with the MPEG-2 video and DVD standards in the late 1990s;[1] Lerner and Tirole (2007) cite one estimate that "sales in 2001 of devices based in whole or in part on pooled patents were at least $100 billion."

When patents in a pool are complements, the pool can lower their combined price and increase licensing revenues—as well as reduce transaction costs (by reducing the number of individual licensing agreements required to make use of the technology) and the risk of holdup by the final patent holders. However, patent pools have also been used to eliminate competition between rival technologies, facilitate collusion, and even administer cartels. In the early 1900s, a pool administered by National Harrow specified which products its licensees could produce and fixed the prices in the downstream market.[2] The Hartford-Empire pool dominated glassware manufacturing in the 1940s and used licensing terms to set production quotas and discourage entry into the market. In the early 1990s, Summit and VISX, which by 1996 were the only two companies with FDA-approved technologies for laser eye surgery, formed a patent pool and set a standard licensing fee of $250 for each use of either firm's technology; according to the FTC, "Instead of competing with each other, the firms placed their competing patents in the patent pool in order to share the proceeds each and every time a Summit or VISX laser was used."

Antitrust treatment of patent pools has evolved significantly over time. In 1902, the Supreme Court upheld the National Harrow pool, noting that "The very object of these [patent] laws is monopoly" (*E. Bement and Sons v. National Harrow*, 186 US 70 (1902)) and that patent law trumped the twelve-year-old Sherman Act. Within ten years, however, the court had backtracked and begun to examine overly restrictive licensing terms. The Hartford-Empire pool mentioned above was ruled to be an antitrust violation because it restricted downstream production; the Supreme Court ruled that the pool itself was legal and Hartford-Empire was free to charge whatever royalty rate it chose, but could not restrict its licensees further. Several other patent pools were found to be antitrust violations in the 1940s and early 50s, and very few new pools formed from the mid-1950s until the 1990s. In 1998, the Summit and VISX pool was dissolved following a settlement with the FTC. The 1995 FTC/DOJ Guidelines for the Licensing of Intellectual Property explicitly recognized the procompetitive possibilities of patent pools, and a number of pools related to technological standards appeared thereafter, beginning with MPEG-2 in 1997.

When evaluating the recent standard-based pools, regulators have drawn a key distinction between patents which are *essential* to comply with the standard and patents for which suitable substitutes exist. The recent pools have all been limited to essential patents, and provide for independent experts to determine which patents should be included on this basis; this is seen as a "competitive safeguard" to ensure that the pool does not have anticompetitive effects.[3]

[1] In addition to the pioneering MPEG-2 pool and two pools related to DVD-ROM and DVD-Video, recent standard-based pools have included Firewire, Bluetooth, RFID, AVC, DVB-T, and MPEG-4.

[2] The pool accounted for 90 percent of the US market for float spring tooth harrows, agricultural devices used to cultivate surface soil before planting.

[3] In business letters issued by the Department of Justice in 1998 and 1999 in response to the proposed DVD pools, Joel Klein writes, "One way to ensure that the proposed pool will integrate only complementary patent rights is to limit the pool to patents that are essential to compliance with the Standard Specifications. Essential patents by definition have no substitutes; one needs licenses to each of them in order to comply with the standard."

As I discuss in the next section, the existing literature on patent pools fails to distinguish between essential and nonessential patents, treating only the polar cases where every patent is essential (users need licenses to all of the patents) or no patent is essential (any set of patents of a particular size is sufficient). The aim of this paper is to understand the effects of a patent pool in an environment in which essential and nonessential patents coexist; given the emphasis placed by regulators on distinguishing essential from nonessential, this seems to be an empirically important case. I find the following:

- Patent pools containing *only essential patents* lead to lower prices for every product using the technology; greater consumer surplus in the downstream market; and higher licensing revenue for all patent holders outside the pool. Such pools are always welfare increasing. As in earlier models, however, such pools may be inefficiently small when individual patent holders can opt out without disrupting pool formation.
- Patent pools containing patents required only for one of several competing products, or *nonessential patents which are perfect complements,* will reduce the price of one product, but can increase or decrease the prices of others. Such pools increase total consumer surplus, but may harm some individual consumers; they increase revenues of outsiders whose patents are complementary to the pool, but decrease revenues to others. The overall welfare effect can be either positive or negative. This is in contrast to previous papers, which found that complementarity of the patents within the pool was sufficient for a pool to be welfare increasing. (The difference is that my model allows for the patents in a pool to be complements to each other, yet have substitutes outside the pool; in earlier models, this was impossible.)
- As in earlier models, pools containing patents which are *substitutes* will tend to reduce consumer surplus and overall welfare.
- In contrast to previous models, robustness to *compulsory individual licensing* (the forced availability of pooled patents individually as well as through the pool) is *not* a sufficient screen for efficiency: pools of complementary nonessential patents are robust to individual licensing, and may be welfare decreasing.
- To the extent that a pool is less flexible than individual patent holders and therefore acts as a *Stackelberg-style leader*, the welfare gains are reduced, and may even be reversed.

This suggests that, from a policy point of view, pools of essential patents should generally be allowed, and encouraged to be as inclusive as possible; pools which include complementary nonessential patents should be considered more cautiously, though not necessarily ruled out; and terms of a pooling arrangement which make the pool less flexible with respect to pricing (terms which "bind the pool's hands") should be eyed suspiciously, since (*provided the same pool would have formed without these terms*) they lead to higher prices and lower welfare.

The rest of this paper proceeds as follows. Section I discusses existing literature on patent pools. Section II introduces a model of price competition in a differentiated-products setting and characterizes its equilibrium. Section III presents the results following from this model. Section IV concludes with a discussion of the limitations of the model and avenues for future work. All results in the text are proved in the Appendix.

I. Related Literature

A few recent papers have presented theoretical models of patent pools. Gilbert (2004) (along with a detailed history of antitrust litigation) and Shapiro (2001) give simple models of competition with perfect substitutes and perfect complements, emphasizing the double-marginalization problem in the latter case. Lerner and Tirole (2004) model a world with n identical patents which need not be perfect substitutes nor perfect complements. They show that a pool containing all the patents is more likely welfare increasing when patents are more complementary; and that forcing pool participants to also offer their patents individually (compulsory individual licensing) destabilizes "bad" pools without affecting "good" ones. Brenner (2009) extends the Lerner and Tirole (2004) framework to consider smaller ("incomplete") pools containing only some of the patents. He explicitly models the fact that some patent holders may do better by remaining outside of the pool, and examines which pools will form under different formation procedures. Brenner compares the welfare achieved under a particular formation protocol to the outcome without a pool, and shows that compulsory individual licensing is a good screen for efficiency under this formation rule. Aoki and Nagaoka (2005) use a coalition-formation model (based on the framework of coalition games with externalities in Maskin 2003) to show that even when the patents are all essential and the grand coalition is therefore desireable, it will not form when the number of patents is large. Kim (2004) shows that in the presence of a patent pool, vertical integration—the presence of patent-holding firms in the downstream market—always lowers the price of the final product. Dequiedt and Versaevel (2008) show that allowing pool formation increases firms' R&D investments prior to the pool being formed.

In all of these models, patents are assumed to be interchangeable; that is, users derive value based on the number of patents they license, not which ones. This means that either all or none of the patents are essential. Under this assumption, the authors generally find that as long as the patents are complements, pools are socially desirable. (One exception is Choi 2003, who observes that pooling of weak patents (patents which are unlikely to be upheld in court) may reduce the incentive for producers to challenge them, leading to a loss in welfare even though the patents may be complements.) My model differs from these in allowing a given patent to have some complements and some substitutes.

In addition to this theoretical work, there have been several recent empirical examinations of patents pools. Lerner, Strojwas, and Tirole (2007) examine the licensing rules of various patent pools, finding that pools of complementary patents are more likely to allow independent licensing and require grantbacks, consistent with their theory. Layne-Farrer and Lerner (2011) examine the rules by which revenues are allocated and the effect this has on participation. Lampe and Moser (2010, 2011) find that patent pools appear to suppress further patenting and innovation related to the pool technology.

The discrete-choice logit model I use for consumer demand is similar to the one considered in Anderson, de Palma, and Thisse (1992) and Nevo (2000), among many others. These papers embed the various products in a multi-dimensional space of product characteristics, assuming that consumers have differentiated (generally linear) preferences over these characteristics. I focus on the simpler case where consumer preferences are over the products themselves, rather than their characteristics. (Also see Berry and Pakes 1993 for a discussion of the use of these and other techniques in merger analysis.) Also related is the problem of tying and bundling of consumer goods—see Kobayashi (2005) for a recent survey of the bundling literature, and Chen and Nalebuff (2006) for a recent contribution. The key difference between my model and all of these is that here, products are overlapping sets of different components, which may be produced (and priced) by different firms.

II. Model

In this section, I introduce my model. Patent holders license their patents to downstream manufacturers, who sell products based on the patented technologies to consumers. The products are substitutes, but consumers have differentiated tastes for them, so competition between patent holders is imperfect; I focus on this upstream competition by assuming the downstream sector is perfectly competitive. (Under perfect competition, the results are the same as if patent holders licensed their patents directly to end users, as in the previous literature.) The model is analogous to a model of two-level production: patent holders correspond to upstream (wholesale) suppliers of product components, which are assembled into products and sold by downstream retailers.

A. *Patents, Products, and Patent Holders*

There is a finite set $\mathcal{K} = \{1, 2, \ldots, K\}$ of distinct *products*. There is a finite set $\mathcal{J} = \{1, 2, \ldots, J\}$ of *patents*, each blocking one or more of the K products. For $k \in \mathcal{K}$, let $\mathcal{J}_k \subset \mathcal{J}$ denote the set of patents blocking product k, which must therefore be licensed in order to legally manufacture that product.

Define a patent to be *essential* if it is required for *every* product, that is, if it is contained in $\mathcal{J}_k$ for every $k \in \mathcal{K}$; and let $\mathcal{J}^E = \mathcal{J}_1 \cap \mathcal{J}_2 \cap \cdots \cap \mathcal{J}_K$ denote the set of essential patents. I will refer to patents outside of $\mathcal{J}^E$ as *nonessential.*

Next, we make the assumption that every nonessential patent blocks only a single product, that is, that the *only* overlap in the patents blocking any two products is the patents blocking all the products:

ASSUMPTION 1: *For any two products* $k' \neq k$, $(\mathcal{J}_k - \mathcal{J}^E) \cap (\mathcal{J}_{k'} - \mathcal{J}^E) = \emptyset$.

I will let $\mathcal{J}_k^N = \mathcal{J}_k - \mathcal{J}^E$ denote the set of nonessential patents blocking a particular product; Assumption 1 is the assumption that $\{\mathcal{J}_k^N\}_{k \in \mathcal{K}}$ forms a partition of the set of nonessential patents.

Assumption 1 is admittedly strong, but buys us a lot. It will lead to a supermodular structure on the pricing game, giving a number of sharp comparative statics. For motivation, one can think of the different products as competing applications or implementations of a common core technology—different cell phones that work on a particular network, for example. Essential patents are patents on the core technology (or the technological standard the products are built to comply with); nonessential patents might be tied to other aspects of product design (a touch-screen interface for a cell phone).[4]

I assume each patent is owned by a separate patent holder, and thus identify each patent holder $1, \ldots, J$ with the patent he or she owns.[5] Patent holders face no marginal costs, and set licensing fees p_j to maximize licensing revenue

$$\pi_j(p_j) = \begin{cases} p_j Q_k & \text{for} \quad j \in \mathcal{J}_k^N \\ p_j \left(\sum_{k' \in \mathcal{K}} Q_{k'} \right) & \text{for} \quad j \in \mathcal{J}^E, \end{cases} \tag{1}$$

where Q_k is the demand for product k.

B. *Manufacturers*

Manufacturers (or retailers) license patents from patent holders and sell the products $k \in \mathcal{K}$ to downstream consumers. I assume that manufacturers are perfectly competitive, with no fixed costs and constant marginal costs for each product which are the same across manufacturers (but may vary across the K products). Under perfect competition, manufacturers will charge consumers the patent holders' licensing fees plus their own marginal costs; for ease of exposition, I will assume that consumer utility terms (described below) are net of these marginal costs, so that the retail price of each product k is

$$P_k \equiv \sum_{j \in \mathcal{J}_k} p_j. \tag{2}$$

[4] One recent example that fits Assumption 1 is the intellectual property related to third-generation, or 3G, mobile telephony, which was designed to use five different radio interfaces, each backward-compatible with one second-generation network. Some patents are tied to one of these interfaces, while others relate to the 3G network as a whole.

[5] A patent holder with multiple patents of the same "type" (multiple essential patents, or multiple nonessential patents for the same product) would always license them together in equilibrium, so there is no loss in treating the entire portfolio as a single patent. A patent holder with different "types" of patents, however, would not fit well within this framework.

C. *Consumers*

Demand for the products comes from a measure 1 of consumers $l \in L$, who have idiosyncratic preferences for the various products. If consumer l buys product k at price P_k, her utility is

$$v_k^l - P_k. \tag{3}$$

Each consumer can buy at most one product; if consumer l declines to buy any of the products, her utility is v_0^l.

While many of the comparative statics in this paper hold for more general demand systems,[6] I will make a strong assumption here about the demand system in order to give sharper results.

ASSUMPTION 2 [*Logit Demand*]: *For* $k \in \mathcal{K} \cup \{0\}$, $v_k^l = v_k + \epsilon_k^l$, *where* $v_0 = 0$, $(v_1, \ldots, v_K)$ *are constants, and* ϵ_k^l *are i.i.d. draws (across k and l) from the standard double-exponential distribution.*[7]

Under Assumption 2, a fraction

$$Q_k = \frac{e^{v_k - P_k}}{1 + \sum_{k' \in \mathcal{K}} e^{v_{k'} - P_{k'}}} \tag{4}$$

of consumers demand product k, and consumer surplus can be written as $CS = E\max\left\{v_0^l, \max_{k \in \mathcal{K}}\{v_k^l - P_k\}\right\} = \log\left(1 + \sum_{k' \in \mathcal{K}} e^{v_k - P_k}\right)$.

D. *Equilibrium*

Manufacturers (retailers) and consumers are nonstrategic, so the licensing fees set by patent holders constitute the equilibrium, although consumer surplus will be included in welfare calculations. Define the following additional notation:

- $n_E \equiv |\mathcal{J}^E|$ the number of essential patents, and $P^E \equiv \sum_{j \in \mathcal{J}^E} p_j$ their combined price
- $n_k \equiv |\mathcal{J}_k^N|$ the number of nonessential patents covering product k, and $P_k^N \equiv \sum_{j \in \mathcal{J}_k^N} p_j$ their combined price.

The price of product k can be written as $P_k = P^E + P_k^N$.

[6] Specifically, Theorems 1, 2, 7, and 8—the price and welfare effects of pools of complements, and the effects of compulsory individual licensing and Stackelberg pools—rely mainly on the signs of the marginal effects of the number of each type of patent holder ($|\mathcal{J}^E|$ and $|\mathcal{J}_k^N|$) on equilibrium outcomes; they therefore extend almost completely to more general demand systems satisfying certain regularity conditions. See Quint (2013) for a discussion of the required conditions.

[7] $F(x) = \exp(-\exp(-(x + \gamma)))$, where γ is Euler's constant. See McFadden (1974) or Anderson, de Palma, and Thisse (1992) for more on logit demand.

LEMMA 1: *Under Assumptions* 1 *and* 2, *equilibrium exists*, *and is unique. Equilibrium values of "aggregate prices"* $(P_1^N, P_2^N, \ldots, P_K^N, P^E)$ *are the unique solution to the system of equations*

$$\begin{aligned} P_1^N(1 - Q_1) &= n_1 \\ P_2^N(1 - Q_2) &= n_2 \\ &\vdots \\ P_K^N(1 - Q_K) &= n_K \\ P^E(1 - Q_1 - Q_2 - \ldots - Q_K) &= n_E \end{aligned} \tag{5}$$

and individual patent holder prices are $p_j = \frac{1}{n_k} P_k^N$ *for* $j \in \mathcal{J}_k^N$ *and* $p_j = \frac{1}{n_E} P^E$ *for* $j \in \mathcal{J}^E$.

The proof is in the Appendix.

A pool of n essential patent holders behaves as if it were a single essential patent holder, so its formation has the same effect on equilibrium prices as a reduction in n_E by $n - 1$. Similarly, a pool of n patent holders in $\mathcal{J}_k^N$ has the same effect on aggregate prices as a reduction in n_k by $n - 1$. Thus, the effects of patent pools can be understood via the effects n_E and n_k have on equilibrium outcomes, through their effects on the solution to equation (5). While $(n_E, n_1, \ldots, n_K)$ are integers, the solution to equation (5) varies continuously with these parameters; by differentiating these equations with respect to each of these parameters, we can solve explicitly for the marginal effect of n_E or n_k on equilibrium prices, allowing us to sign many of the effects of pool formation.

III. Results

A. *Welfare Effects of Pools of Complements*

Pools of Essential Patents.—Previous papers, such as Shapiro (2001) and Lerner and Tirole (2004), have suggested that pools are generally welfare increasing when the patents being pooled are sufficiently strong complements. In my setting, this is not always the case even when patents within the pool are *perfect* complements: pools of essential patents are always welfare increasing, but pools of complementary nonessential patents can be welfare decreasing. First, the result on pools of essential patents:

THEOREM 1: *A pool containing only essential patents will:*

(*i*) *Lower the price* P_k *of every product and increase the surplus to each consumer.*

(*ii*) *Increase the demand* Q_k *for every product.*

(*iii*) *Increase the profits of every patent holder (essential and nonessential) outside the pool.*

(*iv*) *Increase total welfare*.

If the pool is profitable for its members, it therefore represents a Pareto improvement.

The proof, in the Appendix, is mostly by brute-force calculation specific to the logit demand system, but intuition for what drives the results can be gained from Quint (2013). In that paper, I show that in a demand setting like this, the equilibrium values of aggregate prices correspond to the equilibrium of a separate auxiliary game, which is a supermodular game in $(P_1^N, P_2^N, \ldots, P_K^N, -P^E)$, indexed by $(n_1, n_2, \ldots, n_K, -n_E)$. A pool of essential patents corresponds to a reduction in n_E; this therefore leads to a decrease in the equilibrium value of P^E, but increases in P_k^N for each k. The decrease in P^E can be shown to be greater, leading to lower overall prices $P_k = P_k^N + P^E$ for every product. Such a pool can then be shown to always increase total welfare.

Pools of Complementary Nonessential Patents.—A pool of nonessential patents, corresponding to a decrease in n_k, has more ambiguous effects. A decrease in n_k reduces the equilibrium level of P_k^N, as well as (by supermodularity) the level of $P_{k'}^N$ for every $k' \neq k$; and increases P^E. For $k' \neq k$, either effect could dominate, so the price $P_{k'}$ of a competing good could rise or fall in response to the pool. Likewise, the welfare effects are ambiguous:

THEOREM 2: *Choose a product k, and consider a pool of patent holders in $\mathcal{J}_k^N$. For each $k' \neq k$, such a pool will:*

(*i*) *Decrease P_k^N and $P_{k'}^N$ and increase P^E.*

(*ii*) *Increase Q_k and decrease $Q_{k'}$.*

(*iii*) *Decrease P_k and have an ambiguous effect on $P_{k'}$.*

(*iv*) *Increase total consumer surplus, although some individual consumers may be made worse off.*

(*v*) *Increase profits of essential patent holders and outsiders in $\mathcal{J}_k^N$ and decrease profits of patent holders in $\mathcal{J}_{k'}^N$.*

(*vi*) *Have an ambiguous effect on total welfare*.

To better understand the welfare effect, think of n_k as a continuous parameter, affecting prices through its effect on the solution to the equilibrium conditions (equation (5)). A pool of m of the original n patent holders in $\mathcal{J}_k^N$ corresponds to a reduction of n_k from n to $n - m + 1$; its effect on total welfare can be written as

$$\int_{n-m+1}^{n} \left[\sum_{k' \in \mathcal{K} - \{k\}} w_{k'}(P_k - P_{k'}) + w_0 P_k \right] dn_k, \tag{6}$$

where $w_{k'}$ and w_0 are positive weights (varying with n_k) with

$$(7) \qquad w_{k'} \propto \frac{Q_{k'}}{1 + P_{k'}^N \frac{Q_{k'}}{1 - Q_{k'}}} \qquad \text{and} \qquad w_0 \propto \frac{1 - \sum_{k \in \mathcal{K}} Q_k}{1 + P^E}.$$

For intuition about equation (6), recall that payments from consumers to patent holders are welfare neutral, so total surplus is simply the average value of v_k^l realized by consumers' product choices. When consumer l switches from product k' to k in response to a change in relative prices, the change in total surplus is $v_k^l - v_{k'}^l$. On the margin, consumers who switch are indifferent, meaning $v_k^l - P_k = v_{k'}^l - P_{k'}$, so the welfare change is $P_k - P_{k'}$. Similarly, a consumer who was demanding nothing and switches to product k, increases welfare by P_k. The weights $w_{k'}$ and w_0 represent the relative masses of consumers who move across each threshold in response to a marginal reduction in n_k. A decrease in n_k reduces P_k both absolutely and relative to $P_{k'}$, so these weights are all positive.

Writing the welfare change, as in equation (6), gives some idea of when a pool of complementary, nonessential patents will be welfare increasing:

- If product k is expensive relative to the other products, due either to a high average value v_k or a large number of nonessential patents n_k, then a pool containing patents in $\mathcal{J}_k^N$ is likely welfare increasing.

  In that case, consumers switching to k from other products increase welfare, as do consumers switching to k from nothing. (Prices $P_{k'}$ and $P_{k''}$ do not change identically, so there will also be some consumers switching between other products, but these effects will be overwhelmed by those consumers switching to product k.) If product k is still the most expensive product after the pool forms, then all terms in the integrand in equation (6) are positive over the whole range of integration, and the pool will be welfare increasing.[8]
- If most of the market is unserved—each $Q_{k'}$ is small and $1 - \sum_{k'} Q_{k'}$ large—then regardless of the relative prices of the products, a pool of patents in $\mathcal{J}_k^N$ is likely welfare increasing.

  In that case, the last term of the integrand dominates; most of the "action" consists of consumers switching from demanding nothing to demanding product k, which increases total surplus. This means, for example, that when n_E is large, a pool of patents in $\mathcal{J}_k^N$ will be welfare increasing.
- On the other hand, if most of the market is being served and product k is already relatively cheap, then a pool of patents in $\mathcal{J}_k^N$ is likely to be welfare decreasing.

To illustrate the contrast between the last two cases, I offer the following example. For all the examples in this paper, a full description of the equilibrium in each case, both before and after the pool in question forms, is given in the Appendix.

[8] This can be seen as a loose analog to the "concertina theorem" in international trade (originally posited by Meade 1955): that in the absence of complementarities, when different goods face different tariffs, reducing the highest tariff is unambiguously welfare increasing. See Bertrand and Vanek (1971) for a discussion.

Example 1: Let $K = 3$. Let $(v_1, v_2, v_3) = (10, 10, 5)$, and $n_1 = n_2 = n_3 = 3$. Since product 3 is lower-value than 1 and 2, in the absence of a patent pool, its price will be lower in equilibrium. Consider a pool consisting of the three patents in $\mathcal{J}_3^N$.

First, suppose that $n_E = 3$. In the absence of a pool, equilibrium prices are $P_1 = P_2 = 10.557$ and $P_3 = 9.488$, leading to a significant fraction of the market (46.3 percent) not being served. Even though product 3 is cheaper than the other two products, the pool in question is still welfare increasing: when the three patent holders in $\mathcal{J}_3^N$ form a pool, total welfare rises from 6.426 to 6.428.

Second, however, suppose that $n_E = 1$. This leads to lower equilibrium prices—$P_1 = P_2 = 9.439$ and $P_3 = 7.626$ before pool formation—as well as a bigger gap between P_3 and the other prices, and a smaller fraction of the market (21.8 percent) being unserved. In that case, a pool of patents in $\mathcal{J}_3^N$ decreases total welfare, from 8.869 to 8.775.

Pools of Essential and Nonessential Patents.—The addition of patents in $\mathcal{J}_k^N$ to an existing pool of essential patents will have the same effects as a decrease in n_k; the effects will therefore be the same as in Theorem 2. The formation of a pool containing patents in both $\mathcal{J}^E$ and $\mathcal{J}_k^N$ will correspond to decreases in both n_E and n_k. The results will be a hybrid of Theorems 1 and 2: the pool will result in a decrease in P_k, an increase in Q_k, an increase in total consumer surplus, and an increase in profits to outsiders in $\mathcal{J}^E$ and $\mathcal{J}_k^N$; the other effects will be ambiguous.

B. *Profitability of Pools of Complements*

When Will a Pool Be Profitable?—A trade-off between two forces determines when a pool of complementary patents will increase the profit of its members. Working in favor of the pool is the usual benefit of merger: the individual patent holders internalize the effect their pricing decisions have on each other (the double-marginalization problem among pool members is solved). Working again the pool, however, are the price responses of the patent holders outside the pool. When the pool contains essential patents, all outsiders (nonessential patent holders and any excluded essential patent holders) respond to the pool by raising their own licensing fees, reducing demand and lowering pool profit. When the pool contains nonessential patents for one product, nonessential patent holders for competing products lower their licensing fees, and essential patent holders again raise theirs, again lowering pool profitability.

However, when the double-marginalization problem is particularly severe, the first effect will dominate. Thus, fixing the other primitives of the model, when n_E is sufficiently large, a pool of essential patents will always be profitable; and when n_k is sufficiently large, a pool of nonessential patents for product k will always be profitable. This holds both for pools of a fixed size, and for pools whose size increases with n_E or n_k.

THEOREM 3: *Fix* $m \geq 0$. *If* n_E *is sufficiently large*,

(*i*) *a pool of* $m + 2$ *essential patents increases the profits of its members;*

(ii) *a pool containing* all but *m essential patents increases the profits of its members.*

Similarly, fixing $m \geq 0$ *and* $k \in \mathcal{K}$, *if* n_k *is sufficiently large,*

(iii) *a pool of* $m + 2$ *patents in* $\mathcal{J}_k^N$ *increases the profits of its members;*

(iv) *a pool containing all but m patents in* $\mathcal{J}_k^N$ *increases the profits of its members.*

Outsiders and the Incentive to Free Ride.—Thus, when the number of complementary patents is large, a pool will generally be profitable. However, as in many merger situations,[9] much of the benefit of a pool goes not to the insiders but the outsiders, that is, the patent holders who remain outside the pool. This creates a free rider problem which may prevent pools from reaching their optimal size. Aoki and Nagaoka (2005) find that when all patents are essential, the grand coalition (which is both profit-maximizing and efficient) will only occur when the number of patent holders is small; for large numbers of patent holders, "the emergence of [an] outsider is inevitable, so… voluntary negotiation cannot secure the socially efficient outcome." Brenner (2009) proposes an "exclusionary" formation rule, under which a patent pool proposed by one player must be unanimously accepted by its members or fail to form; this allows the grand coalition to be achieved when it is efficient but could not be reached by sequential negotiations.

In both these papers, the grand coalition maximizes total patent-holder profits but still may fail to form. In my setting, the problem is more severe, as a pool of all essential patents, while more efficient than a smaller pool, does not necessarily maximize the joint profits of the essential patent holders. This is due to strategic effects: as the pool grows and P^E falls, nonessential patent holders raise their prices in response. To illustrate this formation problem, consider the same example as before, but this time with $n_E = 6$:

Example 2: As before, let $K = 3$, $(v_1, v_2, v_3) = (10, 10, 5)$, and $n_1 = n_2 = n_3 = 3$. This time, let $n_E = 6$, and consider patent pools containing some or all of the patents in $\mathcal{J}^E$.

Table 1 shows the equilibrium profit level of each essential patent holder under three scenarios: no pool, a pool containing five of the six essential patents, and a pool containing all six.

Relative to no pool, a pool of five essential patents does not increase the profits of the participating patent holders. A pool of all six increases their combined profits by 66 percent; but the addition of the sixth patent to the pool actually *decreases* the combined profits of the essential patent holders.

[9] See, for example, Deneckere and Davidson (1985) for a discussion.

Table 1—Effect of Pools on Essential Patent-Holder Profits

	Per firm profits		Combined profits, all firms
	Firms 1–5	Firm 6	
No pool	0.371	0.371	2.224
Pool of $\{1,2,3,4,5\}$	0.369	1.843	3.685
Pool of $\{1,2,3,4,5,6\}$	0.596	0.596	3.578

Thus, as in previous models, pools of essential patents are welfare increasing, but may fail to achieve their efficient size (or form at all) if individual patent holders can opt out without destroying the pool.

Linking Profitability to Welfare When $K = 2$.—Theorem 2 showed that a pool of complementary, nonessential patents could have either positive or negative welfare effects. Theorem 3, and the discussion following, suggest that such pools will sometimes, but not always, be profitable for their members (and therefore likely to form in the absence of regulatory opposition).

When there are only two competing products, the two questions turn out to be linked: welfare-decreasing pools are unprofitable, and therefore endogenously occurring pools can be assumed to be welfare increasing overall:

THEOREM 4: *Suppose $K = 2$, and fix $k \in \{1,2\}$. If a pool contains only patents in $\mathcal{J}_k^N$ and is profitable, then it is welfare increasing.*

The converse, however, does not hold: not all welfare-increasing pools are profitable. In addition, as Example 1 showed, Theorem 4 does not extend to $K > 2$, as the second pool considered in Example 1 was welfare decreasing, but nearly doubled its members' profits.

Adding Nonessential Patents to an Essential Pool.—Adding nonessential patents to an existing pool of essential patents can be either profitable or unprofitable; but if all the essential patents are already in the pool, and all the nonessential patents blocking a particular product are added, this is always profitable:

THEOREM 5: *Fixing $k \in \mathcal{K}$,*

(i) *the addition of one patent in $\mathcal{J}_k^N$ to a pool of essential patents may or may not increase the profits of its members;*

(ii) *the addition of* all *the patents in $\mathcal{J}_k^N$ to a pool already containing all essential patents always increases the joint profits of the pool members.*

Effects of One Pool on the Profitability of Another.—If the formation of one patent pool alters the incentives of other patent holders to form a different pool, it may not always suffice to consider the welfare effects of one pool in isolation. Unfortunately, the effect that one pool has on the profitability of another pool is ambiguous. To see how difficult it would be to make general statements about these effects, consider the following example, in which a seemingly unrelated change in primitives reverses the direction of the relationship between two pools.

Example 3: As before, let $K = 3$ and $(v_1, v_2, v_3) = (10, 10, 5)$. Let $n_E = 3$ and $n_1 = 5$, and consider two possible pools: a pool of all five patents in $\mathcal{J}_1^N$, and a pool of all three patents in $\mathcal{J}^E$.

First, suppose that $n_2 = 5$ *and* $n_3 = 3$. Regardless of whether the second pool forms, the first pool—containing the five patents in $\mathcal{J}_1^N$—is always profitable for its members, increasing their combined profit by more than 10 percent. Thus, it is at least plausible to imagine that it might form. The second pool—the pool containing all the essential patents—turns out to be profitable for its members if the first pool does not form, but unprofitable if the first pool does form. Thus, there appears to be strategic substitutability between the two patent pools.

On the other hand, switch n_2 and n_3, so that $n_2 = 3$ and $n_3 = 5$. Now the result is reversed. The pool of essential patents is now unprofitable if the first pool does not form, and profitable if the first pool forms, suggesting there is strategic complementarity between the two pools. (The pool of patents in $\mathcal{J}_1^N$ remains profitable, regardless of whether the essential patent-holders pool.)

C. *Pools Containing Substitutes*

Shapiro (2001) and Lerner and Tirole (2004) showed that when pooled patents are not available to be licensed separately, pools of substitute patents lead to higher prices and lower welfare. I give two examples along the same lines in the current setting.

THEOREM 6: (*i*) *A pool containing two nonessential patents blocking different products always decreases total consumer surplus.*

(*ii*) *A pool containing one nonessential patent blocking each product always decreases total consumer surplus and total welfare.*

A pool containing several nonessential patents blocking each product would, in essence, be replacing many nonessential patents with a single essential patent. If enough nonessential patents were included, this would likely lower all prices P_k and increase total welfare. However, one can imagine decomposing the formation of such a pool into two steps:

(i) First, K separate pools are created, each containing nonessential patents for a single product.

(ii) Second, these K pools are combined into a single pool.

The second part of Theorem 6 implies that the second step is welfare decreasing. Thus, even if the overall welfare effect of such a pool (relative to no pool) is positive, it is less positive than if separate pools were maintained for each product. In fact, this is effectively what was done in the 3G case. The general structure was originally intended to be a single "patent platform," with a single body determining standard (a la carte) licensing fees for all relevant patents; antitrust concerns led instead to the establishment of five separate Licensing Administrators to oversee selection of patents and licensing fees separately for each of the five radio interfaces making up the 3G standard.

D. *Compulsory Individual Licensing*

Lerner and Tirole (2004) propose compulsory individual licensing—mandating that participants in a patent pool also offer their patents individually—as a solution to welfare-decreasing pools. Define a pool to be *stable* to individual licensing if equilibrium prices are the same with and without individual licensing, and *weakly unstable* to individual licensing if in at least some equilibria, after the pool names its profit-maximizing price, individual licensing causes prices to revert to their pre-pool levels. Lerner and Tirole show that in their setting, welfare-increasing pools are stable to compulsory individual licensing, while welfare-decreasing pools are weakly unstable. Thus, they claim that compulsory individual licensing functions as a screen for efficient pools.

In Brenner's (2009) setting (an extension of Lerner and Tirole's 2004 model where pools smaller than the grand coalition are considered), all welfare-increasing pools are stable to individual licensing, but some welfare-decreasing pools may be as well. However, pools which form in equilibrium under an "exclusive" formation rule—where participants must agree unanimously to a proposed pool or it does not form, rather than patent holders being able to opt in or out individually—are stable to individual licensing if and only if they are welfare increasing.

In my setting, compulsory individual licensing does not always distinguish welfare increasing from welfare-decreasing pools. In particular:

THEOREM 7: *Any pool containing only essential patents is stable to individual licensing.*

Any pool containing only complementary nonessential patents is stable to individual licensing.

Both these types of pools are unaffected by compulsory individual licensing; but Theorem 2 established that pools of complementary nonessential patents can be welfare decreasing. Thus, compulsory individual licensing is not a sufficient screen against socially inefficient pools.

Like in Lerner and Tirole (2004), a pool containing substitute patents in my setting would be unstable to individual licensing. Consider a pool containing a single nonessential patent blocking each product. Each patent holder in the pool has an incentive to slightly undercut the pool's price, to capture the entire revenue from manufacturers building his product rather than sharing it with the rest of the pool; prices would collapse to those set in the absence of the pool. Thus, compulsory individual licensing in my setting is disruptive to *some* welfare-decreasing pools, but is not a perfect screen for efficiency.

E. *Pools as Stackelberg Leaders*

Since a patent pool involves joint action by multiple firms, a pool might be less flexible than an individual patent holder in its ability to adjust the licensing fees it demands. This would give a pool commitment power, allowing it to act like a Stackelberg leader, expecting the outside patent holders to best respond.[10] Such commitment power would make any pool weakly profitable (since at worst it could commit to its pre-pool prices), and most pools strictly profitable; but it would also typically result in the pool demanding a higher price than the static model suggests, reducing and sometimes reversing the welfare results.

THEOREM 8: *Define a* **Stackelberg pool** *as a patent pool which commits to prices before individual patent holders (and cannot adjust them afterwards). Fix all primitives but n_E, and fix the number n_O of essential patent holders who remain outside the pool.*

(*i*) *A Stackelberg pool of all but n_O of the essential patent holders is always more profitable than a regular pool of the same patent holders, but the latter is always more efficient.*

(*ii*) *There exists an integer $n^* \geq n_O + 2$ such that*
- *If $n_E \geq n^*$, a Stackelberg pool of all but n_O of the essential patent holders is a Pareto improvement over no pool.*
- *If $n_E < n^*$, a Stackelberg pool of all but n_O of the essential patent holders is less efficient than no pool, raises all prices P_k, and reduces the payoffs to everyone (consumers and patent holders) outside the pool.*

The following example illustrates Theorem 8:

Example 4: As before, let $K = 3$ and $(v_1, v_2, v_3) = (10, 10, 5)$. Let $n_1 = n_2 = 6$ and $n_3 = 9$. Table 2 shows the combined profit of the essential patent holders; the combined profit of all nonessential patent holders; consumer surplus; and total welfare, for various values of n_E in the absence of a pool, as well as in the presence of an ordinary patent pool containing all essential patents and a Stackelberg pool containing all essential patents.

In this case, with $n_O = 0$ (the Stackelberg pool considered contains all essential patents), $n^* = 3$: relative to no pool, the Stackelberg pool represents a Pareto improvement when $n_E = 3$ or 4, but lowers total welfare when $n_E = 2$. The regular patent pool gives higher total welfare than any of the other alternatives, but only increases essential patent-holder profits when $n_E \geq 4$.

The analogous result holds for pools of nonessential complements as well. Choosing k and fixing everything but n_k, there exists an n^* such that when $n_k \geq n^*$, the effects of a Stackelberg pool containing all but n_O patent holders in $\mathcal{J}_k^N$ are

[10] One example of this might be a standard-setting organization which was able to commit to licensing policies at the time the standard was being chosen.

Table 2—Effects of Regular and Stackelberg Pool on Total Welfare

	Essential patent-holder profits	Nonessential patent-holder profits	Consumer surplus	Total welfare
No pool, $n_E = 4$	1.165	1.525	0.256	2.946
No pool, $n_E = 3$	1.367	2.227	0.375	3.969
No pool, $n_E = 2$	1.438	3.173	0.542	5.154
Regular pool	1.223	4.550	0.799	6.572
Stackelberg pool	1.440	3.033	0.517	4.990

described by Theorem 2; when $n_k < n^*$, the effects are the opposite of those in Theorem 2; but for any n_k, the pool is profitable.

IV. Conclusion

In this paper, I have introduced a differentiated-products model for price competition in settings where overlapping sets of patents block competing products, and applied the model to understand the static effects of patent pools on pricing. The same framework could also be used to model competition among certain “aggregate” goods such as personal computers, which are made up of components built by different wholesalers, and to understand the effect of mergers between such wholesale suppliers. (The results extend easily to the case where each component is built at a constant but positive marginal cost.)

As it pertains to patent pools, the model does have certain limitations. I ignore uncertainty about patent scope and enforceability, treating each patent as unassailable and having bright-line boundaries. Firms that own nonessential patents blocking different products do not fit within my framework. If the downstream manufacturing sector were not assumed to be perfectly competitive, we would need to consider the separate incentives of vertically integrated firms and patent holders who do not participate in downstream production.

In this paper, I do not attempt to explicitly model pool formation or predict which pool or pools will form in a given setting. My model defines a mapping from coalition structures (a particular pool or set of pools) to payoffs (patent-holder profit and consumer surplus); any cooperative or noncooperative model of coalition formation could therefore be applied, taking these payoffs as given, to attempt to answer this question.

Like most of the existing literature on patent pools, this paper treats both the technology and the set of patents as exogenous; that is, it abstracts away from the question of innovation (and with it the endogeneity of which patents are essential). Again, this paper can be thought of as specifying the final-stage payoffs, this time of a dynamic, multi-firm patent or R&D race; a formal model of this race, taking the “endgame” profits as given, would allow examination of the effects of pools on innovation.

Appendix

Throughout, let $\mathbf{n} = (n_E, n_1, \ldots, n_K)$ denote the number of patents of each type, and $Q_0 = 1 - \sum_{k \in \mathcal{K}} Q_k$ the fraction of consumers who demand none of the products.

PROOF OF LEMMA 1:

The logit demand system satisfies Assumptions 1 and 3 of Quint (2013), so by Lemma 5 of that paper, an equilibrium exists, and is unique, and corresponds to the simultaneous solution to the J patent holders' first-order conditions.[11] Given the payoff functions in equation (1), these first-order conditions are $\frac{1}{p_j} = -\frac{\partial \ln Q_k}{\partial p_j} = -\frac{\partial \ln Q_k}{\partial P_k}$ for $j \in \mathcal{J}_k^N$, and $\frac{1}{p_j} = -\frac{\partial \ln (\Sigma_k Q_k)}{\partial p_j} = -\sum_{k'} \frac{\partial \ln (\Sigma_k Q_k)}{\partial P_{k'}}$ for $j \in \mathcal{J}^E$. One of the key features of the logit demand system is that $\frac{\partial Q_k}{\partial P_k} = -Q_k(1 - Q_k)$ and (for $k' \neq k$) $\frac{\partial Q_k}{\partial P_{k'}} = Q_k Q_{k'}$, so these first-order conditions simplify to $p_j = \frac{1}{1 - Q_k}$ for $j \in \mathcal{J}_k^N$ and $p_j = \frac{1}{1 - \sum_k Q_k} = \frac{1}{Q_0}$ for $j \in \mathcal{J}^E$.

Summing over $j \in \mathcal{J}_k^N$ gives $P_k^N = \sum_{j \in \mathcal{J}_k^N} p_j = \frac{n_k}{1 - Q_k}$, or $P_k^N(1 - Q_k) = n_k$; likewise, summing over $\mathcal{J}^E$ gives $P^E = \sum_{j \in \mathcal{J}^E} p_j = \frac{n_E}{Q_0}$, or $P^E Q_0 = n_E$. Since first-order conditions are necessary and sufficient for best responses, equilibrium therefore corresponds to solutions to the $K + 1$ simultaneous equations $n_k = P_k^N(1 - Q_k)$ and $n_E = P^E Q_0$, along with $p_j = \frac{1}{n_k} P_k^N$ if $j \in \mathcal{J}_k^N$ and $\frac{1}{n_E} P^E$ if $j \in \mathcal{J}^E$.

PROOF OF THEOREM 1:

Marginal Effect of n_E on Prices.—Payoffs to patent holders and consumers can be written as functions of aggregate prices $\left(P^E, P_1^N, \dots, P_K^N\right)$. A pool of essential patents affects aggregate prices like a decrease in n_E. From the supermodular-games approach in Quint (2013), we know that P^E is increasing in n_E and P_k^N decreasing in n_E. To calculate the exact effects, we can treat n_E as a continuous variable, affecting equilibrium prices through the simultaneous equations $P_k^N(1 - Q_k) = n_k$ and $P^E Q_0 = n_E$, and differentiate these with respect to n_E:

$$
\text{(A1)} \qquad \begin{aligned}
P_k^N(1 - Q_k) &= n_k \\
(1 - Q_k)\frac{\partial P_k^N}{\partial n_E} - P_k^N\left(\sum_{k' \in \mathcal{K}} \frac{\partial Q_k}{\partial P_{k'}^N}\frac{\partial P_{k'}^N}{\partial n_E} + \frac{\partial Q_k}{\partial P^E}\frac{\partial P^E}{\partial n_E}\right) &= 0 \\
(1 - Q_k)\frac{\partial P_k^N}{\partial n_E} - P_k^N\left(-Q_k(1 - Q_k)\frac{\partial P_k^N}{\partial n_E} + \sum_{k' \neq k} Q_k Q_{k'}\frac{\partial P_{k'}^N}{\partial n_E} - Q_k Q_0 \frac{\partial P^E}{\partial n_E}\right) &= 0 \\
(1 - Q_k + P_k^N Q_k)\frac{\partial P_k^N}{\partial n_E} - P_k^N Q_k\left(\sum_{k' \in \mathcal{K}} Q_{k'}\frac{\partial P_{k'}^N}{\partial n_E} - Q_0 \frac{\partial P^E}{\partial n_E}\right) &= 0
\end{aligned}
$$

[11] An earlier version of this paper included a separate proof of equilibrium existence and uniqueness under logit demand even in the absence of Assumption 1. Even with patents blocking arbitrary subsets of products and patent holders owning multiple patents of different "types," one can show that under logit demand, patent holders optimally offer their entire portfolio at a single price, and strategies can therefore be assumed to be one-dimensional. Taking the log of patent-holders' payoff functions, the resulting game can then be shown to be a potential game (à la Monderer and Shapley 1996), with a strictly concave potential function; the unique maximizer of that function then corresponds to the unique equilibrium of the game among patent holders.

and

$$P^E Q_0 = n_E$$

$$Q_0 \frac{\partial P^E}{\partial n_E} + P^E\left(\frac{\partial Q_0}{\partial P^E}\frac{\partial P^E}{\partial n_E} + \sum_{k' \in \mathcal{K}} \frac{\partial Q_0}{\partial P^N_{k'}}\frac{\partial P^N_{k'}}{\partial n_E}\right) = 1$$

$$Q_0 \frac{\partial P^E}{\partial n_E} + P^E\left(Q_0(1-Q_0)\frac{\partial P^E}{\partial n_E} + \sum_{k' \in \mathcal{K}} Q_0 Q_{k'}\frac{\partial P^N_{k'}}{\partial n_E}\right) = 1$$

$$(Q_0 + P^E Q_0)\frac{\partial P^E}{\partial n_E} - P^E Q_0\left(Q_0\frac{\partial P^E}{\partial n_E} - \sum_{k' \in \mathcal{K}} Q_{k'}\frac{\partial P^N_{k'}}{\partial n_E}\right) = 1.$$

Letting $\Delta = Q_0 \frac{\partial P^E}{\partial n_E} - \sum_{k' \in \mathcal{K}} Q_{k'} \frac{\partial P^N_{k'}}{\partial n_E}$, which we know from supermodularity is positive, gives

$$\text{(A2)} \qquad \frac{\partial P^N_k}{\partial n_E} = -\frac{P^N_k Q_k}{1 - Q_k + P^N_k Q_k}\Delta \quad \textit{and} \quad \frac{\partial P^E}{\partial n_E} = \frac{1}{Q_0 + P^E Q_0} + \frac{P^E}{1+P^E}\Delta.$$

Plugging these terms back into the definition of Δ and rearranging yields

$$\text{(A3)} \qquad \frac{1}{1+P^E} = \Delta\left(\frac{Q_0}{1+P^E} + \sum_{k' \in \mathcal{K}} Q_{k'}\frac{1-Q_{k'}}{1-Q_{k'}+P^N_{k'}Q_{k'}}\right),$$

and therefore

$$\text{(A4)} \qquad \frac{\partial P^N_k}{\partial n_E} = -\frac{P^N_k Q_k}{1-Q_k+P^N_k Q_k}\frac{\delta}{1+P^E}$$

and

$$\frac{\partial P^E}{\partial n_E} = \frac{1}{Q_0 + P^E Q_0} + \frac{P^E}{1+P^E}\frac{\delta}{1+P^E},$$

where $\frac{1}{\delta} = \frac{Q_0}{1+P^E} + \sum_{k' \in \mathcal{K}} Q_{k'} \frac{1-Q_{k'}}{1-Q_{k'}+P^N_{k'}Q_{k'}}$. We can rearrange $\frac{\partial P^E}{\partial n_E}$ to

$$\text{(A5)} \qquad \frac{\partial P^E}{\partial n_E} = \left(1 + \frac{1}{Q_0}\sum_{k' \in \mathcal{K}} Q_{k'}\frac{1-Q_{k'}}{1-Q_{k'}+P^N_{k'}Q_{k'}}\right)\frac{\delta}{1+P^E}$$

to show that $\frac{\partial(P^N_k + P^E)}{\partial n_E} = \frac{\delta}{1+P^E}\left(-\frac{P^N_k Q_k}{1-Q_k+P^N_k Q_k} + 1 + \frac{1}{Q_0}\sum_{k' \in \mathcal{K}} \frac{Q_{k'}(1-Q_{k'})}{1-Q_{k'}+P^N_{k'}Q_{k'}}\right)$ > 0. Finally, since $P^N_k(1 - Q_k) = n_k$ does not change with n_E, $\frac{\partial\left(\log P^N_k + \log(1-Q_k)\right)}{\partial n_E} = \frac{1}{P^N_k}\frac{\partial P^N_k}{\partial n_E} - \frac{1}{1-Q_k}\frac{\partial Q_k}{\partial n_E} = 0$, which implies $\frac{\partial Q_k}{\partial n_E} = \frac{1-Q_k}{P^N_k}\frac{\partial P^N_k}{\partial n_E}$ $= -\frac{(1-Q_k)Q_k}{1-Q_k+P^N_k Q_k}\frac{\delta}{1+P^E}$.

Welfare.—First, note that consumer surplus can be written as

$$\text{(A6)} \quad CS = \mathbf{E}_{\epsilon_0^l, \epsilon_1^l, \epsilon_2^l, \ldots, \epsilon_K^l} \max\left\{\epsilon_0^l, \max_{k' \in \mathcal{K}} \{v_{k'} - P_{k'} + \epsilon_{k'}^l\}\right\} = \log\left(1 + \sum_{k' \in \mathcal{K}} e^{v_{k'} - P_{k'}}\right)$$

and differentiating, $\frac{\partial CS}{\partial P_k} = -Q_k$. Total patent-holder profits are $P^E\left(\sum_{k' \in \mathcal{K}} Q_{k'}\right) + \sum_{k' \in \mathcal{K}} P_{k'}^N Q_{k'} = \sum_{k' \in \mathcal{K}} P_{k'} Q_{k'}$, so the effect of one price on total welfare is

$$\text{(A7)} \quad \begin{aligned} \frac{\partial Welfare}{\partial P_k} &= \frac{\partial CS}{\partial P_k} + \frac{\partial}{\partial P_k} \sum_{k' \in \mathcal{K}} P_{k'} Q_{k'} \\ &= -Q_k - P_k Q_k (1 - Q_k) + Q_k + \sum_{k' \neq k} P_{k'} Q_{k'} Q_k \\ &= -P_k Q_k + \sum_{k' \in \mathcal{K}} P_{k'} Q_{k'} Q_k \\ &= Q_k (\overline{P} - P_k), \end{aligned}$$

where $\overline{P} \equiv \sum_{k' \in \mathcal{K}} Q_{k'} P_{k'}$. Summing over k,

$$\text{(A8)} \quad \begin{aligned} \frac{\partial Welfare}{\partial P^E} &= \sum_{k \in \mathcal{K}} \frac{\partial Welfare}{\partial P_k} = \sum_{k \in \mathcal{K}} Q_k (\overline{P} - P_k) \\ &= \overline{P} \sum_{k \in \mathcal{K}} Q_k - \overline{P} = -Q_0 \overline{P}. \end{aligned}$$

Using these,

$$\text{(A9)} \quad \begin{aligned} \frac{\partial Welfare}{\partial n_E} &= \frac{\partial Welfare}{\partial P^E} \frac{\partial P^E}{\partial n_E} + \sum_{k' \in \mathcal{K}} \frac{\partial Welfare}{\partial P_{k'}^N} \frac{\partial P_{k'}^N}{\partial n_E} \\ &= -Q_0 \overline{P} \frac{\partial P^E}{\partial n_E} + \sum_{k' \in \mathcal{K}} Q_{k'} (\overline{P} - P_{k'}) \frac{\partial P_{k'}^N}{\partial n_E} \\ &= -Q_0 \overline{P} \left(1 + \frac{1}{Q_0} \sum_{k' \in \mathcal{K}} Q_{k'} \frac{1 - Q_{k'}}{1 - Q_{k'} + P_{k'}^N Q_{k'}}\right) \frac{\delta}{1 + P^E} \\ &\quad - \sum_{k' \in \mathcal{K}} Q_{k'} (\overline{P} - P_{k'}) \frac{P_{k'}^N Q_{k'}}{1 - Q_{k'} + P_{k'}^N Q_{k'}} \frac{\delta}{1 + P^E}, \end{aligned}$$

which simplifies to

$$\text{(A10)} \quad \frac{\partial Welfare}{\partial n_E} = \frac{\delta}{1 + P^E} \sum_{k' \in \mathcal{K}} Q_{k'} P_{k'} \left(\frac{P_{k'}^N Q_{k'}}{1 - Q_{k'} + P_{k'}^N Q_{k'}} - 1\right) < 0.$$

Pools of essential patents have the same impact on equilibrium prices as a decrease in n_E; we can integrate the marginal effect to calculate the effect of a pool.

We showed above that each price $P_k = P_k^N + P^E$ is increasing in n_E, so each price P_k is lower following the formation of a pool; so total consumer surplus (as well as each individual consumer's payoff) is increased by a pool.

We showed that for each k, P_k^N and Q_k are both decreasing in n_E, and therefore higher after a pool. For $j \in \mathcal{J}_k^N$, $\pi_j = \frac{1}{n_k} P_k^N Q_k$, so π_j is higher after a pool for every $j \in \mathcal{J} - \mathcal{J}^E$. As for essential patent holders who remain outside of a pool, they earn profits of $\frac{1}{n_E} P^E(1 - Q_0)$ both before and after the pool forms. Since $P^E Q_0 = n_E$, we can rewrite this as $\frac{1}{Q_0}(1 - Q_0) = \frac{1}{Q_0} - 1$. Since each Q_k is decreasing in n_E, Q_0 is increasing in n_E, so $\frac{1}{n_E} P^E(1 - Q_0) = \frac{1}{Q_0} - 1$ is decreasing in n_E and therefore higher after a pool.

Finally, we showed that total welfare is decreasing in n_E, and therefore higher after a pool.

PROOF OF THEOREM 2:

Marginal Effect of n_1 on Equilibrium Prices.—Again, we begin by calculating the marginal effect, this time of n_1, on equilibrium prices through the simultaneous equations characterizing them. Differentiating, we find

$$\text{(A11)} \qquad P_1^N(1 - Q_1) = n_1$$

$$(1 - Q_1)\,\frac{\partial P_1^N}{\partial n_1} - P_1^N\left(\sum_{k' \in \mathcal{K}} \frac{\partial Q_1}{\partial P_{k'}^N}\,\frac{\partial P_{k'}^N}{\partial n_1} + \frac{\partial Q_1}{\partial P^E}\,\frac{\partial P^E}{\partial n_1}\right) = 1$$

$$(1 - Q_1)\,\frac{\partial P_1^N}{\partial n_1} - P_1^N\left(-Q_1(1 - Q_1)\,\frac{\partial P_1^N}{\partial n_1} + \sum_{k' \neq 1} Q_1 Q_{k'}\,\frac{\partial P_{k'}^N}{\partial n_1} - Q_1 Q_0\,\frac{\partial P^E}{\partial n_1}\right) = 1$$

$$(1 - Q_1 + P_1^N Q_1)\,\frac{\partial P_1^N}{\partial n_1} - P_1^N Q_1\left(\sum_{k' \in \mathcal{K}} Q_{k'}\,\frac{\partial P_{k'}^N}{\partial n_1} - Q_0\,\frac{\partial P^E}{\partial n_1}\right) = 1$$

$$P_k^N(1 - Q_k) = n_k$$

$$(1 - Q_k)\,\frac{\partial P_k^N}{\partial n_1} - P_k^N\left(\sum_{k' \in \mathcal{K}} \frac{\partial Q_k}{\partial P_{k'}^N}\,\frac{\partial P_{k'}^N}{\partial n_1} + \frac{\partial Q_k}{\partial P^E}\,\frac{\partial P^E}{\partial n_1}\right) = 0$$

$$(1 - Q_k)\,\frac{\partial P_k^N}{\partial n_1} - P_k^N\left(-Q_k(1 - Q_k)\,\frac{\partial P_k^N}{\partial n_1} + \sum_{k' \neq k} Q_k Q_{k'}\,\frac{\partial P_{k'}^N}{\partial n_1} - Q_k Q_0\,\frac{\partial P^E}{\partial n_1}\right) = 0$$

$$(1 - Q_k + P_k^N Q_k)\,\frac{\partial P_k^N}{\partial n_1} - P_k^N Q_k\left(\sum_{k' \in \mathcal{K}} Q_{k'}\,\frac{\partial P_{k'}^N}{\partial n_1} - Q_0\,\frac{\partial P^E}{\partial n_1}\right) = 0$$

$$P^E Q_0 = n_E$$

$$Q_0\,\frac{\partial P^E}{\partial n_1} + P^E\left(\sum_{k' \in \mathcal{K}} \frac{\partial Q_0}{\partial P_{k'}^N}\,\frac{\partial P_{k'}^N}{\partial n_1} + \frac{\partial Q_0}{\partial P^E}\,\frac{\partial P^E}{\partial n_1}\right) = 0$$

$$Q_0\,\frac{\partial P^E}{\partial n_1} + P^E\left(\sum_{k' \in \mathcal{K}} Q_0 Q_{k'}\,\frac{\partial P_{k'}^N}{\partial n_1} + Q_0(1 - Q_0)\,\frac{\partial P^E}{\partial n_1}\right) = 0$$

$$(Q_0 + P^E Q_0)\,\frac{\partial P^E}{\partial n_1} + P^E Q_0\left(\sum_{k' \in \mathcal{K}} Q_{k'}\,\frac{\partial P_{k'}^N}{\partial n_1} - Q_0\,\frac{\partial P^E}{\partial n_1}\right) = 0.$$

Let $\Lambda = \sum_{k' \in \mathcal{K}} Q_{k'} \frac{\partial P_{k'}^N}{\partial n_1} - Q_0 \frac{\partial P^E}{\partial n_1}$, which (as before) we know from supermodularity is positive; then

$$\text{(A12)} \qquad \begin{aligned} \frac{\partial P_1^N}{\partial n_1} &= \frac{1}{1 - Q_1 + P_1^N Q_1} + \frac{P_1^N Q_1}{1 - Q_1 + P_1^N Q_1} \Lambda \\ \frac{\partial P_k^N}{\partial n_1} &= \frac{P_k^N Q_k}{1 - Q_k + P_k^N Q_k} \Lambda \\ \frac{\partial P^E}{\partial n_1} &= -\frac{P^E}{1 + P^E} \Lambda \end{aligned}$$

and, plugging these into the definition of Λ and solving for Λ, $\Lambda = \frac{Q_1}{1 - Q_1 + P_1^N Q_1} \delta$, with (as before) $\frac{1}{\delta} = \sum_{k' \in \mathcal{K}} \frac{Q_{k'}(1 - Q_{k'})}{1 - Q_{k'} + P_{k'}^N Q_{k'}} + \frac{Q_0}{1 + P^E}$. We can also rewrite $\frac{\partial P_1^N}{\partial n_1}$ as

$$\text{(A13)} \qquad \frac{\partial P_1^N}{\partial n_1} = \left(1 + \frac{1}{Q_1}\left(\sum_{k' \neq 1} \frac{Q_{k'}(1 - Q_{k'})}{1 - Q_{k'} + P_{k'}^N Q_{k'}} + \frac{Q_0}{1 + P^E}\right)\right) \Lambda$$

to show $\frac{\partial P_1^N}{\partial n_1} > \Lambda$, and therefore that $P_1 = P_1^N + P^E$ is increasing in n_1. For $k \neq 1$,

$$\text{(A14)} \qquad \frac{\partial\left(\log P_k^N + \log(1 - Q_k)\right)}{\partial n_1} = \frac{\partial \log n_k}{\partial n_1} = 0$$

$$\rightarrow \frac{\partial Q_k}{\partial n_1} = \frac{1 - Q_k}{P_k^N} \frac{\partial P_k^N}{\partial n_1} = \frac{Q_k(1 - Q_k)}{1 - Q_k + P_k^N Q_k} \Lambda.$$

As for Q_1, $\frac{\partial Q_1}{\partial n_1} = \frac{\partial Q_1}{\partial P_1^N} \frac{\partial P_1^N}{\partial n_1} + \sum_{k' \neq 1} \frac{\partial Q_1}{\partial P_{k'}^N} \frac{\partial P_{k'}^N}{\partial n_1} + \frac{\partial Q_1}{\partial P^E} \frac{\partial P^E}{\partial n_1}$, which simplifies to

$$\text{(A15)} \qquad \frac{\partial Q_1}{\partial n_1} = -\left(\sum_{k' \neq 1} \frac{Q_{k'}(1 - Q_{k'})}{1 - Q_{k'} + P_{k'}^N Q_{k'}} + \frac{Q_0}{1 + P^E}\right) \Lambda < 0.$$

Consumer Surplus.—As noted above, $\frac{\partial CS}{\partial P_k} = -Q_k$, so $\frac{\partial CS}{\partial n_1} = -Q_1 \frac{\partial P_1^N}{\partial n_1} - \sum_{k' \neq 1} Q_{k'} \frac{\partial P_{k'}^N}{\partial n_1} - \sum_{k' \in \mathcal{K}} Q_{k'} \frac{\partial P^E}{\partial n_1}$; plugging in the expressions above and simplifying gives $\frac{\partial CS}{\partial n_1} = -\frac{1}{1 + P^E} \Lambda < 0$.

Welfare.—As noted above, $\frac{\partial \mathit{Welfare}}{\partial P_k} = Q_k(\overline{P} - P_k)$ and $\frac{\partial \mathit{Welfare}}{\partial P^E} = -Q_0 \overline{P}$, so

$$\text{(A16)} \qquad \frac{\partial \mathit{Welfare}}{\partial n_1} = Q_1(\overline{P} - P_1) \frac{\partial P_1^N}{\partial n_1} + \sum_{k' \neq 1} Q_{k'}(\overline{P} - P_{k'}) \frac{\partial P_{k'}^N}{\partial n_1} - Q_0 \overline{P} \frac{\partial P^E}{\partial n_1}.$$

Plugging in the expressions above and simplifying gives

$$\text{(A17)}\quad \frac{\partial\, Welfare}{\partial n_1}$$

$$= \left(\sum_{k' \neq 1} P_{k'} \frac{Q_{k'}(1-Q_{k'})}{1-Q_{k'}+P^N_{k'}Q_{k'}} - \sum_{k' \neq 1} P_1 \frac{Q_{k'}(1-Q_{k'})}{1-Q_{k'}+P^N_{k'}Q_{k'}} - P_1 \frac{Q_0}{1+P^E} \right) \Lambda.$$

Proof of Theorem.—A pool of patent holders in $\mathcal{J}^N_1$ has the same effect on equilibrium prices as a decrease in n_1. We showed above that P^N_k is increasing in n_1 for every k (including 1); P^E is decreasing in n_1; Q_1 is decreasing in n_1; Q_k $(k \neq 1)$ is increasing in n_1; and P_1 is increasing in n_1. Examples in the text show that P_k can increase or decrease in n_1. We showed above that total consumer surplus is decreasing in n_1; but a pool which increases P_k $(k \neq 1)$ will harm some individual consumers (at a minimum, those who still purchase product k after the pool).

For $k \neq 1$, P^N_k and Q_k are both increasing in n_1, so $\frac{1}{n_k}P^N_k Q_k$ is increasing in n_1, so profits of patent holders in $\mathcal{J}^N_k$ are lower following the pool. Since $P^E Q_0 = n_E$ and P^E is decreasing in n_1, Q_0 must be increasing in n_1, so $\frac{1}{n_E}P^E(1-Q_0)$ is decreasing in n_1, so profits of patent holders in $\mathcal{J}^E$ are higher after the pool. As for outsiders within $\mathcal{J}^N_1$, they earn $\frac{1}{n_1}P^N_1 Q_1$; since $P^N_1(1-Q_1) = n_1$, we can rewrite this as $\frac{Q_1}{1-Q_1}$; since Q_1 is higher after the pool, these outsiders earn more. Examples in the text show both welfare-increasing and welfare-decreasing pools.

PROOF OF THEOREM 3:

Note that claims 1 and 3 are stated slightly differently here from in the text, but are obviously equivalent.

CLAIM 1: *Fix* $m \geq 1$. *If* n_E *is sufficiently large, a pool of* $m+1$ *essential patents increases the profits of its members.*

I will treat n_E as the name of the variable, $\tilde{n}_E$ as its starting value, and $\tilde{n}_E - m$ as its ending value (after pool formation). Let $\Pi_E(\cdot)$ be the equilibrium value of $P^E \sum_{k' \in \mathcal{K}} Q_{k'}$, as a function of n_E. Combined profits of the patent holders forming the pool are $\frac{m+1}{\tilde{n}_E}\Pi^E(\tilde{n}_E)$ before pool formation, and $\frac{1}{\tilde{n}_E - m}\Pi^E(\tilde{n}_E - m)$ after. We can therefore write the gain in profits as

$$\text{(A18)}\qquad d\pi = (m+1)^r \frac{\Pi^E(\tilde{n}_E - (1-r)m)}{\tilde{n}_E - (1-r)m}\Bigg|^{r=0}_{r=1}$$

$$= -\int_0^1 \frac{d}{dr}\left((m+1)^r \frac{\Pi^E(\tilde{n}_E-(1-r)m)}{\tilde{n}_E-(1-r)m} \right) dr.$$

Thus, it will suffice to show that for $\tilde{n}_E$ sufficiently large, $(m+1)^r \frac{\Pi^E(\tilde{n}_E-(1-r)m)}{\tilde{n}_E-(1-r)m}$ is decreasing in r for all $r \in (0,1)$, or

$$\text{(A19)} \qquad 0 > \frac{\partial}{\partial r}\left((m+1)^r \frac{\Pi^E(\tilde{n}_E-(1-r)m)}{\tilde{n}_E-(1-r)m}\right)$$

$$= \frac{d}{dr}\left(e^{r\log(m+1)} \frac{\Pi^E(\tilde{n}_E-m+rm)}{\tilde{n}_E-m+rm}\right).$$

Taking the derivative, plugging in the expressions above, and simplifying, we can show that the derivative is proportional to

$$\text{(A20)} \qquad \frac{\log(m+1)}{m} Q_0^2 \left[\frac{\sum_{k'\in\mathcal{K}} Q_{k'} \frac{P_{k'}^N Q_{k'}}{1-Q_{k'}+P_{k'}^N Q_{k'}}}{\sum_{k'\in\mathcal{K}} Q_{k'} \frac{1-Q_{k'}}{1-Q_{k'}+P_{k'}^N Q_{k'}}}\right]$$

$$+ \frac{\log(m+1)}{m} Q_0 P^E(1-Q_0) - \left(1-\frac{\log(m+1)}{m} Q_0\right),$$

which we can rewrite as

$$\text{(A21)} \qquad \frac{\log(m+1)}{m} Q_0^2 \left[\frac{wtd\,avg\ P_{k'}^N Q_{k'}}{wtd\,avg\ 1-Q_{k'}}\right]$$

$$+ \frac{\log(m+1)}{m} Q_0 P^E(1-Q_0) - \left(1-\frac{\log(m+1)}{m} Q_0\right),$$

where the weighted averages are taken with respect to weights $\frac{Q_{k'}}{1-Q_{k'}+P_{k'}^N Q_{k'}}$, normalized to sum to 1.

Now, $Q_k = \frac{\exp(v_k-P_k)}{1+\sum_{k'\in\mathcal{K}}\exp(v_{k'}-P_{k'})} < e^{v_k}e^{-P^E}$ and $P^E > n_E$, so as n_E grows without bound, $P_k^N Q_k$ and $P^E(1-Q_0)$ go to 0 and $1-Q_k$ does not, so the first two terms vanish. For $m \geq 1$, $\frac{\log(m+1)}{m} < 0.7$, so the term in parentheses is at least 0.3. Pick $\tilde{n}_E$ sufficiently large so that the whole expression is negative for $n_E \geq \tilde{n}_E - m$ and the integrand is negative on $(0,1)$, making the pool profitable.

(Conversely, since the lead term is positive, the integrand will be positive whenever $\frac{\log(m+1)}{m}(Q_0 + P^E(1-Q_0)Q_0) > 1$. For $m = 1$, this will hold for all $r \in (0,1)$ if prior to pool formation, $(\tilde{n}_E-2)(1-Q_0) > \frac{1}{\log 2} - 1 \approx 0.442$.)

CLAIM 2: *Fix $m \geq 0$. If n_E is sufficiently large, a pool containing all but m of the essential patents increases the profits of its members.*

Let π be the equilibrium profits of each essential patent holder in the game with the same primitives except that there are only $m+1$ essential patents. Note that $\pi > 0$. Should a pool form containing all but m of the n_E essential patents, π would be the pool's total revenue; divided up equally, this would give each participant a payoff of $\frac{\pi}{n_E - m}$. In the absence of a pool, each essential patent-holder's profits are $\frac{P^E}{n_E}\sum_{k\in\mathcal{K}} Q_k$. We know that

$$\text{(A22)} \qquad \sum_{k\in\mathcal{K}} Q_k = \frac{\sum_{k\in\mathcal{K}} e^{v_k - P_k^N - P^E}}{1+\sum_{k\in\mathcal{K}} e^{v_k - P_k^N - P^E}} < \sum_{k\in\mathcal{K}} e^{v_k - P_k^N - P^E} < e^{-P^E}\sum_{k\in\mathcal{K}} e^{v_k}$$

and so

$$\text{(A23)} \qquad \frac{P^E}{n_E}\sum_{k\in\mathcal{K}} Q_k < \frac{1}{n_E} P^E e^{-P^E} V,$$

where $V \equiv \sum_{k\in\mathcal{K}} e^{v_k}$. To prove the result, we need to show that for n_E sufficiently large, $\frac{\pi}{n_E - m} > \frac{1}{n_E} P^E e^{-P^E} V$. Since $n_E > n_E - m$, a sufficient condition is $\frac{\pi}{V} > P^E e^{-P^E}$. $\frac{\pi}{V}$ is a constant, and xe^{-x} decreases to 0 as x increases; since $P^E > n_E$ increases without bound as $n_E \to \infty$, the result follows.

CLAIM 3: *Fix $m \geq 1$. If n_k is sufficiently large, a pool of $m+1$ patent holders in $\mathcal{J}_k^N$ increases the profits of its members.*

Like the proof of the first claim above, note that when $m+1$ nonessential patents blocking technology 1 form a pool, the change in their joint profits is

$$\text{(A24)} \qquad d\pi = (m+1)^r \left.\frac{\Pi^1(\tilde{n}_1 - (1-r)m)}{\tilde{n}_1 - (1-r)m}\right|_{r=1}^{r=0}$$

$$= -\int_0^1 \frac{d}{dr}\left((m+1)^r \frac{\Pi^1(\tilde{n}_1 - (1-r)m)}{\tilde{n}_1 - (1-r)m}\right) dr,$$

where Π^1 is the equilibrium value of $P_1^N Q_1$ as a function of n_1. Differentiating and letting $n_1 = \tilde{n}_1 - (1-r)m$, the integrand is

$$\text{(A25)} \qquad \log(m+1) e^{r\log(m+1)} \frac{P_1^N Q_1}{n_1} + e^{r\log(m+1)} m \frac{\partial}{\partial n_1}\left(\frac{P_1^N Q_1}{n_1}\right)$$

which, after a lot of simplification, is proportional to

$$\text{(A26)} \qquad \frac{\log(m+1)}{m}(1-Q_1) - \frac{1}{1-Q_1+P_1^N Q_1}\left(1 - \frac{\frac{Q_1(1-Q_1)}{1-Q_1+P_1^N Q_1}}{\sum_{k\in\mathcal{K}} \frac{Q_k(1-Q_k)}{1-Q_k+P_k^N Q_k} + \frac{Q_0}{1+P^E}}\right).$$

As n_1 increases to infinity, P_1^N grows unboundedly and Q_1 and $P_1^N Q_1$ shrink to 0. The term $\frac{Q_1(1-Q_1)}{1-Q_1+P_1^N Q_1} < Q_1$ vanishes; since Q_0 increases in n_1 and P^E decreases, $\frac{Q_0}{1+P^E}$ does not vanish; so the messy fraction goes to 0 and the term within parentheses goes to 1. $\frac{1}{1-Q_1+P_1^N Q_1}$ goes to 1 as well, so the entire second expression goes to 1, while the first expression is bounded above by $\frac{\log(m+1)}{m} < 0.7$. So when n_1 is sufficiently large, the integrand is negative and the pool is profitable.

CLAIM 4: *Fix $m \geq 0$. If n_k is sufficiently large, a pool containing all but m patents in $\mathcal{J}_k^N$ increases the profits of its members.*

This time, let π denote equilibrium profits to each player in $\mathcal{J}_k^N$ after the pool formed (that is, with $n_k = m+1$). Since there are $n_k - m$ pool members, each earns $\frac{1}{n_k - m}\pi$ given the pool. Without the pool, each earns $\frac{1}{n_k} P_k^N Q_k$. Pick n_k sufficiently large that $\pi e^{-v_k} > n_k e^{-n_k}$. Since $P_k^N > n_k$ and xe^x is decreasing above 1, $\pi e^{-v_k} > P_k^N e^{-P_k^N}$, so $\pi > P_k^N e^{v_k - P_k^N} > P_k^N Q_k$, so

$$\frac{1}{n_k - m}\pi > \frac{1}{n_k}\pi > \frac{1}{n_k} P_k^N Q_k \tag{A27}$$

and the pool is profitable.

PROOF OF THEOREM 4:

Using the results on the effects of n_1 on equilibrium prices given above, we can calculate (for general K)

$$\begin{aligned}\frac{\partial(Welfare - P_1^N Q_1)}{\partial n_1} = &\Bigg(\sum_{k'\neq 1} P_{k'} \frac{Q_{k'}(1-Q_{k'})}{1-Q_{k'}+P_{k'}^N Q_{k'}} - \sum_{k'\neq 1} P_1 \frac{Q_{k'}(1-Q_{k'})}{1-Q_{k'}+P_{k'}^N Q_{k'}} \\ &- P_1 \frac{Q_0}{1+P^E} - P_1^N\left(-\sum_{k'\neq 1} \frac{Q_{k'}(1-Q_{k'})}{1-Q_{k'}+P_{k'}^N Q_{k'}} - \frac{Q_0}{1+P^E}\right) \\ &- Q_1\left(1+\frac{1}{Q_1}\left(\sum_{k'\neq 1} \frac{Q_{k'}(1-Q_{k'})}{1-Q_{k'}+P_{k'}^N Q_{k'}} + \frac{Q_0}{1+P^E}\right)\right)\Bigg)\Lambda,\end{aligned} \tag{A28}$$

which eventually simplifies to $\frac{\partial(Welfare - P_1^N Q_1)}{\partial n_1} = \left(\sum_{k'\neq 1} \frac{P_{k'}^N Q_k}{1-Q_{k'}+P_{k'}^N Q_{k'}} - 1\right)\Lambda$. When $K=2$, the summation has only one term, which is less than 1, so the expression is negative; so a decrease in n_1 (a pool of patent holders in $\mathcal{J}_1^N$) has a net positive externality on everyone outside of $\mathcal{J}_1^N$. Since we already showed that it has a positive externality on outsiders within $\mathcal{J}_1^N$, if it increases the profits of its members, the overall welfare impact must be positive.

PROOF OF THEOREM 5:

Adding All of $\mathcal{J}_k^N$ to a Pool of All $\mathcal{J}^E$.—Consider the effects of the addition of these patents to the pool as occurring in four stages:

(i) The profits of these nonessential patents begin to accrue to the pool, without any changes in any prices.

(ii) The pool reduces the prices of these patents to 0 and adjusts the price of the pool, as a static best response to the prices of the nonessential patents blocking other technologies.

(iii) The nonessential patent holders blocking other technologies change their prices to the new equilibrium levels.

(iv) The pool changes its prices to its new equilibrium level.

Stage 1 does not change the joint profits of the pool and the patents being added, it just allows us to focus on the profits going to essential patent holders in the last three stages.

We showed above that an essential patent holder cannot gain by charging different prices to producers accessing different technologies. Thus, stage 2 increases essential patent-holder (pool) profits, increasing joint profits.

Since the new equilibrium corresponds to a decrease in n_k, the prices of nonessential competing patents $P_{k'}^N$ decrease in equilibrium; this increases essential patent-holder profits. So stage 3 increases the joint profits we're looking at.

Finally, stage 4 increases joint profits, since the pool will only change prices to increase profits.

Thus, each stage increases the joint profits, so the net effect is to increase joint profits.

Adding Some Patents in $\mathcal{J}_k^N$ to a Pool Containing Some Patents in $\mathcal{J}^E$.—If a single essential patent (or a pool) joins with a single essential patent blocking technology 1, we can write the change in their joint profits as

$$d\pi = \int_0^1 \frac{d}{dr} \frac{1}{n_E} \Pi^E(\tilde{n}_1 - r) + (1 - r) \frac{\Pi^1(\tilde{n}_1 - r)}{\tilde{n}_1 - r}\, dr, \tag{A29}$$

where $\tilde{n}_1$ is the starting value of n_1 and Π^E and Π^1 are the equilibrium values of $P^E(1 - Q_0)$ and $P_1^N Q_1$, respectively, as a function of n_1. We find conditions for this to be negative. Taking the derivative and letting $n_1 = \tilde{n}_1 - r$, we can show the integrand is proportional to

$$\begin{aligned} &\frac{1}{Q_0}\frac{1}{1+P^E}\frac{1-Q_1}{1-Q_1+P_1^N Q_1} - \left(\sum_{k\in\mathcal{K}} \frac{Q_k(1-Q_k)}{1-Q_k+P_k^N Q_k} + \frac{Q_0}{1+P^E}\right) \\ &+ (1-r)\frac{1}{1-Q_1}\frac{1}{1-Q_1+P_1^N Q_1}\left(\sum_{k\in\mathcal{K}-\{1\}} \frac{Q_k(1-Q_k)}{1-Q_k+P_k^N Q_k} + \frac{Q_0}{1+P^E}\right). \end{aligned} \tag{A30}$$

If $Q_1 \geq \frac{1}{Q_0} \frac{1}{1+P^E} = \frac{1}{Q_0 + n_E}$ and $1 \geq (1-r)\frac{1}{1-Q_1} \frac{1}{1-Q_1+P_1^N Q_1}$, then the $k=1$ piece of the second term dominates the first term, and the remainder of the second term dominates the third term, making the whole expression negative. The latter requires that

$$\text{(A31)} \qquad 1 \geq \frac{1-r}{(1-Q_1)^2 + P_1^N Q_1(1-Q_1)}$$

$$= \frac{1-r}{1-2Q_1+Q_1^2+n_1 Q_1} = \frac{1-r}{1-2Q_1+Q_1^2+(\tilde{n}_1-r)Q_1}.$$

If $\tilde{n}_1 \geq 2$, the condition $1 \geq \frac{1-r}{1+(\tilde{n}_1-2)Q_1+Q_1^2-rQ_1}$ holds for all $r \in [0,1]$. Thus, $\tilde{n}_1 \geq 2$, along with the condition that $Q_1(Q_0+n_E) \geq 1$ at each $n_1 = \tilde{n}_1 - r$, guarantees all negative integrands. Since Q_1 is decreasing in n_1, if $Q_1 n_E \geq 1$ at $r=0$ (before the pool forms/grows), then $Q_1 n_E \geq 1$ at all r, which will suffice.

So when $n_E > \frac{1}{Q_k}$ (there are at least $\frac{1}{Q_k} - 1$ outsiders to a pool of essential patents) and $n_k \geq 2$, the addition of a single patent holder in $\mathcal{J}_k^N$ to the pool decreases combined profits.

For a concrete example, let $K=3$, $\mathbf{v} = (10,10,5)$, $n_E = 1$ (a pool already exists containing all essential patent holders), and $n_1 = n_2 = n_3 = 3$. Solving numerically for equilibrium prices and other outcomes, adding a single one of the patent holders in $\mathcal{J}_2^N$ to the existing pool decreases their combined profit, as shown in Table A1. Combined profits of the pool and a single patent holder in $\mathcal{J}_2^N$ are $3.578 + \frac{1}{3}(1.861) = 4.198$ to begin, and 4.171 after the patent holder is added to the pool.

PROOF OF THEOREM 6:

For the first claim, suppose the patents block technologies 1 and 2. The pool sets price p^* to maximize $p(Q_1+Q_2)$, leading to first-order condition $p^*(1-Q_1-Q_2) = 1$, which is equivalent to

$$\text{(A32)} \qquad p^*(1-Q_1) = \frac{1-Q_1}{1-Q_1-Q_2} = 1 + \frac{Q_2}{1-Q_1-Q_2}$$

$$p^*(1-Q_2) = \frac{1-Q_2}{1-Q_1-Q_2} = 1 + \frac{Q_1}{1-Q_1-Q_2}.$$

Along with the best-response functions for the other patent holders in T_1^N and T_2^N, this leads to

$$\text{(A33)} \quad P_1^N(1-Q_1) = n_1 + \frac{Q_2}{1-Q_1-Q_2}, \qquad P_2^N(1-Q_2) = n_2 + \frac{Q_1}{1-Q_1-Q_2}$$

Table A1—Equilibrium Outcomes Before and After One Nonessential Patent Is Added to an Essential Pool

n	P^E	P_1^N	P_2^N	P_3^N	Q_1	Q_2	Q_3
$(1,3,3,3)$	4.578	4.861	4.861	3.048	0.383	0.383	0.016
$(1,3,2,3)$	5.171	4.354	3.904	3.024	0.311	0.488	0.008
n	Π_E	Π_1	Π_2	Π_3	CS	W	
$(1,3,3,3)$	**3.578**	1.861	**1.861**	0.048	1.521	8.869	
$(1,3,2,3)$	**4.171**	1.354	1.904	0.024	1.643	9.096	

in equilibrium, along with the usual conditions $P^E\left(1-\sum_{k\in\mathcal{K}} Q_k\right) = n_E$ and $P_k^N(1-Q_k) = n_k$ for $k \in \mathcal{K} - \{1,2\}$. Thus, the equilibrium effect of the pool is equivalent to increases in n_1 and n_2, each of which reduces producer surplus.

As for the second, a pool containing one nonessential patent blocking each technology has the same effect as an increase (by one) of n_E and decreases (by one) of each n_k. Using the expressions calculated earlier for the welfare effects of each parameter, we can calculate that

$$\text{(A34)}\qquad \frac{1}{\delta}\left(\frac{\partial W}{\partial n_E} - \sum \frac{\partial W}{\partial n_k}\right)$$

$$= -\frac{1}{1+P^E}\sum_{k\in\mathcal{K}}\sum_{k'\neq k} Q_{k'}(P_k^N + P^E)\,\frac{Q_k}{1-Q_k+P_k^N Q_k}$$

$$-\sum\sum_{k'<k}(Q_k - Q_{k'})\left(P_{k'}^N - P_k^N\right)\frac{Q_{k'}}{1-Q_{k'}+P_{k'}^N Q_{k'}}\,\frac{Q_k}{1-Q_k+P_k^N Q_k}.$$

Each unordered pair $\{k,k'\}$ contributes two terms to the first double summation and one term to the second; collecting these three terms

$$\text{(A35)}\qquad -\frac{1}{1+P^E}\,Q_{k'}\left(P_k^N + P^E\right)\frac{Q_k}{1-Q_k+P_k^N Q_k}$$

$$-\frac{1}{1+P^E}\,Q_k\left(P_{k'}^N + P^E\right)\frac{Q_{k'}}{1-Q_{k'}+P_{k'}^N Q_{k'}}$$

$$-(Q_k - Q_{k'})\left(P_{k'}^N - P_k^N\right)\frac{Q_{k'}}{1-Q_{k'}+P_{k'}^N Q_{k'}}\,\frac{Q_k}{1-Q_k+P_k^N Q_k}$$

they can be shown to be negative, so summing up over all $\{k,k'\}$ pairs gives $\frac{\partial W}{\partial n_E} - \sum \frac{\partial W}{\partial n_k} < 0$, making the pool welfare-destroying.

As for consumer surplus, we calculated earlier that

$$\text{(A36)}\qquad \frac{\partial CS}{\partial n_E} = -\frac{\delta}{1+P^E}\sum_{k\in\mathcal{K}}\frac{1-Q_k}{Q_0}\,\frac{Q_k}{1-Q_k+P_k^N Q_k}$$

and

$$\frac{\partial CS}{\partial n_k} = -\frac{\delta}{1+P^E}\frac{Q_k}{1-Q_k+P_k^N Q_k}.$$

Since $1 - Q_k = Q_0 + \sum_{k' \neq k} Q_{k'} > Q_0$, $\frac{\partial CS}{\partial n_E} - \sum_{k' \in \mathcal{K}} \frac{\partial CS}{\partial n_{k'}} < 0$, so such a pool lowers consumer surplus.

PROOF OF THEOREM 7:

We assume, as in Lerner and Tirole (2004) and Brenner (2009), that in the presence of a patent pool with compulsory individual licensing, pricing occurs in two stages:

(i) The patent pool, and patent holders outside the pool, name prices.

(ii) The members of the patent pool name prices for their individual patents.

Producers then choose a technology, licensing pooled patents as needed, either individually or through the pool, whichever is cheaper.

I will show that if the patents in a pool are perfect complements—each one is only consumed with all the others in the pool—and in stage 1, the pool sets a price which is a static best response to the prices of the patents outside the pool, then no pool member will license his patent outside the pool in stage 2; so individual licensing does not destabilize the equilibrium with that pool.

The logic is as follows. In stage 2, each member of the pool, by naming a sufficiently high price, can "sabotage" individual licensing, that is, ensure that nobody will license any of the pooled patents individually. Thus, if pool members name prices which sum to less than the price of the pool, it must be that each of them is earning weakly higher profits than they would without individual licensing. Since the price of the pool was the unique maximizer of the pooled patent holders' combined profits, this is only possible if the sum of the individual prices of the pooled patents is the same as the price of the pool; so total prices, and each patent-holder's revenue, is the same as without individual licensing. Anticipating this in stage 1, the pool has no reason to play anything other than a static best response, and the same prices emerge as in equilibrium without individual licensing of pooled patents.

PROOF OF THEOREM 8:

A Stackelberg pool names a price p^* to maximize profits $p^* Q_A$ after all other patent holders have best responded. At these equilibrium prices, let $\hat{n} = P^E Q_0 - n_0$. Aggregate prices correspond to the solution to equation (5), with $n_E = \hat{n} + n_0$. A regular pool corresponds to the case of $n_E = 1 + n_0$. Setting n^* to be the lowest whole number greater than or equal to $\hat{n} + n_0$, and the fact that welfare and all outsider payoffs are decreasing in n_E, will complete the proof once we show that $\hat{n} > 1$.

By optimality, the outsider patent holders $j \in \mathcal{J}^E$ set prices $p_j = \frac{1}{Q_0}$, so $P^E = p^* + \frac{n_0}{Q_0}$, so $\hat{n} = P^E Q_0 - n_0 = p^* Q_0 + n_0 - n_0 = p^* Q_0$. Since $\hat{n} + n_0 = P^E Q_0$,

$p^* = \frac{\hat{n}}{\hat{n}+n_0} P^E_{\hat{n}+n_0}$, where P^E_x is the static equilibrium price when $n_E = x$; since P^E is increasing in n_E, $\frac{\hat{n}}{\hat{n}+n_0} P^E_{\hat{n}+n_0}$ is strictly increasing in $\hat{n}$, from 0 at $\hat{n} = 0$ to infinity as $\hat{n} \to \infty$; so instead of choosing a price, envision the pool choosing $\hat{n}$ and then getting payoff $\frac{\hat{n}}{\hat{n}+n_0} P^E_{\hat{n}+n_0} Q_A$.

Now, increasing $\hat{n}$ has two effects: a direct one (it increases the pool's price, which can increase or decrease profits), and a strategic one (it causes other patent holders to adjust their prices). Both $\frac{1}{n_E} P^E$ and P^N_k are decreasing in n_E, so all outside patent holders (both essential and nonessential) demand lower prices as $\hat{n}$ increases; so the strategic effect always favors raising $\hat{n}$. At $\hat{n} < 1$, the pool's price is below the static best response, so raising it is strictly beneficial; at $\hat{n} = 1$, the pool's price is exactly its static best response, so the direct effect of increasing $\hat{n}$ is second order and the strategic effect dominates. Thus, the Stackelberg pool maximizes profits by setting $\hat{n} > 1$.

A. *Examples 1, 2, 3, and 4*

For all four examples discussed in the text, $K = 3$ *and* $(v_1, v_2, v_3) = (10, 10, 5)$. For each example, the tables below show the "effective" value of **n**, both before and after the pool in question forms; equilibrium aggregate prices P^E and P^N_k; demand for each good Q_k; combined profits for all essential patent holders, Π_E, and for all patent holders in $\mathcal{J}^N_k$, Π_k, for each $k \in \{1, 2, 3\}$; consumer surplus CS, and total welfare W. The examples were solved numerically; it is straightforward to verify that each demand Q_k is correct (according to equation (4)) given (v_1, v_2, v_3) and the equilibrium prices, and that prices and demand together satisfy each first-order condition in equation (5) and therefore constitute an equilibirum. Profit is then calculated as $\Pi_E = P^E(Q_1 + Q_2 + Q_3)$ and $\Pi_k = P^N_k Q_k$; consumer surplus as $CS = \log\left(1 + e^{v_1 - P_1} + e^{v_2 - P_2} + e^{v_3 - P_3}\right)$; and welfare as $W = \Pi_E + \Pi_1 + \Pi_2 + \Pi_3 + CS$.

Example 1: Here, $n_1 = n_2 = n_3 = 3$, and $n_E = 3$. Equilibrium outcomes before and after a pool of the three patents in $\mathcal{J}^N_3$ (reducing n_3 from 3 to 1) are shown in Table A2; total welfare increases from 6.426 to 6.428 when the pool forms.

On the other hand, if we consider the same exercise when $n_E = 1$, the results are shown in Table A3; total welfare decreases from 8.869 to 8.775 when the pool forms.

Example 2: Table A4 shows equilibrium effects of various pools of essential patents. Again, $n_1 = n_2 = n_3 = 3$, but this time, $n_E = 6$. The first row represents no pool, the second row a pool of five of the six essential patents, and the third row a pool of all six. Note that in the first and third cases, each essential patent holder earns $\frac{1}{6}\Pi_E$, but in the second case, each member of the pool earns $\frac{1}{5}\frac{\Pi_E}{2}$ and the single outsider earns $\frac{\Pi_E}{2}$.

Example 3: First, let $(n_E, n_1, n_2, n_3) = (3, 5, 5, 3)$. Table A5 shows the profitability of a pool containing the three essential patent holders: the joint profits of the essential patent holders increase from 2.012 to 2.047 when the pool forms.

Table A2—Change in Equilibrium Outcomes when the Three Patents in $\mathcal{J}_3^N$ Form a Pool, $n_E = 3$

n	P^E	P_1^N	P_2^N	P_3^N	Q_1	Q_2	Q_3
$(3,3,3,3)$	6.472	4.085	4.085	3.016	0.266	0.266	0.005
$(3,3,3,1)$	6.562	4.024	4.024	1.035	0.254	0.254	0.034
n	Π_E	Π_1	Π_2	Π_3	CS	W	
$(3,3,3,3)$	3.472	1.085	1.085	0.016	0.769	**6.426**	
$(3,3,3,1)$	3.562	1.024	1.024	0.035	0.783	**6.428**	

Table A3—Change in Equilibrium Outcomes when the Three Patents in $\mathcal{J}_3^N$ Form a Pool, $n_E = 1$

n	P^E	P_1^N	P_2^N	P_3^N	Q_1	Q_2	Q_3
$(1,3,3,3)$	4.578	4.861	4.861	3.048	0.383	0.383	0.016
$(1,3,3,1)$	4.818	4.647	4.647	1.091	0.354	0.354	0.084
n	Π_E	Π_1	Π_2	Π_3	CS	W	
$(1,3,3,3)$	3.578	1.861	1.861	0.048	1.521	8.869	
$(1,3,3,1)$	3.818	1.647	1.647	0.091	1.572	8.775	

Table A4—Change in Equilibrium Outcomes from Pools of Essential Patents

n	P^E	P_1^N	P_2^N	P_3^N	Q_1	Q_2	Q_3
$(6,3,3,3)$	8.224	3.466	3.466	3.004	0.135	0.135	0.001
$(2,3,3,3)$	5.685	4.410	4.410	3.026	0.320	0.320	0.009
$(1,3,3,3)$	4.578	4.861	4.861	3.048	0.383	0.383	0.016
n	Π_E	Π_1	Π_2	Π_3	CS	W	
$(6,3,3,3)$	2.224	0.466	0.466	0.004	0.315	3.477	
$(2,3,3,3)$	3.685	1.410	1.410	0.026	1.045	7.577	
$(1,3,3,3)$	3.578	1.861	1.861	0.048	1.521	8.869	

Table A5—Profitability of a Pool of Three Essential Patents, in the Absence of a Pool of Patents in $\mathcal{J}_1^N$, when $n_2 = 5$ and $n_3 = 3$

n	P^E	P_1^N	P_2^N	P_3^N	Q_1	Q_2	Q_3
$(3,5,5,3)$	5.012	6.151	6.151	3.084	0.187	0.187	0.027
$(1,5,5,3)$	3.047	7.068	7.068	3.285	0.293	0.293	0.087
n	Π_E	Π_1	Π_2	Π_3	CS	W	
$(3,5,5,3)$	2.012	1.151	1.151	0.084	0.513	4.911	
$(1,5,5,3)$	2.047	2.068	2.068	0.285	1.114	7.582	

Table A6 shows the profitability of the same pool, if a pool of the five patent holders in $\mathcal{J}_1^N$ has already formed. In that case, combined profits of the three essential patent holders fall, from 4.397 to 4.235, when they form a pool.

Table A6—Profitability of a Pool of Three Essential Patents, in the Presence of a Pool of Patents in $\mathcal{J}_1^N$, when $n_2 = 5$ and $n_3 = 3$

n	P^E	P_1^N	P_2^N	P_3^N	Q_1	Q_2	Q_3
$(3, 1, 5, 3)$	7.397	2.279	5.162	3.005	0.561	0.031	0.002
$(1, 1, 5, 3)$	5.235	3.452	5.503	3.022	0.710	0.091	0.007
n	Π_E	Π_1	Π_2	Π_3	CS	W	
$(3, 1, 5, 3)$	4.397	1.279	0.162	0.005	0.902	6.745	
$(1, 1, 5, 3)$	4.235	2.452	0.503	0.022	1.655	8.867	

Table A7—Profitability of a Pool of Three Essential Patents, in the Absence of a Pool of Patents in $\mathcal{J}_1^N$, when $n_2 = 3$ and $n_3 = 5$

n	P^E	P_1^N	P_2^N	P_3^N	Q_1	Q_2	Q_3
$(3, 5, 3, 5)$	5.748	5.700	4.641	5.008	0.123	0.354	0.002
$(1, 5, 3, 5)$	3.632	6.530	5.806	5.035	0.234	0.483	0.007
n	Π_E	Π_1	Π_2	Π_3	CS	W	
$(3, 5, 3, 5)$	2.748	0.700	1.641	0.008	0.650	5.747	
$(1, 5, 3, 5)$	2.632	1.530	2.806	0.035	1.290	8.293	

Table A8—Profitability of a Pool of Three Essential Patents, in the Presence of a Pool of Patents in $\mathcal{J}_1^N$, when $n_2 = 3$ and $n_3 = 5$

n	P^E	P_1^N	P_2^N	P_3^N	Q_1	Q_2	Q_3
$(3, 1, 3, 5)$	7.772	1.980	3.405	5.001	0.495	0.119	0.000
$(1, 1, 3, 5)$	6.001	2.675	3.782	5.002	0.626	0.207	0.000
n	Π_E	Π_1	Π_2	Π_3	CS	W	
$(3, 1, 3, 5)$	4.772	0.980	0.405	0.001	0.952	7.109	
$(1, 1, 3, 5)$	5.001	1.675	0.782	0.002	1.792	9.253	

Next, we do the same, but starting with $(n_E, n_1, n_2, n_3) = (3, 5, 3, 5)$. Table A7 shows the effects of the pool before the pool of patent holders in $\mathcal{J}_1^N$; joint profits of the essential patent holders drop from 2.748 to 2.632.

Table A8 shows the same change after the pool of all five patent holders in $\mathcal{J}_1^N$; joint profits of the essential patent holders now increase, from 4.772 to 5.001.

Example 4: This time, $(n_1, n_2, n_3) = (6, 6, 9)$. Table A9 shows equilibrium outcomes for various values of n_E (unpooled), as well as under pool of all essential patents and a Stackelberg pool of all essential patents. The regular pool corresponds to $n_E = 1$. The Stackelberg pool, with the pool moving first and the nonessential patent holders moving second, gives the same outcome as the static equilibrium of the game with the (continuous) value of n_E that maximizes Π_E, which was calculated numerically to be $n_E \approx 2.129$.

Table A9—Welfare Effects of Various Pools of Essential Patents

	P^E	P_1^N	P_2^N	P_3^N	Q_1	Q_2	Q_3
No pool, $n_E = 4$	5.165	6.762	6.762	9.001	0.113	0.113	0.000
No pool, $n_E = 3$	4.367	7.113	7.113	9.001	0.156	0.156	0.000
No pool, $n_E = 2$	3.438	7.585	7.585	9.003	0.209	0.209	0.000
Stackelberg pool	3.569	7.515	7.515	9.003	0.202	0.202	0.000
Regular pool	2.223	8.271	8.271	9.008	0.275	0.275	0.001

	Π_E	Π_1	Π_2	Π_3	CS	W
No pool, $n_E = 4$	1.165	0.762	0.762	0.001	0.256	**2.946**
No pool, $n_E = 3$	1.367	1.113	1.113	0.001	0.375	**3.969**
No pool, $n_E = 2$	1.438	1.585	1.585	0.003	0.542	**5.154**
Stackelberg pool	1.440	1.515	1.515	0.003	0.517	**4.990**
Regular pool	1.223	2.271	2.271	0.008	0.799	**6.572**

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