IMPERFECT COMPETITION WITH COMPLEMENTS AND SUBSTITUTES

Daniel Quint[1]
University of Wisconsin

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Abstract. I study price competition in settings where end products are combinations of components supplied by different monopolists, nesting standard models of perfect complements and imperfect substitutes. I show sufficient conditions for a discrete-choice demand system to yield demand for each product which is log-concave in price, and has log-increasing differences in own and another product's price, leading to strong comparative statics results. Many results familiar from simple models, like the price effects of mergers or changes in marginal costs, extend naturally to this more complex setting.

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[1] Department of Economics, University of Wisconsin. E-mail: dquint@ssc.wisc.edu. Parts of this paper were contained in an earlier working paper, "Economics of Patent Pools When Some (But Not All) Patents Are Essential".

1 Introduction

Much of our intuition for the effects of competition comes from the simple models of Cournot and Bertrand – models featuring just one type of competition, either substitutes or complements. For example, these models tell us that in a world where all products are substitutes, mergers lead to higher prices, while in a world where all products are complements, mergers lead to lower prices. Real-world settings, however, often include both complements and substitutes. An important question is whether the insights of these simple models apply in more complex situations.

Consider the following example, a variation on the setting studied empirically by Busse and Keohane (2007).[2] A single city is served by three coal mines, each connected to the city by a separate, independently-owned railroad. Buyers in the city have four choices: to buy coal from Mine 1 and pay for transport on Railroad 1; to buy from Mine 2 and Railroad 2; to buy from Mine 3 and Railroad 3; or to do without coal. Coal from the different mines may be delivered to different points in the city and have different characteristics, and buyers have heterogeneous preferences among them, making the three "end products" imperfect substitutes.

In this paper, I consider a model of imperfect competition in prices that captures this type of situation. Competing products are non-overlapping sets of necessary elements, each supplied by a different monopolist. I show straightforward sufficient conditions on a discrete choice demand model under which our usual intuitions for complements and substitutes extend: under which a merger between Mine 1 and Railroad 1 would lead to lower equilibrium prices for customers of all three mines, while a merger between two of the railroads would lead to higher prices for everyone.

The model also fits well with the licensing of intellectual property related to third-generation (3G) wireless communication technologies. 3G is not a single standard, but five different ones, each evolved from (and therefore backward-compatible with) one or more second-generation technologies. Quoting a Department of Justice Business Review Letter of 2002, "It is reasonably likely that essential patents associated with a single 3G technology... will be complements rather than substitutes... [But] There is a reasonable possibility that the five 3G radio interface technologies will continue to be substitutes for each other, and we would expect the owners of intellectual property rights essential to these technologies to compete, including through price..." These concerns led the

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[2] Busse and Keohane (2007) study a single coal mine, serving several towns; there are two railroads, one or both serving each town.

DOJ to reject the proposed formation of a single platform governing the licensing of all 3G-related patents, and approve instead the formation of five separate entities, each licensing patents related to one of the five competing technologies. As I discuss later, this decision is fully in line with the findings of this paper.

The model in this paper applies equally well to retail competition among products containing elements supplied by other firms. We can think of different car companies or personal computer manufacturers: a single firm sets each retail price, but that price implicitly includes the prices of the various parts or components that went into the product – tires and windshields, or microchips and DVD drives, purchased from outside suppliers. If we imagine that downstream firms set their markups (over marginal cost) simultaneously with the upstream suppliers, then the same model applies. In settings like these, however, the assumption that wholesale prices and downstream markups are set simultaneously may seem artificial. I therefore modify the model such that wholesale prices are set through bilateral negotiations between retailers and each of their suppliers, while retail prices are set unilaterally by retailers. I show that a merger between a retailer and one of its suppliers, or a merger between two suppliers of the same retailer, leads to lower prices for all products, while a merger between suppliers of different retailers leads to higher prices. I also give comparative statics for changes in marginal costs, quality improvements, and the introduction of new products.

The sufficient condition for all of these results is that log demand for each product is concave in its own price and has increasing differences in its own and a competitor's price. I show that a commonly-used regularity condition (log-concavity of cumulative and tail distribution functions) imposed on a standard discrete-choice demand model (identical consumers with independent random utility for each alternative) is sufficient to guarantee these properties. This means that in such a model, in the absence of complementarities – if only the downstream retailers (or only the coal mines, without the railroads) were being studied – price competition would be a supermodular game. With complementarities, the game is not supermodular – prices of different components of the same product are strategic substitutes. However, I show that the pricing game has the same equilibrium as a different, and simpler, supermodular game, leading to powerful comparative statics as well as intuition for why they hold. I also extend the model to allow for components which are a required part of *every* product – for example, Intel chips or Microsoft Windows in different manufacturers' PCs; and show that one firm's commitment power (the ability to act as a first-mover in setting a price) is unambiguously profitable but leads to higher prices.

2 Baseline Model and Results

There is a set $\mathcal{K} = \{1, 2, \ldots, K\}$ of **products**. Each product $k \in \mathcal{K}$ has one or more **components**; $\mathcal{T}_k$ will denote the components of product k. No two products share a common component – for $k' \neq k$, $\mathcal{T}_k \cap \mathcal{T}_{k'} = \emptyset$. Each component is produced by a separate monopolist, so the set $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2 \cup \ldots \cup \mathcal{T}_K$ of all components is also the set of firms. Each firm $i \in \mathcal{T}$ has zero fixed costs and constant marginal costs c_i, and sets a price p_i for its components. The price of a product is the sum of the prices of its components, which we will denote $P_k \equiv \sum_{i \in \mathcal{T}_k} p_i$.[3]

Demand for the products (and their components) comes from a measure 1 of price-taking consumers with quasilinear utility. Each consumer can consume one unit of one product, or consume nothing. Consumer l gets payoff v_k^l from product k, and v_0^l from consuming nothing. The demand for each component is the demand for the product it is a part of, which is

$$Q_k(P_1, \ldots, P_K) = \left\| \left\{ l : v_k^l - P_k > \max_{k' \in \mathcal{K} \cup \{0\} - \{k\}} (v_{k'}^l - P_{k'}) \right\} \right\|$$

The solution concept is Bertrand-Nash competition in prices; firms simultaneously name prices, with firm $i \in \mathcal{T}_k$ maximizing its profits $\pi_i = (p_i - c_i) Q_k(\cdot)$.

Throughout the paper, I will use i and j to index firms (or components), k and k' to index products, and l to index consumers.

Assumption 1. *$(v_0^l, v_1^l, \ldots, v_K^l)$ are independent across both k and l. For $k \in \mathcal{K} \cup \{0\}$, v_k^l is drawn from a distribution F_k, where F_k has full support on $\Re$ and both F_k and $1 - F_k$ are log-concave.*

Full support is a technical convenience, and will lead to interior best-responses and a unique equilibrium.[4] As I show next, the other assumption, log-concavity of F_k and $1 - F_k$, will ensure

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[3] This model applies equally well if one "retail" firm $i \in \mathcal{T}_k$ sets the retail price P_k for that product, while paying the wholesale prices $\sum_{i' \in \mathcal{T}_k - \{i\}} p_{i'}$ demanded by the suppliers of the other components.

[4] To see what can go wrong without it, consider the simple case of one product with two components ($K = 1$ and $|\mathcal{T}_1| = 2$), and let F be the distribution of $v_1^l - v_0^l$ (consumers' willingness-to-pay for the product). If supp$(F) \subseteq (-\infty, M]$, then any prices (p_1, p_2) with $p_1, p_2 \geq M$ constitute an equilibrium with no trade. If F is the uniform distribution on $[5, 6]$, then any prices $p_1, p_2 \geq 1$ with $p_1 + p_2 = 5$ are an equilibrium.

that each firm's profit function is log-concave, allowing best-responses to be characterized by first-order conditions. It is a standard regularity condition, and is implied (see Bagnoli and Bergstrom (2005)) by log-concavity of each density function f_k, which is satisfied by many familiar probability distributions.

Lemma 1. *Under Assumption 1, for any* $k \in \mathcal{K}$ *and* $k' \in \mathcal{K} - \{k\}$, $\log Q_k$ *is continuous, differentiable, and concave in* P_k*, and has strictly increasing differences in* P_k *and* $P_{k'}$*.*

The proof, in Appendix A.1, is based on techniques from Barlow and Proschan (1975) for analyzing convolutions of probability distributions. Log-concave demand implies that individual firms' profit functions are strictly log-concave, and firms therefore have unique best-responses characterized by first-order conditions. Even though the pricing game itself is not a supermodular game, I show next that log-supermodularity of the demand functions will allow the equilibrium of the pricing game to be characterized in relation to the equilibrium of a different supermodular game.

Let $n_k = |\mathcal{T}_k|$ be the number of components of product k; $C_k = \sum_{i \in \mathcal{T}_k} c_i$ the combined marginal cost of those components; and $P_{-k} = (P_1, \ldots, P_{k-1}, P_{k+1}, \ldots, P_K)$ a vector of aggregate prices of products other than k.

Lemma 2. *Under Assumption 1, the simultaneous-move pricing game has a unique equilibrium. Each firm* $i \in \mathcal{T}_k$ *sets equilibrium price*

$$p_i \quad = \quad c_i + \frac{1}{n_k}\left(\overline{P}_k - C_k\right)$$

where $(\overline{P}_1, \overline{P}_2, \ldots, \overline{P}_K)$ *is the unique equilibrium of a different game with players* $\{1, 2, \ldots, K\}$*, strategies* $P_k \in \Re^+$*, and payoff functions*[5]

$$u_k(P_k, P_{-k}) \quad = \quad n_k \log(P_k - C_k) + \log Q_k(P_k, P_{-k}) \tag{1}$$

This latter K*-player game is a supermodular game, indexed by both* n_k *and* C_k *for every* k*.*

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[5] To be fully formal, u_k is defined as $-\infty$ for $P_k \leq C_k$. Since Assumption 1 ensures $Q_k(\cdot)$ is always strictly positive, strategies $P_k \leq C_k$ are strictly dominated and can safely be ignored.

Proof of Lemma 2. Fix $k \in \mathcal{K}$ and $i \in \mathcal{T}_k$, and let $p_{-i} = \sum_{j \in \mathcal{T}_k - \{i\}} p_j$. Given other firms' prices, firm i solves

$$\max_{p_i} \{\log(p_i - c_i) + \log Q_k(p_i + p_{-i}, P_{-k})\} \tag{2}$$

By Lemma 1, Q_k is log-concave in P_k, so (2) is strictly concave in p_i; so p_i solves (2) if and only if it satisfies the first-order condition

$$\frac{1}{p_i - c_i} \quad = \quad -\frac{\partial}{\partial p_i} \log Q_k(p_i + p_{-i}, P_{-k}) \quad = \quad -\frac{\partial \log Q_k}{\partial P_k}(P_k, P_{-k}) \tag{3}$$

The right-hand side of (3) depends on k but not i, so in equilibrium, every firm $i \in \mathcal{T}_k$ sets the same markup $p_i - c_i$; so $P_k - C_k = n_k(p_i - c_i)$, and (3) becomes

$$\frac{n_k}{P_k - C_k} \quad = \quad -\frac{\partial \log Q_k}{\partial P_k}(P_k, P_{-k}) \tag{4}$$

This is the first-order condition to the maximization problem

$$\max_{P_k} \{n_k \log(P_k - C_k) + \log Q_k(P_k, P_{-k})\} \tag{5}$$

which is strictly concave, so (4) is satisfied if and only if P_k solves (5). So all firms $i \in \mathcal{T}_k$ are simultaneously best-responding to P_{-k} if and only if their markups are equal to each other and their combined price solves (5). Since this holds for every k, the aggregate prices in the equilibrium of the actual pricing game simultaneously maximize (5) for each $k \in \{1, 2, \ldots, K\}$, and therefore correspond to the equilibrium of the K-player game with payoffs given by (5); and the equilibrium prices of this new game, along with each firm $i \in \mathcal{T}_k$ setting markup $p_i - c_i = \frac{1}{n_k}(P_k - C_k)$, satisfy (3) and therefore solve (2), and are therefore an equilibrium of the full game. By Lemma 1, $\frac{\partial u_k}{\partial P_k} = \frac{n_k}{P_k - C_k} + \frac{\partial \log Q_k}{\partial P_k}$ is increasing in $P_{k'}$ for every $k' \neq k$, so the new K-player game is a supermodular game.

Existence of an equilibrium is shown in Appendix A.2. To establish uniqueness, suppose there were two distinct equilibria, with aggregate prices $(\overline{P}_0, \overline{P}_1, \overline{P}_2, \ldots, \overline{P}_K)$ and $(\overline{P}'_0, \overline{P}'_1, \overline{P}'_2, \ldots, \overline{P}'_K)$. Note that we include imaginary prices for the outside option (buying nothing), but $\overline{P}_0 = \overline{P}'_0 = 0$. Fix $k \in \arg\max_{j \in \mathcal{K}} |\overline{P}'_j - \overline{P}_j|$, assume without loss of generality that $\overline{P}'_k > \overline{P}_k$, and let $\epsilon =$

$\overline{P}'_k - \overline{P}_k > 0$. Let $P_{-k} + \epsilon$ denote adding ϵ to every price in P_{-k}. Each consumer's problem, and therefore demand and price elasticity, are unaffected when the same constant is added to every price including the price of the outside good; so

$$\frac{\partial \log Q_k}{\partial P_k}\left(\overline{P}_k, \overline{P}_0, \overline{P}_{-k}\right) \;=\; \frac{\partial \log Q_k}{\partial P_k}\left(\overline{P}_k + \epsilon, \overline{P}_0 + \epsilon, \overline{P}_{-k} + \epsilon\right) \;>\; \frac{\partial \log Q_k}{\partial P_k}\left(\overline{P}'_k, \overline{P}'_0, \overline{P}'_{-k}\right)$$

The latter inequality holds because $\frac{\partial \log Q_k}{\partial P_k}$ is increasing in $P_{k'}$ for every $k' \in \mathcal{K} \cup \{0\} - \{k\}$ (by Lemma 1), $\overline{P}'_0 = 0 < \epsilon = \overline{P} + \epsilon$, and $\overline{P}'_{k'} \leq \overline{P}_{k'} + \epsilon$ for every $k' \in \mathcal{K} - \{k\}$. But since $\overline{P}_k < \overline{P}'_k$, $\frac{n_k}{\overline{P}_k - C_k} > \frac{n_k}{\overline{P}'_k - C_k}$ as well, so the first-order condition (4) cannot hold at both equilibria, giving a contradiction.

Finally, since $\frac{\partial}{\partial P_k}\left(n_k \log\left(P_k - C_k\right) + \log Q_k\right) = \frac{n_k}{P_k - C_k} + \frac{\partial \log Q_k}{\partial P_k}$ is increasing in n_k and C_k, the game is indexed by n_k and C_k for every k. □

The marginal benefit to firm $i \in \mathcal{T}_k$ of raising its price is $\frac{1}{p_i - c_i} + \frac{\partial \log Q_k}{\partial P_k}(p_i + p_{-i}, P_{-k})$, which is decreasing in p_{-i} and increasing in P_{-k}. This means the prices of the other firms $j \in \mathcal{T}_k - \{i\}$ – manufacturers of the other components of the same product – are strategic substitutes for p_i, and the prices of firms $j \in \mathcal{T} - \mathcal{T}_k$ – manufacturers of competing products – are strategic complements. Given the supermodular structure of the equilibrium, these effects are mutually reinforcing, and therefore persist when considered together. Thus, any exogenous change causing an increase or decrease in a single firm's best-response, will have predictable effects on equilibrium prices:

Theorem 1. *Under Assumption 1,*

1. *An increase in any one firm's marginal cost leads to higher prices for all products*
    - *Fix $k \in \mathcal{K}$ and $i \in \mathcal{T}_k$. If firm i's marginal cost c_i is increased by Δc_i, then for any $k' \neq k$, $0 < \Delta P_{k'} < \Delta P_k < \Delta c_i$, where ΔP is the resulting change in the equilibrium price P. An increase in c_i therefore results in lower demand Q_k for product k and lower profits for all firms $i' \in \mathcal{T}_k$.*
2. *A quality improvement to one product (defined as a constant increase in all consumers' willingness to pay for that product, or a parallel right-shift in one distribution F_k) leads to a higher price for that product and lower prices for all other products*

- *If F_k is shifted to the right by Δv, then for $k' \neq k$, $\Delta P_k - \Delta v < \Delta P_{k'} < 0$. A quality improvement to product k therefore results in greater demand Q_k for that product, and greater profits for all firms $i \in \mathcal{T}_k$ making its components.*

3. *The introduction of a new product leads to lower prices for all existing products*

Proof of Theorem 1. *Part 1.* Since the supermodular game described in Lemma 2 has a unique equilibrium and is indexed by C_k, an increase in C_k leads to higher prices for all products. By the same logic as the uniqueness proof in Lemma 2, if $k' \in \arg\max_{k''} \Delta P_{k''}$ and $k' \neq k$, then both $\frac{n_{k'}}{P_{k'}-C_{k'}}$ and $\frac{\partial \log Q_{k'}}{\partial P_{k'}}$ are lower after the change, and the first-order condition $\frac{n_{k'}}{P_{k'}-C_{k'}} = -\frac{\partial \log Q_{k'}}{\partial P_{k'}}$ could therefore not hold both before and after the change; so $\Delta P_k > \Delta P_{k'}$. But since $\Delta P_k > \Delta P_{k'}$ for every k', $\frac{\partial \log Q_k}{\partial P_k}$ is lower after the change, so $\frac{n_k}{P_k - C_k}$ must be higher, meaning $P_k - C_k$ is lower or $\Delta P_k < \Delta C_k$. Since P_k rose, and rose by more than the price of any other product, Q_k must fall; in addition, $p_{i'} - c_{i'} = \frac{1}{n_k}(P_k - C_k)$ fell for any $i' \in \mathcal{T}_k$, giving lower profits $(p_{i'} - c_{i'})Q_k$.

Part 2. Let $t \in \{0,1\}$ indicate whether or not the right-shift in F_k has occurred, and let $\mathbf{Q}^t(P_1, \ldots, P_K) = (Q^t_1(\cdot), \ldots, Q^t_K(\cdot))$ denote the demand system before and after the shift. Given quasilinear consumer preferences, a price reduction would have the same effect on demand as a quality increase, so $\mathbf{Q}^1(P_k, P_{-k}) = \mathbf{Q}^0(P_k - \Delta v, P_{-k})$; or, letting $P^t_k \equiv P_k - t\Delta v$ denote the "quality-adjusted price" of good k, $\mathbf{Q}^t(P_k, P_{-k}) = \mathbf{Q}^0(P^t_k, P_{-k})$. Thus, rather than the demand system changing, we can think of the demand system remaining constant, but as a function of P^t_k. Firms $i \in \mathcal{T} - \mathcal{T}_k$ have the same payoff functions as before, but as a function of P^t_k, and so their aggregate prices solve $\max_{P_{k'}} \{n_{k'} \log(P_{k'} - C_{k'}) + \log Q^0_{k'}(P_{k'}, P^t_k, P_{-k,k'})\}$; the aggregate price of product k now solves

$$\max_{P^t_k} \left\{ n_k \log(P^t_k + t\Delta v - C_k) + \log Q^0_k(P^t_k, P_{-k}) \right\}$$

This relabeled game is as before a supermodular game in (P^t_k, P_{-k}), and is indexed by $-t$; so opposite to Part 1, the increase in t (from 0 to 1) causes P^t_k and $P_{k'}$ to fall, with P^t_k falling by more (thus $\Delta P^t_k = \Delta P_k - \Delta v < \Delta P_{k'} < 0$). Since P^t_k falls by more than each $P_{k'}$, $\frac{\partial \log Q_k}{\partial P^t_k}$ is higher than before, so $\frac{1}{P^t_k + t\Delta v - C_k}$ must be lower than before if the first-order condition is to hold; thus $P_k - C_k = P^t_k + t\Delta v - C_k$ has risen, or $\Delta P_k > 0$. Since P^t_k falls by more than each $P_{k'}$, Q_k rises, and since $p_i - c_i = \frac{1}{n_k}(P_k - C_k)$ also rises for each $i \in \mathcal{T}_k$, profits are higher.

Part 3. The introduction of a new product can be thought of as the limit, as $M \to +\infty$, of a reduction in product k's costs from $C_k + M$ to C_k; by the reverse of part 1, this implies lower prices for all products. □

The supermodular structure of the equilibrium also allows us to characterize the effects of mergers between firms:

Theorem 2. *Under Assumption 1,*

1. *A merger between suppliers of components of the same product leads to lower prices for all products*
   - *If the merger is between firms $i, j \in \mathcal{T}_k$, then $|\Delta P_{k'}| < |\Delta P_k|$ for any $k' \neq k$, resulting in an increase in Q_k and greater profits for all firms $i' \in \mathcal{T}_k - \{i, j\}$*
2. *A merger between suppliers of components of different products leads to higher prices for all products*

Proof of Theorem 2. *Part 1.* Since the supermodular game in Lemma 2 is indexed by n_k, a reduction in n_k while holding C_k constant (a merger between complements with no cost synergies) lowers all prices. By the same logic as in the proof of uniqueness in Lemma 2, P_k must fall more than any of the other prices, since otherwise, whichever product's price fell the most could not have satisfied its first-order condition (4) both before and after the merger. By the same logic, since $\Delta P_k < \Delta P_{k'} < 0$, $\frac{\partial \log Q_k}{\partial P_k}$ is higher than before, so for any $i' \in \mathcal{T}_k - \{i, j\}$, $p_{i'}$ must rise in order for $\frac{1}{p_{i'} - c_{i'}} = -\frac{\partial \log Q_k}{\partial P_k}$ to hold before and after. Since P_k fell by more than any other price, Q_k must rise, so these firms i' sell more at a higher markup and therefore earn higher profits. (However, since P_k fell and $p_{i'}$ rose for all $i' \in \mathcal{T}_k - \{i, j\}$, the merged firm must set a lower combined price than before, and there is no guarantee such a merger will be profitable.)

Part 2. Note that a merger between substitutes destroys the supermodular structure of the game, so the uniqueness proof from earlier no longer holds; however, the second result holds for *any* post-merger equilibrium, provided one exists. Suppose the merger is between firms $i \in \mathcal{T}_1$ and $j \in \mathcal{T}_2$. The merged firm maximizes $(p_i - c_i)Q_1 + (p_j - c_j)Q_2$; the first-order condition with respect to p_i is $Q_1 + (p_i - c_i)\frac{\partial Q_1}{\partial P_1} + (p_j - c_j)\frac{\partial Q_2}{\partial P_1} = 0$, or rearranging,

$$\frac{1}{p_i - c_i}\left(1 + \frac{p_j - c_j}{Q_1}\frac{\partial Q_2}{\partial P_1}\right) \quad = \quad -\frac{\partial \log Q_1}{\partial P_1} \tag{6}$$

Let x be the equilibrium value of $\frac{p_j - c_j}{Q_1}\frac{\partial Q_2}{\partial P_1}$, so (6) becomes

$$p_i - c_i \quad = \quad \frac{1 + x}{-\partial \log Q_1 / \partial P_1}$$

Along with the usual first-order condition $p_{i'} - c_{i'} = \frac{1}{-\partial \log Q_1 / \partial P_1}$ of the firms $i' \in \mathcal{T}_1 - \{i\}$, this establishes

$$P_1 - C_1 \quad = \quad \frac{n_1 + x}{-\partial \log Q_1 / \partial P_1}$$

or

$$P_1 \quad = \quad \arg\max_{P_1} \{(n_1 + x)\log(P_1 - C_1) + \log Q_1\}$$

By identical arguments,

$$P_2 \quad = \quad \arg\max_{P_2} \{(n_2 + y)\log(P_2 - C_2) + \log Q_2\}$$

where y is the equilibrium value of $\frac{p_i - c_i}{Q_2}\frac{\partial Q_1}{\partial P_2}$. Since components i and j are substitutes, nonpositive markups for either product are strictly dominated for the merged firm, and so $x, y > 0$; so the merger corresponds to increases in (n_1, n_2) from their old values to $(n_1 + x, n_2 + y)$. Since the K-player game described in Lemma 2 is supermodular and indexed by n_1 and n_2, this means all prices are higher post-merger. □

Theorems 1 and 2 establish the effect on a firm $i' \in \mathcal{T}_k$ of a reduction in cost of one of its complements $i \in \mathcal{T}_k - \{i\}$, or a merger between two of its complements $i, j \in \mathcal{T}_k - \{i\}$: firm i' gets to charge a higher price but still sell more units, leading to unambiguously higher profits. When a cost reduction or merger involves a different product k', however, the effect on the price charged by firm $i' \in \mathcal{T}_k$ is clear, but understanding the effect on demand Q_k, or profits $\pi_{i'}$, requires an additional assumption.

To see why, first consider the case where product k has only one component ($n_k = 1$). Suppose

$P_{k'}$ increased, due to an increase in the marginal cost of one of its components, and the firm selling product k responded by raising its own price. We would know its profits were higher than before, since the increase in $P_{k'}$ increased the demand for product k at its old price, and the change in P_k was the move to a new best-response and therefore only increased profits further. When $n_k > 1$, however, this logic no longer holds. From the supermodular structure of the game, we know that if $P_{k'}$ goes up, P_k will go up in response; but since the firms $j \in \mathcal{T}_k$ are not working together, there is no guarantee the latter step increases profits. It is theoretically possible that something apparently "good" for the firms in $\mathcal{T}_k$ – an increase in the price of a competing product – could alter the demand curve in a way that makes the double-marginalization problem among these firms more severe, so that after they have adjusted their prices, profits are lower than before.

While it seems like natural regularity conditions on the demand system should rule this out, the sufficient conditions I have found to eliminate this problem are not that intuitive.

Assumption 2. *For every k and every $k' \neq k$, $\varepsilon_{k,k}/\varepsilon_{k,k'}$ is increasing in P_k, where $\varepsilon_{k,k} = -\frac{\partial \log Q_k}{\partial \log P_k} > 0$ and $\varepsilon_{k,k'} = \frac{\partial \log Q_k}{\partial \log P_{k'}} > 0$ are the usual own- and cross-price elasticities.*

This is not an unnatural requirement: since $\frac{\varepsilon_{k,k}}{\varepsilon_{k,k'}} = \frac{P_k}{P_{k'}} \left(-\frac{\partial \log Q_k}{\partial P_k} \right) \Big/ \frac{\partial \log Q_k}{\partial P_{k'}}$, it requires only that $\frac{\partial \log Q_k}{\partial P_{k'}}$ not increase in P_k much faster than $-\frac{\partial \log Q_k}{\partial P_k}$ does, so that the decrease in the ratio $-\frac{\partial \log Q_k}{\partial P_k} \Big/ \frac{\partial \log Q_k}{\partial P_{k'}}$ cannot overwhelm the increase in $\frac{P_k}{P_{k'}}$.[6] If this condition is satisfied, then changes in the price of one product have predictable effects on the firms selling competing products:

Lemma 3. *Under Assumptions 1 and 2, an increase in $P_{k'}$, followed by the adjustment of prices by firms $i \in \mathcal{T}_k$ to their new mutual best-responses, leaves demand for product k higher than before.*

Lemma 3 is proved in Appendix A.3. Combining Lemma 3 with Theorems 1 and 2 above gives the following results:

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[6] If the demand system is logit, as in Anderson, de Palma and Thisse (1992) and many others, then $\frac{\partial \log Q_k}{\partial P_k} \Big/ \frac{\partial \log Q_k}{\partial P_{k'}}$ is constant as P_k changes, so Assumption 2 holds. (For much more on my model under the additional assumption of logit demand, see Quint (2012).) Appendix A.3 also notes how this assumption can be weakened while preserving some or all of Theorem 3.

Theorem 3. *Under Assumptions 1 and 2, any of the following lead to lower equilibrium demand* Q_k *for product* k *and lower profits for all firms* $i' \in \mathcal{T}_k$ *producing its components:*

1. *The introduction of a new competing product* $k' \neq k$
2. *The reduction of the marginal cost* c_i *of any component* $i \in \mathcal{T}_{k'}$ *of any competing product* $k' \neq k$
3. *A quality improvement to any competing product* $k' \neq k$
4. *A merger between two firms* $i, j \in \mathcal{T}_{k'}$ *producing components of the same competing product* $k' \neq k$

On the other hand, a merger between firms $i \in \mathcal{T}_{k'}$ *and* $j \in \mathcal{T}_{k''}$*, with* $k'' \neq k' \neq k$*, leads to increased demand* Q_k *and increased profits for all firms* $i' \in \mathcal{T}_k$*.*

Proof of Theorem 3. Following any of these changes, decompose the move from old equilibrium prices to new prices into $K-1$ steps: in each step, for one $k'' \in \mathcal{K} - \{k\}$, the firms in $\mathcal{T}_{k''}$ change from their old to their new equilibrium prices, and the firms in $\mathcal{T}_k$ move to their new simultaneous best-responses. By Theorems 1 and 2, each step involves a reduction in $P_{k''}$ (or, in the case of firm k' following a right-shift in $F_{k'}$, a reduction in the quality-adjusted price $P_{k'} - \Delta v$); by Lemma 3, then, each step leaves Q_k lower than before. Since Q_k and P_k both end up lower than before, for $j \in \mathcal{T}_k$, firm j's profits $\pi_j = \frac{1}{n_k}\left(P_k - C_k\right) Q_k$ are lower than before. For the merger between firms $i \in \mathcal{T}_{k'}$ and $j \in \mathcal{T}_{k''}$, the same logic holds in reverse. $\square$

Note that between Theorems 2 and 3, we have pinned down the effect of a merger between two firms $i, j \in \mathcal{T}_k$ on every firm other than the two merged firms: the effect on profits is positive for firms $i' \in \mathcal{T}_k - \{i, j\}$, and negative for firms $i' \in \mathcal{T} - \mathcal{T}_k$. However, we have not said anything about the effect on the merged firms. The combined profits of all the firms in $\mathcal{T}_k$ are likely to be higher, but could potentially be lower post-merger (since Q_k rose but P_k fell); and the merged firms now get a fraction $\frac{1}{n_k - 1}$ rather than $\frac{2}{n_k}$ of those joint profits. Whether a particular merger between complementary firms is profitable for them is an empirical question. (Quint (2012) gives conditions under which particular mergers are guaranteed to be profitable within a logit demand system.)

Pure substitutes and pure complements are special cases of this model, about which we can say more:

Special Case 1 (Pure Complements)**.** *Suppose* $K = 1$*, so all components are perfect complements. Then under Assumption 1, cost reductions are Pareto-improving, as they increase both consumer surplus and every firm's profit. And provided a merger is profitable for the merged firm, it represents a Pareto-improvement as well.*

Special Case 2 (Pure Substitutes)**.** *Suppose* $n_1 = n_2 = \ldots = n_K = 1$*, so each product has just one component, and therefore all components are substitutes. Then Assumption 2 is not required for the profit results in Theorem 3: under only Assumption 1, a reduction in one firm's marginal cost, a new product, or a quality improvement to any existing product reduces the profits of all other firms. Additionally, a merger between any two firms is now guaranteed to increase the profits of all firms, including the joint profits of the merging firms.*

Thus, as expected, mergers in a pure complements world are Pareto-improving, as they lessen a double-marginalization problem, while mergers in a pure substitutes world represent a gain to producers but a loss to consumers.

3 Extensions to Baseline Model

Wholesale Pricing through Bilateral Negotiation

If we think of each product representing a distinct vertical supply chain, the assumption that all firms set prices simultaneously, and that all firms producing components of the same product are inherently symmetric, may seem artificial. That is, it might seem strange to suppose that each of Ford's suppliers for tires, brake pads, and so on get to unilaterally name prices, which are tacked on to the price of each Ford vehicle; and similarly strange to suppose that Ford cannot use the threat to change suppliers to force narrower margins on its suppliers. Next, I present an extension to the baseline model that addresses these concerns. Each product will be represented by a single dominant (downstream) firm, with full control over the final price of the product, and one or more upstream suppliers. Wholesale prices will be determined via simultaneous bilateral bargaining between the downstream firm and each of its suppliers. Competition across upstream firms to supply a particular downstream firm will still not be explicitly modeled; however, the extent of such competition can be incorporated, at least roughly, by manipulating the bargaining power ascribed to each upstream firm.

Continue to assume that there are K products labeled $1, 2, \ldots, K$. For each product k, suppose firm i_k^0 is the retail seller, and firms $i_k^1, i_k^2, \ldots, i_k^{m_k}$ are suppliers of components to that retailer. For $j \in \{0, 1, \ldots, m_k\}$, let c_k^j denote the marginal cost of firm i_k^j; for $j \in \{1, \ldots, m_k\}$, let p_k^j denote the price charged by firm i_k^j for its components; and let P_k denote the retail price, set by firm i_k^0.

We will assume that each wholesale price p_k^j is determined via Nash bargaining between that wholesale supplier and the retailer, so that p_k^j is set as the maximizer of the usual Nash product $(\pi_k^j)^{\phi_k^j}(\pi_k^0)^{1-\phi_k^j}$, where ϕ_k^j is the bargaining power of the wholesale firm i_k^j. However, to solve this maximization problem, we need to know how the two parties expect retail prices to respond to the wholesale price p_k^j. That is, the Nash bargaining could occur under the belief that P_k is fixed, regardless of p_k^j; that the retailer's markup $P_k - p_k^j$ is fixed, so P_k would increase one-to-one with an increase in p_k^j; or any other belief. We will assume that during Nash bargaining, competitors' prices $P_{k'}$ $(k' \neq k)$ are taken as fixed, and it is assumed that P_k will increase linearly with p_k^j, with slope $\alpha_k^j \in [0, 1]$. Downstream prices P_k are then set to maximize retailer profits, taking wholesale prices as fixed.

Under these assumptions, prices $(\overline{P}_1, \ldots, \overline{P}_K)$ and $(\overline{p}_k^j)_{k,j}$ constitute an equilibrium if and only if they simultaneously satisfy

$$\overline{P}_k = \arg\max_{P_k} \left\{ \left(P_k - c_k^0 - \sum_{j>0} \overline{p}_k^j \right) Q_k(P_k, \overline{P}_{-k}) \right\}$$

$$\overline{p}_k^j = \arg\max_{p_k^j} \left\{ \begin{array}{l} \left[\left(p_k^j - c_k^j \right) Q_k \left(\overline{P}_k + \alpha_k^j (p_k^j - \overline{p}_k^j), \overline{P}_{-k} \right) \right]^{\phi_k^j} \times \\ \left[\left(\overline{P}_k + \alpha_k^j (p_k^j - \overline{p}_k^j) - c_k^0 - p_k^j - \sum_{j' \neq j} \overline{p}_k^{j'} \right) Q_k \left(\overline{P}_k + \alpha_k^j (p_k^j - \overline{p}_k^j), \overline{P}_{-k} \right) \right]^{1-\phi_k^j} \end{array} \right\}$$

for each $k \in \mathcal{K}$ and each $j \in \{1, 2, \ldots, m_k\}$. While these conditions look complex, they lead to a surprisingly straightforward characterization of equilibrium prices, which are a natural analog to the baseline model:

Lemma 4. *Suppose that* $\phi_k^j \in (0, 1)$ *and* $\alpha_k^j \in [0, 1]$ *for every* $k \in \mathcal{K}$ *and every* $j \in \{1, 2, \ldots, m_k\}$. *Under Assumption 1...*

1. *The game described above – simultaneous Bertrand-Nash competition in the retail market and*

bilateral Nash bargaining in the upstream markets, with exogenously fixed (ϕ_k^j) *and* (α_k^j) *– has a unique equilibrium*

2. *Let* $\beta_k^j \equiv \frac{\phi_k^j}{\alpha_k^j+(1-\alpha_k^j)(1-\phi_k^j)}$ *and* $n_k = 1 + \sum_{j=1}^{m_k} \beta_k^j$. *Then...*
   - *(i) Equilibrium retail prices* $(\overline{P}_1, \overline{P}_2, \ldots, \overline{P}_K)$ *are those described in Lemma 2 – that is, the equilibrium of the* K*-player game with payoffs* $u_k = n_k \log(P_k - C_k) + \log Q_k(P_k, P_{-k})$
   - *(ii) Equilibrium wholesale prices are* $\overline{p}_k^j = c_k^j + \frac{\beta_k^j}{n_k}(\overline{P}_k - C_k)$, *and the equilibrium markup of the retailer is* $\overline{P}_k - \sum_{j>0} \overline{p}_k^j = c_k^0 + \frac{1}{n_k}(\overline{P}_k - C_k)$

This is proved in Appendix A.4. As a result of the similarity between the equilibrium characterizations of this model and the baseline model, most of the previous results apply here as well:

Theorem 4. *Under Assumption 1,*

1. *An increase in the marginal cost* c_k^j *of firm* i_k^j *leads to higher retail prices for all products, lower demand* Q_k *for product* k*, and lower profits for all firms* $i_k^{j'}$. *Under Assumption 2, it also leads to greater demand for all other products* $Q_{k'}$ *and higher profits for all firms* $i_{k'}^{j'}$ $(k' \neq k)$.

2. *A quality increase to product* k *leads to a higher retail price* P_k*, higher demand* Q_k*, and higher profits for each firm* i_k^j $(j \in \{0, 1, \ldots, m_k\})$*, and lower prices* $P_{k'}$ *for all other products* $k' \neq k$. *Under Assumption 2, it also leads to lower demand* $Q_{k'}$ *and lower profits for firms* $i_{k'}^{j'}$ *for* $k' \neq k$. *The introduction of a new product has these same effects on all existing products.*

3. *A merger between a retail firm* i_k^0 *and one of its suppliers* i_k^j*, provided it does not affect the bargaining position* ($\phi_{k'}^{j'}$ *and* $\alpha_{k'}^{j'}$) *of other firms, leads to lower retail prices for all products, greater demand for product* k*, and higher profits for non-merging firms* $i_k^{j'}$ $(j' \neq 0, j)$. *Under Assumption 2, it also leads to lower demand for all other products* $Q_{k'}$ *and lower profits for all firms* $i_{k'}^{j'}$ $(k' \neq k)$.

4. *A merger between two complementary suppliers* i_k^j *and* $i_k^{j'}$ $(j \neq j' \neq 0)$ *has the same effects, provided the bargaining power of the merged firm is such that* $\beta_k^{j,j'}$ *post-merger is no greater than the pre-merger sum* $\beta_k^j + \beta_k^{j'}$.

In light of Lemma 4, the proofs are identical to those of Theorems 1, 2, and 3. The baseline model can therefore be thought of as a reduced form for a more complex model involving bilateral negotiation between upstream suppliers and downstream retailers, who then compete in prices.

Essential Components

Next, we extend the model to allow for components which are common to *all* products. That is, instead of requiring no overlap in the components of the various products – $\mathcal{T}_k \cap \mathcal{T}_{k'} = \emptyset$ – we now allow for a common, nonempty overlap, $\mathcal{T}_k \cap \mathcal{T}_{k'} = \mathcal{T}_1 \cap \mathcal{T}_2 \cap \ldots \cap \mathcal{T}_K \equiv \mathcal{T}^E$. We will refer to these new components required for every product as *essential* – think of these as monopolists in a supply chain (a single national railroad that transports coal for all coal companies), or necessary components with no substitutes (Microsoft Windows for the mainstream PC market). Let $n_E = |\mathcal{T}^E|$ be their number, $C^E = \sum_{i \in \mathcal{T}^E} c_i$ their combined marginal costs, and $P^E = \sum_{i \in \mathcal{T}^E} p_i$ their combined price. Assume that the essential suppliers do not price-discriminate, that is, they set a single price for the entire market.

To differentiate the other components from the essential ones, let $\mathcal{T}_k^N = \mathcal{T}_k - \mathcal{T}^E$ (N for nonessential) now denote the set of components required only for product k, C_k^N their combined marginal costs, and P_k^N their combined price; a consumer wishing to consume product k must therefore buy each of the components in $\mathcal{T}^E \cup \mathcal{T}_k^N$, at a price $P_k = P^E + P_k^N = \sum_{i \in \mathcal{T}^E \cup \mathcal{T}_k^N} p_i$. Let $Q_k(\cdot)$ still denote the demand for product k, this time as a function of the $K+1$ aggregate prices $(P_1^N, \ldots, P_K^N, P^E)$: and let $Q_A(\cdot) = \sum_{k \in \mathcal{K}} Q_k(\cdot)$ denote the combined demand for all the products (and therefore the demand for each essential component).

Similar to Lemma 2, I will show that the equilibrium of the pricing game here coincides with the equilibrium of a different, now $K+1$-player game, which is a supermodular game in $(P_1^1, \ldots, P_K^N, -P^E)$. Under Assumption 1, $\log Q_k$ is still concave in P_k^N and supermodular in $(P_k^N, P_{k'}^N)$ for $k' \neq k$, and also turns out to be supermodular in $(P_k^N, -P^E)$. We will similarly require concavity of $\log Q_A$, and supermodularity of $\log Q_A$ in $(P^E, -P_k^N)$; unfortunately, these do not follow directly from Assumption 1. I have not found a simple condition on model primitives which guarantees these conditions; instead, I explicitly assume what I need, and then argue (in Appendix A.5) that it's a reasonable assumption. Both Assumptions 3 and 4 below turn out to hold under logit demand, a model commonly used in the empirical literature.

Assumption 3. $\frac{\partial \log Q_A}{\partial P^E}$ *is decreasing in* P^E *and in* P_k^N.

In Appendix A.5, I discuss why this is a reasonable assumption. As with the baseline model, Assumptions 1 and 3 guarantee log-concave profit functions and supermodularity of a transformed game characterizing equilibrium prices; and as with the baseline model, an additional assumption is required for some of the comparative statics:

Assumption 4. *For any* $k, k' \in \mathcal{K}$, $\frac{\varepsilon_{k,k}}{\varepsilon_{k,k'}}$ *and* $\frac{\varepsilon_{k,k}}{\varepsilon_{k,E}}$ *are increasing in* P_k^N *and* $\frac{\varepsilon_{A,E}}{\varepsilon_{A,k}}$ *is increasing in* P^E*, where* $\varepsilon_{k,E} = -\frac{\partial \log Q_k}{\partial \log P^E}$, $\varepsilon_{A,E} = -\frac{\partial \log Q_A}{\partial \log P^E}$*, and* $\varepsilon_{A,k} = -\frac{\partial \log Q_A}{\partial \log P_k^N}$.

These serve the exact same purpose as Assumption 2 did in the previous model. The effect of these assumptions – the analog of Lemma 3 in the setting with some essential components – is stated and proved in Appendix A.6, as Lemma 7. The equilibrium characterization for the model with essential components is nearly identical to the characterization of the baseline model:

Lemma 5. *Under Assumptions 1 and 3, the simultaneous-move pricing game with essential components has a unique equilibrium. Firm i sets equilibrium price*

$$p_i = \begin{cases} c_i + \frac{1}{n_k}\left(\overline{P}_k^N - C_k^N\right) & if \quad i \in \mathcal{T}_k^N \\ c_i + \frac{1}{n_E}\left(\overline{P}^E - C^E\right) & if \quad i \in \mathcal{T}^E \end{cases}$$

where $\left(\overline{P}_1^N, \ldots, \overline{P}_K^N, \overline{P}^E\right)$ *is the unique equilibrium of a different,* $K+1$ *player game with players* $\mathcal{K} \cup \{E\}$ *and payoff functions*[7]

$$u_k(\,\cdot\,) = \begin{cases} n_k \log(P_k^N - C_k^N) + \log Q_k(\,\cdot\,) & for \quad k \in \mathcal{K} \\ n_E \log(P^E - C^E) + \log Q_A(\,\cdot\,) & for \quad k = E \end{cases}$$

Further, this latter game is a supermodular game in $\left(P_1^N, \ldots, P_K^N, -P^E\right)$*, indexed by* C_k^N *and* n_k *(*$k \in \mathcal{K}$*) and* $-C^E$ *and* $-n_E$.

The proof is similar to that of Lemma 2, and can be found in Appendix A.7. The supermodular characterization leads to comparative statics analogous to Theorems 1, 2, and 3. Results for essential firms are similar to those for firms in a pure-complements model; results for non-essential

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[7] Again, to be complete, u_k is defined as $-\infty$ for $P_k^N \leq C_k^N$, and u_E as $-\infty$ for $P^E \leq C^E$.

firms are similar to those in the baseline model. For simplicity, I do not distinguish those results which rely only on Assumptions 1 and 3 from those which also require Assumption 4.

Theorem 5. *Under Assumptions 1, 3, and 4,*

1. *A decrease in costs* c_i *for* $i \in \mathcal{T}_k^N$*, or a merger between two firms* $i, j \in \mathcal{T}_k^N$*, both lead to...*
    - *a lower price* P_k *for product* k*, but an ambiguous effect on the prices* $P_{k'}$ *of other products*
    - *higher markups and profits for (nonmerging) firms* $i' \in \mathcal{T}_k^N$ *and greater demand for product* k
    - *lower prices and profits for firms* $i' \in \mathcal{T}_{k'}^N$ *for* $k' \neq k$*, and lower demand for product* k'
    - *higher prices and profits for firms* $i' \in \mathcal{T}^E$*, and greater combined demand for all products*
2. *A decrease in costs* c_i *for* $i \in \mathcal{T}^E$*, or a merger between two firms* $i, j \in \mathcal{T}^E$*, both lead to lower prices for every product, higher demand for every product, and higher markups and profits for every (nonmerging) firm*

Theorem 5 is proved in Appendix A.8. As before, with suitable adjustments to wording, part 1 applies as well to quality improvements and new products.

First-Mover Advantage

In any static game with a unique, pure-strategy equilibrium, a player can never be worse off when he is suddenly allowed to act as a first-mover, anticipating that all remaining players will then play mutual best-responses to his move. This is because, if he cannot do better, he can always simply play his static equilibrium strategy, leading to the same outcome as before.

In this particular game, however, there are always strict gains from the ability to be a Stackelberg leader, and the leader always prices strictly higher than in the static game. This is because strategic complementarities and profit externalities go in the same direction. I benefit from higher prices from firms producing substitutes for my product; but these firms' prices are strategic complements to mine, and they therefore go up when I rase my price. Similarly, I benefit from lower prices from firms producing complements, but these firms lower their prices when I raise mine. And the supermodular structure of the game ensures that the adjustments by all the other players reinforce these moves. Thus, raising my price (and committing to not lower it) leads to only positive indirect effects, leading to the following result:

Theorem 6. *1. In either the baseline model or the model with essential components, a firm which prices as a Stackelberg first-mover (committing to its own price first, knowing all other firms will play mutual best-responses) will set a strictly higher price, and earn strictly higher profits, than it would have in the simultaneous-move equilibrium.*

2. In the baseline model, if one firm prices as a Stackelberg first-mover, the equilibrium prices of all products will be higher than the prices in the simultaneous-move game.

Proof of Theorem 6. *Part 1.* I show this for the model with essential components (under Assumptions 1 and 3), since the baseline model is simply the special case $n_E = 0$. Consider a firm $i \in \mathcal{T}_k^N$ which prices as a first-mover, and consider a small increase in p_i above its static-equilibrium level. Since p_i is close to the optimum, the increase will have only a second-order direct effect on firm i's profit. But by the same logic as in the proof of Theorem 5, this will lead to lower prices from the other firms in $\mathcal{T}_k^N$, as well as the firms in $\mathcal{T}^E$, and higher prices from the firms in $\mathcal{T}_{k'}^N$ – all of which have first-order positive effects on firm i's profits. So firm i can increase its profits beyond the static-equilibrium level with a small increase in price.

On the other hand, if firm i priced lower than in the static equilibrium, we could break the move from the static to the Stackelberg equilibrium prices into two steps: first, firm i lowers its price away from its old static best-response, and second, everyone else adjusts their prices to the new level, with firms $j \in \mathcal{T}_k^N \cup \mathcal{T}^E - \{i\}$ raising their prices and firms $j \in \mathcal{T}_{k'}^N$ $(k' \neq k)$ lowering theirs. Each of these steps reduces firm i's profits. So firm i prices higher as a first-mover, and earns strictly higher profits.

For a firm $i \in \mathcal{T}^E$, the same logic would hold: an increase in p_i, by the same logic as in the proof of Theorem 5, would lead to lower prices by all other firms, strictly increasing i's profits, while a decrease in p_i would lead to higher prices from all other firms and therefore lower profits for firm i.

Part 2. If we treat the first-mover's price as an exogenous parameter, we can show that the K-player supermodular game characterizing aggregate prices is indexed by that parameter. (If the first-mover is firm $i \in \mathcal{T}_k$, and we let $C_k^{-i} = \sum_{i' \in \mathcal{T}_k - \{i\}} c_{i'}$, then by the same logic as in Lemma

2, P_k solves $\max_{P_k} (n_k - 1) \log(P_k - p_i - C_k^{-1}) + \log Q_k(P_k, P_{-k})$, which has increasing differences in P_k and p_i; combining this with the other $K - 1$ conditions, $(P_1, \ldots, P_K)$ are the equilibrium of a supermodular game indexed by p_i.) Since p_i is higher in the Stackelberg game than in the simultaneous-move game, all aggregate prices are higher. □

4 Related Literature

The discrete-choice demand general differentiated-products framework I use is closely related to work done by others. Deneckere and Davidson (1985) consider mergers of firms producing substitutes in a linear demand system, and show that such mergers are profitable for both the merging firms and for outsiders. Perloff and Salop (1985) consider the symmetric case ($F_k = F_{k'}$) of the discrete choice model I use, and show existence and uniqueness of a "single-price" equilibrium. Chen and Riordan (2008) consider the two-firm case, with a more general (symmetric) joint distribution of valuations. Gabaix et. al. (2010) characterize equilibrium prices in the limit as the number of (identical) firms in the market goes to infinity. A "special case" of the model I use is the logit model used in Anderson, de Palma and Thisse (1992) and many subsequent papers. The most general model I'm aware of is that of Caplin and Nalebuff (1991). They situate each product in an m-dimensional space of product attributes over which consumers have preferences. They allow preferences over these different product dimensions to be correlated; they show that a condition similar to log-concavity of the density function (but slightly weaker) is sufficient to guarantee existence of an equilibrium, although they show uniqueness and log-supermodularity only for special cases. Berry, Levinsohn and Pakes (1995) use a similar model as the framework for empirical estimation.

A number of other recent papers examine settings with both complements and substitutes. Casadesus-Masanell, Nalebuff and Yoffie (2007) consider a downstream monopolist (Microsoft) and perfect competition among two upstream suppliers (Intel and AMD); Chen and Nalebuff (2006) consider a monopolist in one market (Microsoft as the supplier of an operating system) who also competes in a complementary market (Microsoft and Netscape in the browser market). The recent literature on two-sided markets (such as newspapers, which must attract both advertisers and readers) and competition among platforms (such as XBox and PlayStation, which may be substitutes for consumers but are each accompanied by a collection of complementary products) also considers both complementarities and substitutes, although the focus is different (see, for example, Carrillo and Tan (2006), Rochet and Tirole (2006), Armstrong (2006), and Weyl (2010)). Coexistence of substitutes and complements is also explicitly allowed in the recent extension of the two-sided matching literature to supply chains and other settings: see Ostrovsky and (2008) and Hatfield and Kominers (2012).

5 Conclusion

Certain received wisdom about price competition – for example, that mergers lead to higher prices when firms produce substitutes, but lower prices when firms produce complements – comes from simple models where only one type of competition is considered. I show that these effects persist in a setting where a given good has both complements and substitutes. In particular, when competing supply chains do not overlap, vertical mergers are consumer-friendly, while horizontal mergers between levels of competing value chains are not.

The Department of Justice ruling on 3G patent licensing was fully in line with these insights. The DOJ rejected a proposal to create a single Patent Platform (similar to a traditional patent pool, but with more flexibility) which would handle licensing of all 3G-related patents; instead, five separate Patent Platforms were formed, one for each competing radio interface technology. In other words, full vertical integration was allowed – for pricing purposes, each set of complementary firms was replaced by a single entity – while the proposed horizontal merger was not. This is consistent with the DOJ's mandate to promote competition, as this policy would be expected to lead to the lowest possible licensing costs. (Quint (2012) applies the model in this paper to patent pools more generally, under stronger conditions on the demand system, focusing on the distinction between essential and non-essential patents.)

Two other markets that might fit this model reasonably well have already been mentioned above. The market for Windows-compatible personal computers involves many competing retailers (Acer, Dell, Gateway, HP, Lenovo, etc.); some essential components (Intel and Microsoft); and lots of nonessential manufacturers (sources for hard drives, optical drives, and other components). Similarly, to the extent that competing car manufacturers have non-overlapping supply chains, the model presented above could apply. The market for delivered coal, discussed in Busse and Keohane

(2007), does not fit the model perfectly, as many of the power plants purchasing coal are serviced by only one railroad, and therefore do not face the full menu of available "products", and railroads each deliver coal from many mines; but the model could potentially be adapted to this type of market. A similar modification might apply the model to the cell phone market, where phone manufacturers (Nokia, Samsung, LG, Motorola, Sony Ericsson, and more recently Apple) partner with service providers (AT&T, Verizon, Sprint).

One significant limitation of the model considered in this paper is the exclusive focus on single-product firms. When each firm produces multiple components, or a single retailer offers several different products, the supermodular equilibrium structure demonstrated above does not hold, as a single firm's log-profit function need not have increasing differences in any two of its own prices. Finding conditions under which comparable results can be achieved for multi-product firms is a significant challenge left for future work.

Appendix – Omitted Proofs

A.1 Proof of Lemma 1

The claim was that under Assumption 1, Q_k is log-concave and log-supermodular in $(P_k, P_{k'})$. Let G_k be the distribution of

$$v_k^l - \max_{j \in \mathcal{K} \cup \{0\} - \{k\}} (v_j^l - P_j)$$

(with $P_0 = 0$), so $Q_k = 1 - G_k(P_k)$. Let F_{-k} be the distribution of $\max_{j \in \mathcal{K} \cup \{0\} - \{k\}} (v_j^l - P_j)$. Since v_j^l are all independent and each is drawn from a distribution F_j,

$$\begin{aligned} F_{-k}(t) &= \Pr\left(\max_{j \in \mathcal{K} \cup \{0\} - \{k\}} (v_j^l - P_j) < t\right) \\ &= \Pr\left(v_j^l < t + P_j \;\; \forall j \in \mathcal{K} \cup \{0\} - \{k\}\right) \\ &= \textstyle\prod_{j \in \mathcal{K} \cup \{0\} - \{k\}} F_j(t + P_j) \end{aligned}$$

and, differentiating,

$$\begin{aligned}
f_{-k}(t) &= \textstyle\sum_{j\in\mathcal{K}\cup\{0\}-\{k\}}\left(f_j(t+P_j)\prod_{i\in\mathcal{K}\cup\{0\}-\{k\}-\{j\}}F_i(t+P_i)\right)\\
&= \textstyle\sum_{j\in\mathcal{K}\cup\{0\}-\{k\}}\left(\frac{f_j(t+P_j)}{F_j(t+P_j)}\prod_{i\in\mathcal{K}\cup\{0\}-\{k\}}F_i(t+P_i)\right)\\
&= \left(\textstyle\sum_{j\in\mathcal{K}\cup\{0\}-\{k\}}\frac{f_j(t+P_j)}{F_j(t+P_j)}\right)F_{-k}(t)
\end{aligned}$$

so $\frac{f_{-k}(t)}{F_{-k}(t)} = \sum_{j\in\mathcal{K}\cup\{0\}-\{k\}}\frac{f_j(t+P_j)}{F_j(t+P_j)}$. Since each F_j is log-concave by assumption, this is decreasing in both t and P_j for any j. Letting $X_1 = v_k^l$ and $X_2 = \max_{j\in\mathcal{K}\cup\{0\}-\{k\}}(v_j^l - p_j)$ in Lemma 6 below completes the proof.

Lemma 6. *Let X_1 and X_2 be independent random variables with full support on on $(-\infty,\infty)$ and continuous, differentiable distribution functions F_1 and F_2. Suppose that the distribution F_2 is parameterized by some parameter a which does not affect F_1. Let F_3 be the distribution of $X_1 - X_2$.*

1. *F_3 is continuous and differentiable, with full support.*
2. *If $1 - F_1$ and F_2 are log-concave, then $1 - F_3$ is log-concave.*
3. *If $1 - F_1$ is log-concave and $\frac{f_2}{F_2}$ is decreasing in a, then $\frac{f_3}{1-F_3}$ is decreasing in a*

Proof. Following Barlow and Proschan (1975), write $\overline{F}(\cdot)$ for $1 - F(\cdot)$. Since $\Pr(X_1 - X_2 > t) = \Pr(X_1 > t + X_2)$, we can write

$$\overline{F}_3(t,a) = \int_{-\infty}^{\infty}\overline{F}_1(t+s)f_2(s,a)ds$$

Continuity, differentiability, and full support follow.

Barlow and Proschan (p. 100) use techniques from Karlin (1968) to show that if F is the distribution of $X_1 + X_2$ and $1 - F_1$ and $1 - F_2$ are log-concave, so is $1 - F$; flipping the sign on X_2 gives the second result above. They show $\overline{F}$ is log-concave – or, equivalently, that $\overline{F}(t-u)$ is log-supermodular in (t,u) – by letting

$$D \equiv \begin{vmatrix} \overline{F}(t_1-u_1) & \overline{F}(t_1-u_2) \\ \overline{F}(t_2-u_1) & \overline{F}(t_2-u_2) \end{vmatrix}$$

rewriting it as

$$D = \iint_{s_1<s_2} \begin{vmatrix} \overline{F}_1(t_1-s_1) & \overline{F}_1(t_1-s_2) \\ \overline{F}_1(t_2-s_1) & \overline{F}_1(t_2-s_2) \end{vmatrix} \begin{vmatrix} f_2(s_1-u_1) & f_2(s_1-u_2) \\ f_2(s_2-u_1) & f_2(s_2-u_2) \end{vmatrix} ds_2 ds_1$$

integrating by parts to get

$$D = \iint_{s_1<s_2} \begin{vmatrix} \overline{F}_1(t_1-s_1) & f_1(t_1-s_2) \\ \overline{F}_1(t_2-s_1) & f_1(t_2-s_2) \end{vmatrix} \begin{vmatrix} f_2(s_1-u_1) & f_2(s_1-u_2) \\ \overline{F}_2(s_2-u_1) & \overline{F}_2(s_2-u_2) \end{vmatrix} ds_2 ds_1$$

and showing that both of these determinants are positive. (When $\overline{F}_1$ and $\overline{F}_2$ are strictly log-concave, both determinants are strictly positive, so the result becomes strict.)

To prove our last claim, we follow the same outline. We want to show that $\overline{F}_3$ is log-supermodular in t and a, since this means $\frac{d}{dt}\overline{F}_3(t,a) = -\frac{f_3(t,a)}{\overline{F}_3(t,a)}$ is increasing in a, or $\frac{f_3(t,a)}{1-F_3(t,a)}$ is decreasing in a. For $t_1 > t_2$ and $a_1 > a_2$, then, we want to show that

$$\overline{F}_3(t_1,a_1)\overline{F}_3(t_2,a_2) > \overline{F}_3(t_1,a_2)\overline{F}_3(t_2,a_1)$$

which is the same as showing the positivity of the determinant

$$D = \begin{vmatrix} \overline{F}_3(t_1,a_1) & \overline{F}_3(t_1,a_2) \\ \overline{F}_3(t_2,a_1) & \overline{F}_3(t_2,a_2) \end{vmatrix} = \begin{vmatrix} \int_{-\infty}^{\infty} \overline{F}_1(t_1+s) f_2(s,a_1) ds & \int_{-\infty}^{\infty} \overline{F}_1(t_1+s) f_2(s,a_2) ds \\ \int_{-\infty}^{\infty} \overline{F}_1(t_2+s) f_2(s,a_1) ds & \int_{-\infty}^{\infty} \overline{F}_1(t_2+s) f_2(s,a_2) ds \end{vmatrix}$$

Write the latter as

$$\begin{aligned} D &= \int_{-\infty}^{\infty} \overline{F}_1(t_1+s_1) f_2(s_1,a_1) ds_1 \int_{-\infty}^{\infty} \overline{F}_1(t_2+s_2) f_2(s_2,a_2) ds_2 \\ &\quad - \int_{-\infty}^{\infty} \overline{F}_1(t_2+s_2) f_2(s_2,a_1) ds_2 \int_{-\infty}^{\infty} \overline{F}_1(t_1+s_1) f_2(s_1,a_2) ds_1 \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{F}_1(t_1+s_1)\overline{F}_1(t_2+s_2)\left(f_2(s_1,a_1)f_2(s_2,a_2) - f_2(s_2,a_1)f_2(s_1,a_2)\right) ds_2 ds_1 \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{s_1} \overline{F}_1(t_1+s_1)\overline{F}_1(t_2+s_2)\left(f_2(s_1,a_1)f_2(s_2,a_2) - f_2(s_2,a_1)f_2(s_1,a_2)\right) ds_2 ds_1 \\ &\quad + \int_{-\infty}^{\infty}\int_{s_1}^{\infty} \overline{F}_1(t_1+s_1)\overline{F}_1(t_2+s_2)\left(f_2(s_1,a_1)f_2(s_2,a_2) - f_2(s_2,a_1)f_2(s_1,a_2)\right) ds_2 ds_1 \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{s_1} \overline{F}_1(t_1+s_1)\overline{F}_1(t_2+s_2)\left(f_2(s_1,a_1)f_2(s_2,a_2) - f_2(s_2,a_1)f_2(s_1,a_2)\right) ds_2 ds_1 \\ &\quad - \int_{-\infty}^{\infty}\int_{s_1}^{\infty} \overline{F}_1(t_1+s_1)\overline{F}_1(t_2+s_2)\left(f_2(s_2,a_1)f_2(s_1,a_2) - f_2(s_1,a_1)f_2(s_2,a_2)\right) ds_2 ds_1 \end{aligned}$$

Switching the order of integration in the second integral, and then exchanging the names of the variables s_1 and s_2 in the second integral, gives

$$\begin{aligned} D &= \textstyle\int_{-\infty}^{\infty}\int_{-\infty}^{s_1} \overline{F}_1(t_1+s_1)\overline{F}_1(t_2+s_2)\left(f_2(s_1,a_1)f_2(s_2,a_2)-f_2(s_2,a_1)f_2(s_1,a_2)\right)ds_2ds_1 \\ &\quad -\textstyle\int_{-\infty}^{\infty}\int_{-\infty}^{s_2} \overline{F}_1(t_1+s_1)\overline{F}_1(t_2+s_2)\left(f_2(s_2,a_1)f_2(s_1,a_2)-f_2(s_1,a_1)f_2(s_2,a_2)\right)ds_1ds_2 \\ &= \textstyle\int_{-\infty}^{\infty}\int_{-\infty}^{s_1} \overline{F}_1(t_1+s_1)\overline{F}_1(t_2+s_2)\left(f_2(s_1,a_1)f_2(s_2,a_2)-f_2(s_2,a_1)f_2(s_1,a_2)\right)ds_2ds_1 \\ &\quad -\textstyle\int_{-\infty}^{\infty}\int_{-\infty}^{s_1} \overline{F}_1(t_1+s_2)\overline{F}_1(t_2+s_1)\left(f_2(s_1,a_1)f_2(s_2,a_2)-f_2(s_2,a_1)f_2(s_1,a_2)\right)ds_2ds_1 \end{aligned}$$

Recalling that $d\overline{F}_1 = -f_1$ and $dF_2 = f_2$, evaluate both inner (ds_2) integrals by parts, giving

$$\begin{aligned} D &= \textstyle\int_{-\infty}^{\infty}\int_{-\infty}^{s_1} \overline{F}_1(t_1+s_1)\overline{F}_1(t_2+s_2)\left(f_2(s_1,a_1)f_2(s_2,a_2)-f_2(s_2,a_1)f_2(s_1,a_2)\right)ds_2ds_1 \\ &\quad -\textstyle\int_{-\infty}^{\infty}\int_{-\infty}^{s_1} \overline{F}_1(t_1+s_2)\overline{F}_1(t_2+s_1)\left(f_2(s_1,a_1)f_2(s_2,a_2)-f_2(s_2,a_1)f_2(s_1,a_2)\right)ds_2ds_1 \\ &= \textstyle\int_{-\infty}^{\infty}\Big[\overline{F}_1(t_1+s_1)\overline{F}_1(t_2+s_2)\left(f_2(s_1,a_1)F_2(s_2,a_2)-F_2(s_2,a_1)f_2(s_1,a_2)\right)\Big|_{s_2=-\infty}^{s_2=s_1} \\ &\quad +\textstyle\int_{-\infty}^{s_1} \overline{F}_1(t_1+s_1)f_1(t_2+s_2)\left(f_2(s_1,a_1)F_2(s_2,a_2)-F_2(s_2,a_1)f_2(s_1,a_2)\right)ds_2\Big]\,ds_1 \\ &\quad -\textstyle\int_{-\infty}^{\infty}\Big[\overline{F}_1(t_1+s_2)\overline{F}_1(t_2+s_1)\left(f_2(s_1,a_1)F_2(s_2,a_2)-F_2(s_2,a_1)f_2(s_1,a_2)\right)\Big|_{s_2=-\infty}^{s_2=s_1} \\ &\quad +\textstyle\int_{-\infty}^{s_1} f_1(t_1+s_2)\overline{F}_1(t_2+s_1)\left(f_2(s_1,a_1)F2(s_2,a_2)-F_2(s_2,a_1)f_2(s_1,a_2)\right)ds_2\Big]\,ds_1 \end{aligned}$$

Plugging $s_2 = -\infty$ into $F_2(s_2, a)$ in the first and third lines gives 0; plugging in $s_2 = s_1$ in the first and third gives the four terms

$$\begin{aligned} &\textstyle\int_{-\infty}^{\infty} \overline{F}_1(t_1+s_1)\overline{F}_1(t_2+s_1)\left(f_2(s_1,a_1)F_2(s_1,a_2)-F_2(s_1,a_1)f_2(s_1,a_2)\right)ds_1 \\ &-\textstyle\int_{-\infty}^{\infty} \overline{F}_1(t_1+s_1)\overline{F}_1(t_2+s_1)\left(f_2(s_1,a_1)F_2(s_1,a_2)-F_2(s_1,a_1)f_2(s_1,a_2)\right)ds_1 \end{aligned}$$

which conveniently cancel, leaving

$$\begin{aligned} D &= \textstyle\int_{-\infty}^{\infty}\int_{-\infty}^{s_1} \overline{F}_1(t_1+s_1)f_1(t_2+s_2)\left(f_2(s_1,a_1)F_2(s_2,a_2)-F_2(s_2,a_1)f_2(s_1,a_2)\right)ds_2ds_1 \\ &\quad -\textstyle\int_{-\infty}^{\infty}\int_{-\infty}^{s_1} f_1(t_1+s_2)\overline{F}_1(t_2+s_1)\left(f_2(s_1,a_1)F_2(s_2,a_2)-F_2(s_2,a_1)f_2(s_1,a_2)\right)ds_2ds_1 \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{s_1} \begin{vmatrix} \overline{F}_1(t_1+s_1) & f_1(t_1+s_2) \\ \overline{F}_1(t_2+s_1) & f_1(t_2+s_2) \end{vmatrix} \begin{vmatrix} f_2(s_1,a_1) & f_2(s_1,a_2) \\ F_2(s_2,a_1) & F_2(s_2,a_2) \end{vmatrix} ds_2ds_1 \end{aligned}$$

The first determinant, $\overline{F}_1(t_1+s_1)f_1(t_2+s_2) - \overline{F}_1(t_2+s_1)f_1(t_1+s_2)$, has the same sign as

$$\frac{f_1(t_2+s_2)}{\overline{F}_1(t_2+s_1)} - \frac{f_1(t_1+s_2)}{\overline{F}_1(t_1+s_1)} = \frac{f_1(t_2+s_2)}{\overline{F}_1(t_2+s_2)}\frac{\overline{F}_1(t_2+s_2)}{\overline{F}_1(t_2+s_1)} - \frac{f_1(t_1+s_2)}{\overline{F}_1(t_1+s_2)}\frac{\overline{F}_1(t_1+s_2)}{\overline{F}_1(t_1+s_1)}$$

Since (by assumption) $\frac{f_1}{\overline{F}_1}$ is increasing in its argument and $t_1 > t_2$, $\frac{f_1(t_2+s_2)}{\overline{F}_1(t_2+s_2)} < \frac{f_1(t_1+s_2)}{\overline{F}_1(t_1+s_2)}$. And since $\overline{F}_1$ is log-concave, $\overline{F}_1(t+s)$ is log-submodular in (t,s); so $\overline{F}_1(t_1+s_1)\overline{F}_1(t_2+s_2) < \overline{F}_1(t_1+s_2)\overline{F}_1(t_2+s_1)$, or $\frac{\overline{F}_1(t_2+s_2)}{\overline{F}_1(t_2+s_1)} < \frac{\overline{F}_1(t_1+s_2)}{\overline{F}_1(t_1+s_1)}$. Since all our terms are positive, this means the first determinant is strictly negative.

As for the second, $f_2(s_1,a_1)F_2(s_2,a_2) - f_2(s_1,a_2)F_2(s_2,a_1)$ has the same sign as

$$\frac{f_2(s_1,a_1)}{F_2(s_2,a_1)} - \frac{f_2(s_1,a_2)}{F_2(s_2,a_2)} = \frac{f_2(s_1,a_1)}{F_2(s_1,a_1)}\frac{F_2(s_1,a_1)}{F_2(s_2,a_1)} - \frac{f_2(s_1,a_2)}{F_2(s_1,a_2)}\frac{F_2(s_1,a_2)}{F_2(s_2,a_2)}$$

Since $F_2(t,a)$ is log-submodular in (t,a), $\frac{f_2(s_1,a_1)}{F_2(s_1,a_1)} < \frac{f_2(s_1,a_2)}{F_2(s_1,a_2)}$, and $F_2(s_1,a_1)F_2(s_2,a_2) < F_2(s_1,a_2)F_2(s_2,a_1)$, or $\frac{F_2(s_1,a_1)}{F_2(s_2,a_1)} < \frac{F_2(s_1,a_2)}{F_2(s_2,a_2)}$. So the second determinant is strictly negative as well, proving the integrand is everywhere strictly positive and therefore $D > 0$, finishing the proof.

A.2 Proof of Equilibrium Existence in Lemma 2

We need to show that the K-player game with strategies $P_k \in \Re^+$ and payoff functions $n_k \log(P_k - C_k) + \log Q_k$ has an equilibrium. We know it's a supermodular game, so if we can restrict best-responses to a compact subset of $(\Re^+)^K$, existence is guaranteed.

Since the game is supermodular, player k's best-response is bounded above by the limit of his best-responses as $P_{k'} \to +\infty$ for $P_{k'}$. Luckily, this is finite, since product k would still be competing against the (free) outside option: even without any other products available, Q_k is still strictly log-concave under Assumption 1; $-\frac{\partial \log Q_k}{\partial P_k}$ is strictly positive and increasing, so $\frac{n_k}{P_k - C_k} = -\frac{\partial \log Q_k}{\partial P_k}$ has a finite solution. Letting P_k^* denote this solution, it's easy to show that prices above P_k^* are strictly dominated by P_k^*, and can therefore be eliminated without loss.

Given supermodularity, best-responses are similarly bounded below by the best-response to zero prices by all competitors, which will be strictly above C_k. Thus, we can eliminate strategies for player k outside of some range $[\underline{P}_k, P_k^*]$ with $\underline{P}_k > C_k$ and $P_k^* < \infty$. (The lower bound is needed because we focus on log-profits, and $\log(P_k - C_k)$ is not continuous at $P_k = C_k$.) Continuous supermodular games on bounded strategy spaces are guaranteed to have an equilibrium, and one can be found by iterating best-responses from either the "lower-left" or "upper-right" corner of the strategy space.

A.3 Proof of Lemma 3

What we actually need is for

$$-\frac{\partial \log Q_k}{\partial \log(P_k - C_k)} \cdot \frac{1}{\varepsilon_{k,k'}}$$

to be increasing in P_k, which is weaker than Assumption 2 because

$$-\frac{\partial \log Q_k}{\partial \log(P_k - C_k)} \cdot \frac{1}{\varepsilon_{k,k'}} = -(P_k - C_k)\frac{\partial \log Q_k}{\partial P_k} \cdot \frac{1}{\varepsilon_{k,k'}} = \frac{P_k - C_k}{P_k}\frac{\varepsilon_{k,k}}{\varepsilon_{k,k'}}$$

and $\frac{P_k - C_k}{P_k}$ is increasing in P_k. Letting $\overline{\varepsilon}_{k,k} \equiv -\frac{\partial \log Q_k}{\partial \log(P_k - C_k)}$, rearranging (4) shows that the mutual best-responses of the n_k firms $i \in \mathcal{T}_k$ are the unique solution to $\overline{\varepsilon}_{k,k} = n_k$. Consider an incremental increase in $\log P_{k'}$ of $d \log P_{k'}$, followed by the resulting change in $\log(P_k - C_k)$ of $d\log(P_k - C_k)$. Since $\overline{\varepsilon}_{k,k} = n_k$ holds both before and after,

$$\frac{\partial \overline{\varepsilon}_{k,k}}{\partial \log P_{k'}} d\log P_{k'} + \frac{\partial \overline{\varepsilon}_{k,k}}{\partial \log(P_k - C_k)} d\log(P_k - C_k) = 0$$

Defining $\Delta = d\log P_{k'} \Big/ \frac{\partial \overline{\varepsilon}_{k,k}}{\partial \log(P_k - C_k)}$, which we know is positive, we get

$$d\log P_{k'} = \frac{\partial \overline{\varepsilon}_{k,k}}{\partial \log(P_k - C_k)}\Delta \quad \textit{and} \quad d\log(P_k - C_k) = -\frac{\partial \overline{\varepsilon}_{k,k}}{\partial \log P_{k'}}\Delta$$

The net effect on $\log Q_k$, then, is

$$\begin{aligned} d\log Q_k &= \frac{\partial \log Q_k}{\partial \log P_{k'}} d\log P_{k'} + \frac{\partial \log Q_k}{\partial \log(P_k - C_k)} d\log(P_k - C_k) \\ &= \frac{\partial \log Q_k}{\partial \log P_{k'}} \frac{\partial \overline{\varepsilon}_{k,k}}{\partial \log(P_k - C_k)}\Delta - \frac{\partial \log Q_k}{\partial \log(P_k - C_k)} \frac{\partial \overline{\varepsilon}_{k,k}}{\partial \log P_{k'}}\Delta \\ &= \varepsilon_{k,k'} \frac{\partial \overline{\varepsilon}_{k,k}}{\partial \log(P_k - C_k)}\Delta - (-\overline{\varepsilon}_{k,k}) \frac{\partial^2(-\log Q_k)}{\partial \log P_{k'} \partial \log(P_k - C_k)}\Delta \end{aligned}$$

Switching the order of the two partial derivatives in the last term gives

$$\begin{aligned}
d\log Q_k &= \varepsilon_{k,k'}\frac{\partial\overline{\varepsilon}_{k,k}}{\partial\log(P_k-C_k)}\Delta-\overline{\varepsilon}_{k,k}\frac{\partial}{\partial\log(P_k-C_k)}\left(\frac{\partial\log Q_k}{\partial\log P_{k'}}\right)\Delta \\
&= \varepsilon_{k,k'}\overline{\varepsilon}_{k,k}\Delta\left[\frac{1}{\overline{\varepsilon}_{k,k}}\frac{\partial\overline{\varepsilon}_{k,k}}{\partial\log(P_k-C_k)}-\frac{1}{\varepsilon_{k,k'}}\frac{\partial\varepsilon_{k,k'}}{\partial\log(P_k-C_k)}\right] \\
&= \varepsilon_{k,k'}\overline{\varepsilon}_{k,k}\Delta\left[\frac{\partial\log\overline{\varepsilon}_{k,k}}{\partial\log(P_k-C_k)}-\frac{\partial\log\varepsilon_{k,k'}}{\partial\log(P_k-C_k)}\right] \\
&= \varepsilon_{k,k'}\overline{\varepsilon}_{k,k}\Delta\frac{\partial}{\partial\log(P_k-C_k)}\log\frac{\overline{\varepsilon}_{k,k}}{\varepsilon_{k,k'}}
\end{aligned}$$

Under Assumption 2, $\overline{\varepsilon}_{k,k}/\varepsilon_{k,k'}$ is increasing in P_k, so $\log\left(\overline{\varepsilon}_{k,k}/\varepsilon_{k,k'}\right)$ is increasing in $\log(P_k-C_k)$, so $d\log Q_k>0$. For a "large" change in $P^N_{k'}$, $\Delta Q_k=\int dQ_k>0$, so Q_k ends up higher than it started.

Also note that we can similarly calculate the change in log-profits $d\left(\log(P_k-C_k)+\log Q_k\right)$; this turns out to be positive if $\left(\overline{\varepsilon}_{k,k}-1\right)/\varepsilon_{k,k'}$ is increasing in P_k. So under this weaker condition, the changes considered in Lemma 3 leave the profits of the firms in $\mathcal{T}_k$ higher, though not necessarily the demand for product k.

A.4 Proof of Lemma 4

As noted in the text, by assumption, prices $(\overline{P}_1,\ldots,\overline{P}_K)$ and $(\overline{p}^j_k)_{k,j}$ are an equilibrium if and only if they satisfy

$$\begin{aligned}
\overline{P}_k &= \arg\max_{P_k}\left\{\left(P_k-c^0_k-\sum_{j>0}\overline{p}^j_k\right)Q_k(P_k,\overline{P}_{-k})\right\} \\
\overline{p}^j_k &= \arg\max_{p^j_k}\left\{\begin{array}{l}\left[\left(p^j_k-c^j_k\right)Q_k(\overline{P}_k+\alpha^j_k(p^j_k-\overline{p}^j_k),\overline{P}_{-k}\right]^{\phi^j_k}\times \\ \left[\left(\overline{P}_k+\alpha^j_k(p^j_k-\overline{p}^j_k)-c^0_k-p^j_k-\sum_{j'\neq j}\overline{p}^{j'}_k\right)Q_k(\overline{P}_k+\alpha^j_k(p^j_k-\overline{p}^j_k),\overline{P}_{-k}\right]^{1-\phi^j_k}\end{array}\right\}
\end{aligned}$$

as the former is the profit-maximization problem of downstream firm i^0_k, and the latter is the Nash product $(\pi^j_k)^{\phi^j_k}(\pi^0_k)^{1-\phi^j_k}$, where profits are evaluated at the price $P_k=\overline{P}_k+\alpha^j_k(p^j_k-\overline{p}^j_k)$ and all prices besides P_k and p^j_k are assumed fixed at their equilibrium values. The derivatives of the logs of these maximands are $\frac{1}{P_k - c_k^0 - \sum_{j>0} \overline{p}_k^j} + \frac{\partial \log Q_k}{\partial P_k}(P_k, \overline{P}_{-k})$, which is decreasing in P_k, and

$$\begin{aligned} &\phi_k^j \left[\frac{1}{p_k^j - c_k^j} + \alpha_k^j \frac{\partial \log Q_k}{\partial P_k}(\overline{P}_k + \alpha_k^j (p_k^j - \overline{p}_k^j), \overline{P}_{-k}) \right] \\ +(1-\phi_k^j) &\left[-\frac{1-\alpha_k^j}{\overline{P}_k + \alpha_k^j (p_k^j - \overline{p}_k^j) - c_k^0 - p_k^j - \sum_{j' \neq j} \overline{p}_k^{j'}} + \alpha_k^j \frac{\partial \log Q_k}{\partial P_k}(\overline{P}_k + \alpha_k^j (p_k^j - \overline{p}_k^j), \overline{P}_{-k}) \right] \end{aligned}$$

which is decreasing in p_k^j, so both maximization problems are log-concave. Equilibrium is therefore exactly equivalent to the first-order conditions

$$\begin{aligned} \frac{1}{\overline{P}_k - c_k^0 - \sum_{j>0} \overline{p}_k^j} &= -\frac{\partial \log Q_k}{\partial P_k}(\overline{P}_k, \overline{P}_{-k}) \\ \frac{\phi_k^j}{\overline{p}_k^j - c_k^j} - \frac{(1-\phi_k^j)(1-\alpha_k^j)}{\overline{P}_k - c_k^0 - \sum_{j>0} \overline{p}_k^j} &= -\alpha_k^j \frac{\partial \log Q_k}{\partial P_k}(\overline{P}_k, \overline{P}_{-k}) \end{aligned}$$

Using the first to substitute for the middle term of the second gives

$$\frac{\phi_k^j}{\overline{p}_k^j - c_k^j} = -\Big((1-\phi_k^j)(1-\alpha_k^j) + \alpha_k^j \Big) \frac{\partial \log Q_k}{\partial P_k}(\overline{P}_k, \overline{P}_{-k})$$

Defining $\beta_k^j = \frac{\phi_k^j}{\alpha_k^j + (1-\phi_k^j)(1-\alpha_k^j)}$ as in the text, we get

$$\overline{p}_k^j - c_k^j = \frac{\beta_k^j}{-\partial \log Q_k / \partial P_k}$$

and $\overline{P}_k - c_k^0 - \sum_{j>0} \overline{p}_k^j = \frac{1}{-\partial \log Q_k / \partial P_k}$. Summing over $j \geq 0$, the wholesale prices cancel and we get

$$\overline{P}_k - C_k = \frac{1 + \sum_{j>0} \beta_k^j}{-\partial \log Q_k / \partial P_k}$$

Defining $n_k = 1 + \sum_{j>0} \beta_k^j$, this is $\frac{\overline{P}_k - C_k}{n_k} = \left(-\frac{\partial \log Q_k}{\partial P_K} \right)^{-1}$, which is equivalent to (4), so equilibrium values of $(\overline{P}_1, \ldots, \overline{P}_K)$ are unique and as characterized in Lemma 2; and $\overline{p}_k^j - c_k^j = \beta_k^j \Big/ \left(-\frac{\partial \log Q_k}{\partial P_k} \right) = \beta_k^j \frac{\overline{P}_k - C_k}{n_k}$, completing the proof.

A.5 On Assumption 3

Since an increase in P^E affects demand in exactly the same way as equal increases in each of the prices P_k^N, it's straightforward to show that if $\frac{\partial \log Q_A}{\partial P^E}$ is decreasing in P_k^N for every $k \in \mathcal{K}$, it's decreasing in P^E. Let F^* be the CDF of $\max_{k\in\mathcal{K}}\{v_k^l - P_k^N\}$. Since $F^*(t) = \prod_{j\in\mathcal{K}} F_j(t + P_j^N)$, differentiating and rearranging gives

$$\frac{f^*(t)}{1-F^*(t)} \quad = \quad X(t)\left[\frac{F_k(t+P_k^N)\sum_{j\neq k}\frac{f_j(t+P_j^N)}{F_j(t+P_j^N)}}{1-F_k(t+P_k^N)X(t)} + \frac{f_k(t+P_k^N)}{1-F_k(t+P_k^N)X(t)}\right] \tag{7}$$

where $X(t) = \prod_{j\neq k} F_j(t+P_j^N)$. If (7) is increasing in P_k^N, then $1 - F^*(t)$ is log-submodular in (t, P_k^N); then, similar to the proof of Lemma 6, the distribution F^{**} of $\max_{k\in\mathcal{K}}\{v_k^l - P_k^N\} - v_0^l$ inherits the same property, which would make $Q_A = 1 - F^{**}(P^E)$ log-submodular in P^E and P_k^N. Thus, a sufficient condition for Assumption 3 is for (7) to be increasing in P_k^N.

Now, $X(t)$ is a constant with respect to P_k^N. Examining the first term in the square brackets, the numerator is increasing in P_k^N and the denominator is decreasing, so the first term is increasing. Examining the second term, the denominator is decreasing in P_k^N, but the numerator could be increasing or decreasing. Further, we've already assumed that $\frac{f_k}{1-F_k}$ is increasing, so without the extra $X(t)$ term in the denominator, the entire second term would be increasing and we'd be done. But Assumption 1 does not guarantee that $\frac{f_k(t+P_k^N)}{1-F_k(t+P_k^N)\prod_{j\neq k}F_j(t+P_j^N)}$ is increasing in P_k^N. So we explicitly assume what we need – that $\frac{\partial \log Q_A}{\partial P^E}$ is increasing in each P_k^N, which also implies increasing in P^E – and move on.

A.6 On Assumption 4

Lemma 7. *Under Assumptions 1, 3, and 4,*

- *An increase in $P_{k'}^N$, followed by adjustments of prices p_i for $i \in \mathcal{T}_k^N$ to their new mutual best-responses, leaves Q_k higher than before*
- *A decrease in P^E, followed by adjustments of prices p_i for $i \in \mathcal{T}_k^N$ to their new mutual best-responses, leaves Q_k higher than before*
- *A decrease in P_k^N, followed by adjustments of prices p_i for $i \in \mathcal{T}^E$ to their new mutual*

best-responses, leaves Q_A higher than before

Proof. The proof of the first part is identical to the proof of Lemma 3, in Appendix A.3.

For the second part, $\frac{\varepsilon_{k,k}}{\varepsilon_{k,E}}$ increasing in P_k^N implies $\frac{\bar{\varepsilon}_{k,k}}{\varepsilon_{k,E}} = \frac{P_k^N - C_k^N}{P_k^N}\frac{\varepsilon_{k,k}}{\varepsilon_{k,E}}$ increasing in P_k^N. Recalling that mutual best-responses by firms $i \in \mathcal{T}_k^N$ solve $\bar{\varepsilon}_{k,k} = n_k$, it must be that

$$\frac{\partial \bar{\varepsilon}_{k,k}}{\partial \log P^E} d\log P^E + \frac{\partial \bar{\varepsilon}_{k,k}}{\partial \log(P_k^N - C_k^N)} d\log(P_k^N - C_k^N) = 0$$

so letting $\Delta = -d\log P^E \Big/ \frac{\partial \bar{\varepsilon}_{k,k}}{\partial \log(P_k^N - C_k^N)} > 0$, $d\log(P_k^N - C_k^N) = \frac{\partial \bar{\varepsilon}_{k,k}}{\partial \log P^E}\Delta$, so

$$\begin{aligned}
d\log Q_k &= \tfrac{\partial \log Q_k}{\partial \log P^E} d\log P^E + \tfrac{\partial \log Q_k}{\partial \log(P_k^N - C_k^N)} d\log(P_k^N - C_k^N) \\
&= \tfrac{\partial \log Q_k}{\partial \log P^E}\left(-\tfrac{\partial \bar{\varepsilon}_{k,k}}{\partial \log(P_k^N - C_k^N)}\Delta\right) + \tfrac{\partial \log Q_k}{\partial \log(P_k^N - C_k^N)} \tfrac{\partial \bar{\varepsilon}_{k,k}}{\partial \log P^E}\Delta \\
&= -\varepsilon_{k,E}\left(-\tfrac{\partial \bar{\varepsilon}_{k,k}}{\partial \log(P_k^N - C_k^N)}\Delta\right) - \bar{\varepsilon}_{k,k}\tfrac{\partial \bar{\varepsilon}_{k,k}}{\partial \log P^E}\Delta \\
&= \varepsilon_{k,E}\bar{\varepsilon}_{k,k}\Delta\left(\tfrac{1}{\bar{\varepsilon}_{k,k}}\tfrac{\partial \bar{\varepsilon}_{k,k}}{\partial \log(P_k^N - C_k^N)} + \tfrac{1}{\varepsilon_{k,E}}\tfrac{\partial^2 \log Q_k}{\partial \log(P_k^N - C_k^N)\partial P^E}\right) \\
&= \varepsilon_{k,E}\bar{\varepsilon}_{k,k}\Delta\left(\tfrac{\partial \log \bar{\varepsilon}_{k,k}}{\partial \log(P_k^N - C_k^N)} - \tfrac{1}{\varepsilon_{k,E}}\tfrac{\partial \varepsilon_{k,E}}{\partial \log(P_k^N - C_k^N)}\right) \\
&= \varepsilon_{k,E}\bar{\varepsilon}_{k,k}\Delta\tfrac{\partial \log\left(\bar{\varepsilon}_{k,k}\middle/\varepsilon_{k,E}\right)}{\partial \log(P_k^N - C_k^N)} \quad > \quad 0
\end{aligned}$$

For the last part, $\frac{\varepsilon_{A,E}}{\varepsilon_{A,k}}$ increasing in P^E implies $\frac{\bar{\varepsilon}_{A,E}}{\varepsilon_{A,k}} = \frac{P^E - C^E}{P^E}\frac{\varepsilon_{A,E}}{\varepsilon_{A,k}}$ increasing in P^E, where $\bar{\varepsilon}_{A,E} = \frac{\partial \log Q_A}{\partial \log(P^E - C^E)}$. Since the mutual best-responses by firms $i \in \mathcal{T}^E$ solve $\bar{\varepsilon}_{A,E} = n_E$,

$$\frac{\partial \bar{\varepsilon}_{A,E}}{\partial \log P_k^N} d\log P_k^N + \frac{\partial \bar{\varepsilon}_{A,E}}{\partial \log(P^E - C^E)} d\log(P^E - C^E) = 0$$

so letting $\Delta \equiv -d\log P_k^N \Big/ \frac{\partial \bar{\varepsilon}_{A,E}}{\partial \log(P^E - C^E)} > 0$, $d\log(P^E - C^E) = \frac{\partial \bar{\varepsilon}_{A,E}}{\partial \log P_k^N}\Delta$, and

$$\begin{aligned}
d\log Q_A &= \tfrac{\partial \log Q_A}{\partial \log P_k^N} d\log P_k^N + \tfrac{\partial \log Q_A}{\partial \log(P^E - C^E)} d\log(P^E - C^E) \\
&= -\varepsilon_{A,k}\left(-\tfrac{\partial \bar{\varepsilon}_{A,E}}{\partial \log(P^E - C^E)}\Delta\right) - \bar{\varepsilon}_{A,E}\tfrac{\partial \bar{\varepsilon}_{A,E}}{\partial \log P_k^N}\Delta \\
&= \varepsilon_{A,k}\bar{\varepsilon}_{A,E}\Delta\left(\tfrac{1}{\bar{\varepsilon}_{A,E}}\tfrac{\partial \bar{\varepsilon}_{A,E}}{\partial \log(P^E - C^E)} + \tfrac{1}{\varepsilon_{A,k}}\tfrac{\partial^2 \log Q_k}{\partial \log P_k^N \partial \log(P^E - C^E)}\right) \\
&= \varepsilon_{A,k}\bar{\varepsilon}_{A,E}\Delta\left(\tfrac{\partial \log \bar{\varepsilon}_{A,E}}{\partial \log(P^E - C^E)} - \tfrac{1}{\varepsilon_{A,k}}\tfrac{\partial \varepsilon_{A,k}}{\partial \log(P^E - C^E)}\right) \\
&= \varepsilon_{A,k}\bar{\varepsilon}_{A,E}\Delta\tfrac{\partial \log\left(\bar{\varepsilon}_{A,E}\middle/\varepsilon_{A,k}\right)}{\partial \log(P^E - C^E)} \quad > \quad 0
\end{aligned}$$

completing the proof.

A.7 Proof of Lemma 5

As in the proof of Lemma 2, the best-responses of all the firms in $\mathcal{T}_k^N$ collectively solve $\frac{n_k}{P_k^N - C_k^N} = -\frac{\partial \log Q_k}{\partial P_k^N}$, and, by the same logic, the best-responses of the firms in $\mathcal{T}^E$ collectively solve

$$\frac{n_E}{P^E - C^E} = -\frac{\partial \log Q_A}{\partial P^E} \tag{8}$$

These are the first-order conditions to the problems $\max_{P_k^N} \left\{ n_k \log(P_k^N - C_k^N) + \log Q_k \right\}$ and $\max_{P^E} \left\{ n_E \log(P^E - C^E) + \log Q_A \right\}$. Under Assumption 1, as before, the former is concave; under Assumption 3, the latter is concave as well; so solutions to these $K+1$ problems correspond to the first-order conditions, and so the equilibria of this latter $K+1$-player game correspond to equilibrium prices in the original pricing game. As before, the former problem has increasing differences in $(P_k^N, P_{k'}^N)$ for $k' \neq k$. Since an increase in P^E is equivalent to the same decrease in P_0 (the imaginary price of the outside option) and $\frac{n_k}{P_k^N - C_k^N} + \frac{\partial \log Q_k}{\partial P_k^N}$ is increasing in the prices of every alternative to k, it is decreasing in P^E; we explicitly assume $\frac{n_E}{P^E - C^E} + \frac{\partial \log Q_A}{\partial P^E}$ is decreasing in P_k^N, which together make the new game a supermodular game when the sign of P^E is reversed.

To show equilibrium existence, first note that by supermodularity, the best-response for player E is bounded above by his best-response to $P_1^N = \ldots = P_K^N = 0$. The best-response for player $k \neq E$ is bounded above by his best-response to $P^E = 0$ and $P_{k'}^N = \infty$ for $k' \neq k$, which, as argued in the proof of Lemma 2, is finite. Player E's best-response to the upper bounds on each P_k^N gives a lower bound above C^E, and player k's best-response to the upper bound on P^E, along with $P_{k'}^N = 0$ for $k' \neq k$, gives a lower bound above C_k^N. So we have a continuous, supermodular game on a bounded strategy space, so equilibrium existence is guaranteed.

The uniqueness proof is likewise similar to that in Lemma 2. Suppose there were two equilibria, $\left(\overline{P}_1^N, \ldots, \overline{P}_K^N, \overline{P}^E\right)$ and $\left(\tilde{P}_1^N, \ldots, \tilde{P}_K^N, \tilde{P}^E\right)$. We treat two cases separately.

First, suppose

$$\left|\tilde{P}^E - \overline{P}^E\right| \geq \max_{k \in \mathcal{K}} \left|\tilde{P}_k^N - \overline{P}_k^N\right|$$

and assume without loss that $\tilde{P}^E > \overline{P}^E$. This means the overall price $P^E + P_k^N$ of each product k is weakly higher at the second equilibrium, so $\frac{\partial Q_A}{\partial P^E}$ is weakly lower. Since $\tilde{P}^E > \overline{P}^E$, $\frac{n_E}{\tilde{P}^E - C^E} < \frac{n_E}{\overline{P}^E - C^E}$, so the first-order condition (8) cannot hold at both equilibria.

For the second case, $\left|\tilde{P}^E - \overline{P}^E\right| < \max_{k \in \mathcal{K}} \left|\tilde{P}^N_k - \overline{P}^N_k\right|$, fix $k \in \arg\max_{j \in \mathcal{K}} \left|\tilde{P}^N_j - \overline{P}^N_j\right|$, and assume without loss that $\tilde{P}^N_k > \overline{P}^N_k$. By assumption, $\tilde{P}^N_k - \overline{P}^N_k > \tilde{P}^E - \overline{P}^E$, so the price of product k is strictly higher at the second equilibrium; and $\tilde{P}^N_k - \overline{P}^N_k \geq \tilde{P}^N_{k'} - \overline{P}^N_{k'}$, so it's gone up by at least as much as any other price. By the same logic as in the proof of Lemma 2, this means $\frac{\partial \log Q_k}{\partial P^N_k}$ is lower at the second equilibrium; $\tilde{P}^N_k > \overline{P}^N_k$ implies $\frac{n_k}{\tilde{P}^N_k - C^N_k} < \frac{n_k}{\overline{P}^N_k - C^N_k}$, so the first-order condition $\frac{n_k}{P^N_k - C^N_k} = -\frac{\partial \log Q_k}{\partial P^N_k}$ cannot hold at both equilibria.

Finally, note that $\frac{n_k}{P^N_k - C^N_k} + \frac{\partial \log Q_k}{\partial P^N_k}$ is increasing in n_k and C^N_k and $\frac{n_E}{P^E - C^E} + \frac{\partial \log Q_A}{\partial P^E}$ is increasing in n_E and C^E, so the $K+1$-player supermodular game characterizing equilibrium prices is indexed by n_k, C^N_k, $-n_E$, and $-C^E$.

A.8 Proof of Theorem 5

Part 1. Since the $K+1$-player supermodular game in Lemma 5 is indexed by C^N_k, the drop in price c_i for $i \in \mathcal{T}^N_k$ leads to an increase in P^E and a decrease in $P^N_{k'}$ for all $k' \in \mathcal{K}$. Following the logic of the uniqueness proof in Lemma 5, if $\max\left\{\Delta P^E, \max_{k' \neq k} \left|\Delta P^N_{k'}\right|\right\} \geq \left|\Delta P^N_k\right|$, then the first-order condition for either P^E or $\arg\max_{k' \neq k} \left|\Delta P^N_{k'}\right|$ cannot hold both before and after the change; which means that both $-\Delta P^N_k > \Delta P^E$ and $\left|\Delta P^N_k\right| > \left|\Delta P^N_{k'}\right|$, meaning $\Delta P_k < 0$ and $|\Delta P_k| > |\Delta P_{k'}|$. Quint (2012) offers examples of mergers (using logit demand, which satisfies all our assumptions) where a competing product's price $P_{k'} = P^N_{k'} + P^E$ can go up or down, and the total welfare effect can be positive or negative.

Since P_k fell, and fell by more than any other price, $\frac{g_k(P_k)}{1 - G_k(P_k)}$ (the hazard rate of the distribution G_k defined in the proof of Lemma 1) must be lower than before, so $p_j - c_j$ must be higher for any $j \in \mathcal{T}^N_k - \{i\}$ for the first-order condition to still hold. Since P_k fell, and fell more than any other price, Q_k must rise, so firms $j \in \mathcal{T}^N_k - \{i\}$ have higher prices and higher demand, hence higher profits. Fixing $k' \neq k$, decompose the change into the following steps: first P^E rises and $P^N_{k'}$ adjusts, then one k'' at a time, $P^N_{k''}$ falls and $P^N_{k'}$ adjusts. Under Assumption 4, Lemma 7 tells us that each step decreases $Q_{k'}$; since $P^N_{k'}$ is also lower, each firm $j \in \mathcal{T}^N_{k'}$ earns lower profits. Similarly, decompose the change into each $P^N_{k''}$ falling and P^E adjusting; under Assumption 4, Lemma 7 tells us that each step increases Q_A, and therefore also the profits of firms $j \in \mathcal{T}^E$. As for a merger, the same logic holds, since the $K+1$-player supermodular game in Lemma 5 is indexed by n_k, which is effectively decreased by 1 by a merger.

As before, a new product is like a reduction in costs C_k^N from $+\infty$ to a new finite level. As in the proof of Theorem 1, a right-shift in the distribution of v_k^l can be seen as a reduction in C_k^N when the "quality-adjusted" price $\tilde{P}_k^N = P_k^N - \Delta v$ is used, so the results are the same.

Part 2. Since the supermodular game in Lemma 5 is indexed by $-C^E$, lower costs mean lower P^E and higher P_k^N for every k, and by the same logic as before, $\left|\Delta P^E\right| > \Delta P_k^N$ for every k; the results follow via the same steps as part 1. Similarly, since the supermodular game is indexed by $-n_E$, the same results follow for a merger between two firms in $\mathcal{T}^E$.

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