Available online at www.sciencedirect.com

ScienceDirect

Journal of Economic Theory 152 (2014) 266–290

JOURNAL OF Economic Theory

www.elsevier.com/locate/jet

# Imperfect competition with complements and substitutes ☆

Daniel Quint

*Department of Economics, University of Wisconsin, 1180 Observatory Drive, Madison, WI 53706, USA*

Received 22 January 2013; final version received 1 May 2014; accepted 8 May 2014

Available online 14 May 2014

**Abstract**

I study price competition in settings where end products are combinations of components supplied by different monopolists, nesting standard models of perfect complements and imperfect substitutes. I show sufficient conditions for a discrete-choice demand system to yield demand for each product which is log-concave in price, and has log-increasing differences in own and another product's price, leading to strong comparative statics results. Many results familiar from simple models, like the price effects of mergers or changes in marginal costs, extend naturally to this more complex setting.


*JEL classification:* D43; L13

*Keywords:* Substitutes; Complements; Mergers; Imperfect competition

## 1. Introduction

Much of our intuition for the effects of competition comes from the simple models of Cournot and Bertrand – models featuring just one type of competition, either substitutes or complements. For example, these models tell us that in a world where all products are substitutes, mergers lead to higher prices, while in a world where all products are complements, mergers lead to lower

☆ Parts of this paper were contained in an earlier working paper, "Economics of Patent Pools When Some (But Not All) Patents Are Essential".

*E-mail address:* dquint@ssc.wisc.edu.

http://dx.doi.org/10.1016/j.jet.2014.05.004
0022-0531/

prices. Real-world settings, however, often include both complements and substitutes. An important question is whether the insights of these simple models apply in more complex situations.

Consider the following example, a variation on the setting studied empirically by Busse and Keohane [8].[1] A single city is served by three coal mines, each connected to the city by a separate, independently-owned railroad. Buyers in the city have four choices: to buy coal from Mine 1 and pay for transport on Railroad 1; to buy from Mine 2 and Railroad 2; to buy from Mine 3 and Railroad 3; or to do without coal. Coal from the different mines may be delivered to different points in the city and have different characteristics, and buyers have heterogeneous preferences among them, making the three "end products" imperfect substitutes.

In this paper, I consider a model of imperfect competition in prices that captures this type of situation. The products demanded by downstream consumers are non-overlapping sets of necessary elements, each supplied by a different monopolist. I show straightforward sufficient conditions on a discrete choice demand model under which our usual intuitions for complements and substitutes extend: under which a merger between Mine 1 and Railroad 1 would lead to lower equilibrium prices for customers of all three mines, while a merger between two of the railroads would lead to higher prices for everyone.

These results follow from two key properties of the demand system: that the log demand for each product is concave in its own price, and has increasing differences in its own and a competing product's price. I show that these two properties hold in a discrete choice setting if consumer preferences are independent across products and drawn from distributions satisfying a commonly-used regularity condition – log-concavity of both the cumulative distribution function and survival function. (A stronger sufficient condition is for each preference distribution to have a log-concave density function.)

Given log-supermodular demand, in the absence of complementarities – if only the coal mines, without the railroads, were being studied – price competition would be a supermodular game. With complementarities, the game is not supermodular – prices of different components of the same product are strategic substitutes. However, I show that the pricing game has the same equilibrium as a different, and simpler, supermodular game, leading to powerful comparative statics as well as intuition for why they hold.

Aside from settings like the coal example above, the model in this paper can also be seen as a model of retail competition among products containing elements supplied by other firms. We can think of different car companies or personal computer manufacturers: a single firm sets each retail price, but that price implicitly includes the prices of the various parts or components that went into the product – tires and windshields, or microchips and DVD drives, purchased from outside suppliers. The model also fits well with the licensing of intellectual property related to third-generation (3G) wireless communication technologies. 3G is not a single standard, but five different ones, each evolved from (and therefore backward-compatible with) one or more second-generation technologies. Quoting a Department of Justice Business Review Letter [20], "It is reasonably likely that essential patents associated with a single 3G technology... will be complements rather than substitutes... [But] There is a reasonable possibility that the five 3G radio interface technologies will continue to be substitutes for each other, and we would expect the owners of intellectual property rights essential to these technologies to compete, including through price..." These concerns led the DOJ to reject the proposed formation of a single plat-

[1] Busse and Keohane [8] study a single coal mine, serving several towns; there are two railroads, one or both serving each town.

form governing the licensing of all 3G-related patents, and approve instead the formation of five separate entities, each licensing patents related to one of the five competing technologies. As I discuss later, this decision is fully in line with the findings of this paper. I also extend the model to allow for components which are a required part of *every* product – for example, Intel chips or Microsoft Windows in different manufacturers' PCs – or for the prices of components to be set through bilateral bargaining between upstream suppliers and downstream retailers prior to downstream competition.

The rest of the paper proceeds as follows. In Section 2, I introduce the model. In Section 3, I establish the main results of the paper: I show two key properties of the demand system that hold under my assumptions, use them to characterize the equilibrium of the pricing game, and establish comparative statics. In Section 4, I compare these results to those that would hold under other models of demand. In Section 5, I consider two extensions to the baseline model: one where downstream retailers negotiate wholesale prices through bilateral negotiations with suppliers, and another to accommodate components required for every product. In Section 6, I relate my findings to the existing literature; Section 7 concludes. All proofs are in Appendix A; two examples referenced in the text, and material relating to the extensions, are contained in a second, online-only appendix.

## 2. Model

There is a finite set of products $\mathcal{K} = \{1, 2, \ldots, K\}$. Each product $k \in \mathcal{K}$ has one or more components; $\mathcal{T}_k$ will denote the components of product $k$. No two products share a common component: for $k' \neq k$, $\mathcal{T}_k \cap \mathcal{T}_{k'} = \emptyset$.

Each component is produced by a separate monopolist, so the set $\mathcal{T} \equiv \mathcal{T}_1 \cup \mathcal{T}_2 \cup \ldots \cup \mathcal{T}_K$ of all components is also the set of firms. Each firm $i \in \mathcal{T}$ has zero fixed costs and constant marginal costs $c_i$, and sets a price $p_i$ for its component. The price of a product is the sum of the prices of its components, $P_k = \sum_{i \in \mathcal{T}_k} p_i$.[2]

Demand for the products comes from a measure 1 of price-taking consumers with unit demand and quasilinear utility. Each consumer can consume a unit of one product, or nothing. Consumer $l$ gets payoff $v_k^l$ from product $k$, and $v_0^l$ from consuming nothing. The demand for each product is the fraction of consumers who prefer that product (given its price) to any of the others or the outside option; letting $P_0 = 0$, this demand is

$$Q_k(P_1, \ldots, P_K) = \Pr\Big(v_k^l - P_k = \max_{j \in \mathcal{K} \cup \{0\}} \{v_j^l - P_j\}\Big) \tag{1}$$

(Assumption 1 below will imply that the set of consumers indifferent between two choices will have measure zero, so ties can safely be ignored.)

The solution concept is Bertrand–Nash competition in prices; firms set prices simultaneously, with firm $i \in \mathcal{T}_k$ seeking to maximize

$$\pi_i = (p_i - c_i) Q_k(p_i, \mathbf{p}_{-i}) \tag{2}$$

where $\mathbf{p}_{-i}$ are the prices set by firms $j \in \mathcal{T} - \{i\}$.

[2] This model applies equally well if one "retail" firm $i \in \mathcal{T}_k$ sets the retail price $P_k$ for each product, by choosing the profit-maximizing markup at the same time as its suppliers choose their wholesale prices.

Table 1
Distributions with log-concave density, which therefore satisfy Assumption 1(c).

| Name of distribution | Support | Density function |
|---|---|---|
| Uniform | [0, 1] | 1 |
| Normal | $(-\infty, \infty)$ | $\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ |
| Exponential | $(0, \infty)$ | $\lambda e^{-\lambda x}$ |
| Logistic | $(-\infty, \infty)$ | $\frac{e^{-x}}{(1+e^{-x})^2}$ |
| Extreme value | $(-\infty, \infty)$ | $e^{-x}e^{-e^{-x}}$ |
| Laplace (double exponential) | $(-\infty, \infty)$ | $\frac{1}{2}e^{-\lvert x\rvert}$ |
| Power function, $c \geq 1$ | $(0, 1]$ | $cx^{c-1}$ |
| Weibull, $c \geq 1$ | $[0, \infty)$ | $cx^{c-1}e^{-x^c}$ |
| Gamma, $c \geq 1$ | $[0, \infty)$ | $\frac{x^{c-1}e^{-x}}{\Gamma(c)}$ |
| Chi-squared, $c \geq 2$ | $[0, \infty)$ | $\frac{x^{(c-2)/2}e^{-x/2}}{2^{c/2}\Gamma(c/2)}$ |
| Chi, $c \geq 1$ | $[0, \infty)$ | $\frac{x^{c-1}e^{-x^2/2}}{2^{(c-2)/2}\Gamma(c)}$ |
| Beta, $\nu \geq 1$ and $\omega \geq 1$ | [0, 1] | $\frac{x^{\nu-1}(1-x)^{\omega-1}}{B(\nu,\omega)}$ |
| Maxwell | Chi with $c = 3$ | |
| Rayleigh | Weibull with $c = 2$ | |

Throughout the paper, I will maintain the following set of assumptions about the distribution of individual product-specific tastes $v_k^l$:

**Assumption 1.** For $k \in \mathcal{K} \cup \{0\}$ and $l \in [0, 1]$,

(a) $\{v_k^l\}$ are independent across $k$ and $l$.
(b) For each $k$, $\{v_k^l\}$ are i.i.d. draws from a massless distribution $F_k$ with density $f_k$.
(c) For each $k$, $F_k$ and $1 - F_k$ are both log-concave.

Note that there is no symmetry assumption: each product may have a different distribution of valuations $F_k$. While independence of $\{v_k^l\}$ is of course a strong assumption, Assumption 1(c) – log-concavity of each CDF and survival function[3] – holds for a large number of familiar probability distributions. Bagnoli and Bergstrom [3, Theorems 1 and 3] show that it holds for any distribution with a differentiable and log-concave density function; Table 1, reproduced from Bagnoli and Bergstrom, therefore gives a long list of well-known distributions for which Assumption 1(c) is known to hold. (In the next section, I will focus on distributions with unbounded support in the positive direction, which still includes nearly all of this list.)

[3] Following the reliability theory literature, I refer to the function $1 - F$ as the survival function of a given distribution $F$.

## 3. Results

### 3.1. Properties of demand under Assumption 1

Assumption 1 leads to two key properties of the demand system $\{Q_k\}$, viewed as a function of aggregate prices $\{P_k\}$:

**Theorem 1.** *Under Assumption 1, for each $k, k' \in \mathcal{K}$, demand $Q_k$ is continuous, differentiable, and log-concave in $P_k$, and* $\log Q_k$ *has strictly increasing differences in $P_k$ and $P_{k'}$.*

For my purposes, log-concavity of demand will ensure that firm best-responses are unique, and characterized by first-order conditions, and that certain comparative statics results hold. Weyl and Fabinger [33] discuss the fact that the log-curvature of demand – whether demand is log-concave or log-convex – is closely related to the *pass-through rate*, the fraction of a cost increase or tax which is borne by the consumer: under a variety of demand models with constant marginal costs, pass-through is below 1 if demand is log-concave, and above 1 if demand is log-convex.[4]

Increasing differences, which I will abuse terminology slightly and refer to as log-supermodularity of demand,[5] will in some sense make prices of competing products strategic complements. This is a slight oversimplification, because in the model of competition I consider, each product's price is not determined by a single firm but by the collective actions of several firms, but the intuition is roughly that of a standard supermodular game: an increase in one product's price will put upward pressure on the prices of the other products, which will lead to intuitive comparative statics.

In Section 4, I examine whether other demand specifications lead to similar, or markedly different, results to Theorem 1, and the implications this has for the conclusions of this section.

### 3.2. Characterization of equilibrium

Let $n_k = |\mathcal{T}_k|$ be the number of components of product $k$, and $C_k = \sum_{i \in \mathcal{T}_k} c_i$ their combined marginal costs. To establish comparative statics results on equilibrium prices, I will relate the equilibrium of the pricing game described above to the equilibrium of a simpler auxiliary game. In particular, I will replace each group of firms $\mathcal{T}_k$ with a single player, whose payoff function causes him to set $P_k$ exactly as the $n_k$ firms in $\mathcal{T}_k$ would have if they were acting separately. While a monopolist firm would maximize $\log(p - c) + \log Q$, double-marginalization causes the $n_k$ firms in $\mathcal{T}_k$ to collectively set the price that maximizes $n_k \log(P_k - C_k) + \log Q_k$.[6] Log-concavity of $Q_k$ ensures that this is a valid exercise, as best-responses are fully characterized by first-order conditions; log-supermodularity of $Q_k$ ensures that the resulting $K$-player game is a

[4] In Fabinger and Weyl [16], the same authors suggest that pass-through rates less than 1 are likely a reasonable expectation in settings where the distribution of consumer willingness-to-pay is unimodal; but that in settings where resale is possible, the resulting floor on willingness-to-pay introduces a convexity, and pass-through rates above 1 might be expected in those settings, citing Einav et al. [15] as an example.

[5] The abuse of terminology is because $\log Q_k$ need only have increasing differences in $P_k$ and $P_j$, not in $P_j$ and $P_{j'}$ ($j' \neq j \neq k$). Still, this is supermodularity in the same sense as payoffs in a supermodular game.

[6] This is analogous to the observation that while a monopolist sets price such that the own-price elasticity of demand is 1, $n$ complementary monopolists collectively set prices such that the elasticity is $n$.

supermodular game, which will lead to an easy proof of equilibrium existence and a number of powerful comparative statics.

First, I impose one additional assumption about the demand system:

**Assumption 2.** For each $k \in \mathcal{K}$, the support of $F_k$ has no upper bound.

This assumption eliminates certain "no-trade"-type equilibria that would otherwise exist given the type of perfect complementarities considered here,[7] leading to uniqueness of equilibrium. Let $P_{-k} = (P_1, \dots, P_{k-1}, P_{k+1}, \dots, P_K)$ denote the vector of aggregate prices of products other than $k$.

**Lemma 1.** *Under Assumptions 1 and 2, the simultaneous-move pricing game has a unique equilibrium. Each firm $i \in \mathcal{T}_k$ sets equilibrium price*

$$p_i = c_i + \frac{1}{n_k}(\overline{P}_k - C_k) \tag{3}$$

*where $(\overline{P}_1, \overline{P}_2, \dots, \overline{P}_K)$ is the unique equilibrium of a different game with players $\{1, 2, \dots, K\}$, strategies $P_k \in \Re^+$, and payoff functions*[8]

$$u_k(P_k, P_{-k}) = n_k \log(P_k - C_k) + \log Q_k(P_k, P_{-k}) \tag{4}$$

*This latter $K$-player game is a supermodular game, indexed by both $n_k$ and $C_k$ for every $k$.*

### 3.3. Comparative statics on prices

Next, I consider the implications the supermodular structure established above has for equilibrium prices. Fix a product $k$ and a firm $i \in \mathcal{T}_k$, and let $p_{-i} = \sum_{j \in \mathcal{T}_k - \{i\}} p_j$ denote the combined price of the other components of $k$. Differentiating firm $i$'s log-profit function, the marginal benefit of raising its price is $\frac{1}{p_i - c_i} + \frac{\partial \log Q_k}{\partial P_k}(p_i + p_{-i}, P_{-k})$; since $Q_k$ is log-concave and log-supermodular, this is decreasing in $p_{-i}$ and increasing in $P_{-k}$. This means the prices of the other firms $j \in \mathcal{T}_k - \{i\}$ (manufacturers of the same product's other components) are strategic substitutes for $p_i$, and the prices of firms $j \in \mathcal{T} - \mathcal{T}_k$ (manufacturers of competing products) are strategic complements. Given the supermodular structure of the equilibrium, these effects are mutually reinforcing, and therefore persist when considered together. Thus, any exogenous change causing an increase or decrease in a single firm's best-response, will have predictable effects on equilibrium prices:

**Theorem 2.** *Under Assumptions 1 and 2,*

1. *An increase in any one firm's marginal cost leads to higher prices for all products.*
   - *Fix $k \in \mathcal{K}$ and $i \in \mathcal{T}_k$. If firm $i$'s marginal cost $c_i$ is increased by $\Delta c_i$, then for any $k' \neq k$, $0 < \Delta P_{k'} < \Delta P_k < \Delta c_i$, where $\Delta P$ is the resulting change in the equilibrium price $P$. An*

[7] For example, if $n_1 > 1$, $\text{supp}(F_1) = [a, b]$, and $\text{supp}(F_0) \subseteq \Re^+$, then it is an equilibrium for every firm $i \in \mathcal{T}_1$ to price above $b$, giving $Q_1 = 0$. Assumption 2 ensures that demand is positive at any price, eliminating this sort of equilibrium.

[8] To be fully formal, $u_k$ is defined as $-\infty$ for $P_k \leq C_k$. Since Assumption 2 ensures $Q_k(\cdot)$ is always strictly positive, strategies $P_k \leq C_k$ are strictly dominated and can safely be ignored.

*increase in $c_i$ therefore results in lower demand $Q_k$ for product k and lower profits for all firms $i' \in \mathcal{T}_k$.*

2. *A quality improvement to one product (defined as a constant increase in all consumers' willingness to pay for that product, or a parallel right-shift in one distribution $F_k$) leads to a higher price for that product and lower prices for all other products.*
   - *If $F_k$ is shifted to the right by $\Delta v$, then for $k' \neq k$, $\Delta P_k - \Delta v < \Delta P_{k'} < 0$. A quality improvement to product k therefore results in greater demand $Q_k$ for that product, and greater profits for all firms $i \in \mathcal{T}_k$ making its components.*
3. *The introduction of a new product leads to lower prices for all existing products.*

The supermodular structure of the equilibrium also allows us to characterize the effects of mergers between firms. A vertical merger between two firms in $\mathcal{T}_k$ lessens the double-marginalization problem, lowering $P_k$ and, as a result, the other products' prices as well. A horizontal merger, on the other hand, causes the merged firm to raise its price; once again, the supermodular structure of equilibrium prices then leads to higher prices for all products:

**Theorem 3.** *Under Assumptions 1 and 2,*

1. *A merger between suppliers of components of the same product leads to lower prices for all products.*
   - *If the merger is between firms $i, j \in \mathcal{T}_k$, then $|\Delta P_{k'}| < |\Delta P_k|$ for any $k' \neq k$, resulting in an increase in $Q_k$ and greater profits for all firms $i' \in \mathcal{T}_k - \{i, j\}$.*
2. *A merger between suppliers of components of different products leads to higher prices for all products.*

### *3.4. Comparative statics on profits*

Theorem 2 establishes that if the marginal cost of one of the components of product 2 increases, the equilibrium prices of *every* product rise. It is tempting to assume that this must be good news for the makers of components of the other products. For products with just one component, this would indeed be the case. For example, if $n_1 = 1$, the lone firm in $\mathcal{T}_1$ sees the price of every competing product go up; and if it chooses in equilibrium to raise its own price $P_1$, this can only be to further increase profits.

However, when $n_1 > 1$, it may not follow that an increase in a competing product's cost is good news. The increases in $P_2, \dots, P_K$ certainly increase the profits of each firm in $\mathcal{T}_1$; but at least in principle, it is possible that these increases also change the shape of the residual demand curve $Q_1$ in such a way that the double-marginalization problem among the firms in $\mathcal{T}_1$ gets more severe, leading to $P_1$ increasing "too much" in equilibrium and actually lowering the profits of these firms. (In the online-only appendix, I give an example where this occurs, although not in an environment consistent with this paper's model.)

To rule out this possibility, I introduce a new condition, which ensures that an increase in $P_2$ leads to both greater demand for competing products, and higher profits for the firms making their components, in equilibrium. Absent marginal costs, the change in $Q_1$ resulting from an increase in $P_2$, followed by the equilibrium response of the firms in $\mathcal{T}_1$, can be shown to have the same sign as $\frac{\partial}{\partial P_1}(\frac{\varepsilon_{1,1}}{\varepsilon_{1,2}})$, where $\varepsilon_{1,1}$ and $\varepsilon_{1,2}$ are the usual own- and cross-price elasticities of demand.[9] I therefore introduce the following assumption:

**Assumption 3.** For every $k$ and every $k' \neq k$, $\varepsilon_{k,k}/\varepsilon_{k,k'}$ is increasing in $P_k$, where $\varepsilon_{k,k} = -\frac{\partial \log Q_k}{\partial \log P_k} > 0$ and $\varepsilon_{k,k'} = \frac{\partial \log Q_k}{\partial \log P_{k'}} > 0$ are the usual own- and cross-price elasticities.

**Lemma 2.** *Under Assumptions 1, 2 and 3, an increase in $P_{k'}$, followed by the adjustment of prices by firms $i \in \mathcal{T}_k$ to their new mutual best-responses, leaves demand for product $k$ higher than before.*

Assumption 3 is not so unnatural. We can write $\frac{\varepsilon_{k,k}}{\varepsilon_{k,k'}}$ as $\frac{P_k}{P_{k'}}(-\frac{\partial \log Q_k}{\partial P_k})/\frac{\partial \log Q_k}{\partial P_{k'}}$, and note that Theorem 1 implies that both $-\frac{\partial \log Q_k}{\partial P_k}$ and $\frac{\partial \log Q_k}{\partial P_{k'}}$ are increasing in $P_k$; Assumption 3 then requires that the latter not increase too much faster than the former, so that any decrease in the ratio $-\frac{\partial \log Q_k}{\partial P_k}/\frac{\partial \log Q_k}{\partial P_{k'}}$ cannot overwhelm the increase in $\frac{P_k}{P_{k'}}$. As noted above, it is exactly the condition that is required, since the change in $Q_k$ (in response to a marginal increase in $P_{k'}$ followed by an adjustment by the firms in $\mathcal{T}_k$) is proportional to $\frac{\partial}{\partial P_k}(\frac{\varepsilon_{k,k}}{\varepsilon_{k,k'}})$. Assumption 3 leads to additional comparative statics on competitor demand and profit:

**Theorem 4.** *Under Assumptions 1, 2 and 3, any of the following lead to lower demand $Q_k$ for product $k$ and lower profits for all firms $i' \in \mathcal{T}_k$ producing its components:*

1. *The introduction of a new competing product $k' \neq k$.*
2. *The reduction of the marginal cost $c_i$ of any component $i \in \mathcal{T}_{k'}$ of any competing product $k' \neq k$.*
3. *A quality improvement to any competing product $k' \neq k$.*
4. *A merger between two firms $i, j \in \mathcal{T}_{k'}$ producing components of the same competing product $k' \neq k$.*

*On the other hand, a merger between firms $i \in \mathcal{T}_{k'}$ and $j \in \mathcal{T}_{k''}$, with $k'' \neq k' \neq k$, leads to increased demand $Q_k$ and increased profits for all firms $i' \in \mathcal{T}_k$.*

As discussed in Section A.5, the weaker condition that $\frac{\varepsilon_{k,k}-1}{\varepsilon_{k,k'}}$ is increasing in $P_k$ suffices to ensure that the firms in $\mathcal{T}_k$ end up with higher profits, though not necessarily higher demand, following an increase in $P_{k'}$. This weaker condition would therefore suffice (along with Assumptions 1 and 2) for the results on profit in Theorem 4.

Theorems 3 and 4 together pin down the effect of a merger between two firms $i, j \in \mathcal{T}_k$ on every firm other than the two merged firms: the effect on profits is positive for firms $i' \in \mathcal{T}_k - \{i, j\}$, and negative for firms $i' \in \mathcal{T} - \mathcal{T}_k$. However, I have not said anything about the effect on the merged firms. The combined profits of all the firms in $\mathcal{T}_k$ are likely to be higher, but could potentially be lower, post-merger, since $Q_k$ rose but $P_k$ fell. And post-merger, the merged firms get a fraction $\frac{1}{n_k-1}$, rather than $\frac{2}{n_k}$, of those joint profits. Whether a particular merger between

[9] With positive marginal costs, elasticities of the form $\frac{\partial \log Q_{k'}}{\partial \log(P_k - C_k)}$, rather than $\frac{\partial \log Q_{k'}}{\partial \log P_k}$, need to be used, but as shown in Section A.5, Assumption 3 (as written) is still sufficient for the result.

complementary firms is profitable for them is an empirical question. (Quint [25] shows that under logit demand, such a merger is profitable when $n_k$ is sufficiently large.)

Two special cases of this model are worth mentioning separately:

**Pure Complements.** *If $K = 1$, all components are perfect complements. Then under Assumption 1, cost reductions are Pareto-improving, as they increase both consumer surplus and every firm's profit. Provided a merger is profitable for the merging firms, it represents a Pareto-improvement as well.*

**Pure Substitutes.** *If $n_1 = n_2 = \ldots = n_K = 1$ (each product has just one component), all components are substitutes. Then Assumption 3 is not required for the profit results in Theorem 4: under only Assumptions 1 and 2, a reduction in one firm's marginal cost, a new product, or a quality improvement to any existing product reduces the profits of all other firms. Additionally, a merger between any two firms is now guaranteed to increase the profits of all firms, including the joint profits of the merging firms.*

Thus, as expected, mergers in a pure complements world are Pareto-improving, as they lessen a double-marginalization problem, while mergers in a pure substitutes world represent a gain to producers but a loss to consumers.

## 4. Comparison to other demand models

The proof of Lemma 1 (the supermodular characterization of equilibrium prices) uses the two properties established in Theorem 1, log-concavity and log-supermodularity of demand. Theorems 2 and 3 then follow from Lemma 1; Theorem 4 follows from Lemma 1 and an additional condition, Assumption 3. In this section, we consider what happens when each of these properties – log-concavity, log-supermodularity, and Assumption 3 – is weakened or violated, and whether they hold under other commonly-used demand systems.

### *4.1. Weakening the sufficient conditions*

Log-concave demand, while a feature of my demand model, is a stronger assumption than needed to prove Lemma 1. What is actually needed is strict quasi-concavity of both the individual firm profit functions $(p_i - c_i)Q_k(p_i + p_{-i}, P_{-k})$ and the "virtual firm" payoff functions $n_k \log(P_k - C_k) + \log Q_k$, so that both problems' maximizers are unique and characterized by the corresponding first-order condition. As noted by Vives [30, p. 149], a sufficient condition for quasi-concave profit functions is that the own-price elasticity of demand $\varepsilon_{k,k} = -\frac{\partial \log Q_k}{\partial \log P_k}$ is strictly increasing in $P_k$.[10] Thus, if each $\varepsilon_{k,k}$ is strictly increasing, Lemma 1 holds, with or without log-concave demand. Conveniently, many demand functions (including some discussed

[10] In my setting, if $\varepsilon_{k,k}$ is strictly increasing in $P_k$, then for any $n$ and $a \geq 0$, $n + (P_k - a)\frac{\partial \log Q_k}{\partial P_k}$ is strictly decreasing on $P_k > a$, meaning $\frac{n}{P_k - a} + \frac{\partial \log Q_k}{\partial P_k}(P_k, P_{-k})$, which has the same sign, crosses zero at most once, from above, on $(a, +\infty)$. Letting $n = 1$ and $a = p_{-i} + c_i$, this makes firm $i$'s log-profit function strictly quasiconcave; letting $n = n_k$ and $a = C_k$, this makes the payoff function of the $k$th "virtual firm" in Lemma 1 strictly quasiconcave.

Table 2
Some distributions with log-*convex* densities and their properties.

| Name of distribution | Support | Density function | $F$ | $1-F$ |
|---|---|---|---|---|
| Power ($c<1$) | $(0,1]$ | $cx^{c-1}$ | log-concave | neither |
| Weibull ($c<1$) | $(0,\infty)$ | $cx^{c-1}e^{-x^c}$ | log-concave | log-convex |
| Gamma ($c<1$) | $(0,\infty)$ | $\frac{x^{c-1}e^{-x}}{\Gamma(c)}$ | log-concave | log-convex |
| Arc-sine | $[0,1]$ | $\frac{1}{\pi\sqrt{x(1-x)}}$ | neither | neither |
| Pareto | $[1,\infty)$ | $\beta x^{-\beta-1}$ | log-concave | log-convex |
| Mirror-image Pareto | $(-\infty,-1)$ | $\beta(-x)^{-\beta-1}$ | log-convex | log-concave |

below) that are not log-concave still satisfy this weaker property; most of the results in Theorems 2 and 3 therefore hold for these demand systems as well.[11]

If demand features increasing own-price elasticities but is not log-supermodular, the equilibrium characterization in Lemma 1 is still valid, but the $K$-player game described in the lemma is not a supermodular game, so the comparative statics results in Theorems 2 and 3 do not hold. For $K>2$, analogous general results do not hold for the submodular case: even if each cross-partial $\frac{\partial^2 \log Q_k}{\partial P_k \partial P_{k'}}$ were negative, this would not pin down the relevant comparative statics. (While all the "direct effects" of a single price change would be negative, these would then lead to indirect effects which were positive – an increase in $P_k$ leading to an increase in $P_{k''}$ through a decrease in $P_{k'}$ – and it would be unclear which effects would dominate.)

Finally, as noted earlier, when Lemma 1 holds and demand is log-supermodular, we can relax Assumption 3 and still maintain some, but not all, of the results in Theorem 4. Assumption 3 says that $\frac{\varepsilon_{k,k}}{\varepsilon_{k,k'}}$ must be increasing in $P_k$. Since, under Lemma 1, both $\varepsilon_{k,k}$ and $\varepsilon_{k,k'}$ are increasing in $P_k$, a weaker assumption, which I will refer to as Assumption 3′, is that $\frac{\varepsilon_{k,k}-1}{\varepsilon_{k,k'}}$ is increasing in $P_k$. Under Assumption 3′, the parts of Theorem 4 relating to profits, though not demand, still hold. (See Section A.5 for a discussion.) Two of the demand models discussed below satisfy Assumption 3′ but not Assumption 3.

### 4.2. Distributions with log-convex density

One way to interpret Theorem 1, combined with Theorems 1 and 3 of Bagnoli and Bergstrom [3], is that if preferences are independent across products and drawn from distributions with log-concave densities, demand is log-concave and log-supermodular. A natural question, then, is whether these results would be reversed if product preferences were drawn instead from distributions with log-*convex* densities.

Bagnoli and Bergstrom discuss several distributions with log-convex density functions. Pulling from Tables 2 and 3 of their paper, Table 2 shows several such distributions.[12] Theo-

[11] The exception: if demand were not log-concave, firms $i, j \in \mathcal{T}_k$ would not have prices which were strategic substitutes. Thus, for example, an increase in $c_i$ (and the resulting increase in $p_i$) would not necessarily lead to a lower price $p_j$ and lower profit $\pi_j$.

[12] Several other common distributions – the lognormal, Student's $t$, and Cauchy, as well as the beta and $F$ distributions for some parameter values – have density functions which can be log-concave, log-convex, or neither. For the most part,

rems 2 and 4 of Bagnoli and Bergstrom imply that if a density function is log-convex and has support $(a, \infty)$, then its survival function must be log-concave on its support. (As Table 2 illustrates, this can vary with the support of the distribution. A distribution with log-convex density cannot have full support on $\Re$.) However, as pointed out by Block, Savits and Singh [6], a survival function cannot be log-convex on the entire real line: the inequalities defining log-convexity must be violated at points outside the support of the distribution. For this reason, general results analogous to Theorem 1 cannot be proven in the same way for the log-convex case.[13]

To better understand the log-convex case, I've examined numerical examples using three of the distributions from Table 2 – specifically, Weibull and Gamma distributions with various shape parameters below 1, as well as Pareto distributions. In the cases I've tried, demand has consistently turned out to be log-*convex* and log-*submodular*, the opposite of Theorem 1. While demand has been log-convex, the own-price elasticity has still been increasing, so the equilibrium characterization in Lemma 1 still appears to be valid; but since log-demand is submodular, the comparative statics results do not hold. However, the examples have been limited to a small number of products and a few specific distributions; I do not see a way to prove this as a general result for log-convex distributions.

### *4.3. Demand systems with preferences over attributes*

Assumption 1(a) requires consumer preferences to be uncorrelated across products. While it looks standard, this is embedding an important implicit assumption: that it is the products themselves, not their attributes, over which consumers have preferences. In a random-coefficients-type model like those considered by Caplin and Nalebuff [9], Berry, Levinsohn and Pakes [5], and many others, consumers have heterogeneous preferences over product attributes; even if the individual preference parameters are independent and satisfy the distributional assumptions above, this can still introduce correlation among $\{v_k^l\}$ which causes Theorem 1 to fail.

For example, consider the simple linear random-coefficients model with

$$v_k^l = x_k \cdot \beta^l + \xi_k + \epsilon_k^l \tag{5}$$

where $x_k$ is a vector of characteristics of product $k$ and $\beta^l$ a vector of preference parameters over those characteristics for consumer $l$. With the right assumption, log-concavity of demand can again be shown theoretically: Theorem 1 of Caplin and Nalebuff [9] implies that if the joint distribution of preference parameters has a log-concave density function, the resulting demand functions will be log-concave as well.[14] However, in a similar model with heterogeneous wealth effects, where consumer $l$ gets utility $v_k^l - \alpha^l P_k$ from product $k$, log-concavity cannot be shown

these distributions also typically have CDFs and survival rates which are neither log-concave nor log-convex, giving less hope that results analogous to Theorem 1 could be found.

[13] A key step of the proof of Theorem 1 involves the result (from Karlin [22] via Barlow and Proschan [4]) that if two independent random variables have log-concave survival functions, so does their sum. The proof depends on the sign of an integral over $(-\infty, \infty)$ of an integrand whose sign is determined by the local log-curvature of the survival function. When a survival function is log-concave on the support of the distribution, its extension to the whole real line is log-concave everywhere, so the sign of the integral is pinned down. Since log-convexity cannot hold over the whole real line, the same trick does not work for the log-convex case; Barlow and Proschan note that no analogous result holds for the case of decreasing hazard rates (log-convex survival functions), and give a simple counterexample to illustrate this.

[14] Specifically, apply Theorem 1 of Caplin and Nalebuff with $\rho = 0$, and recall that in the limit $\rho = 0$, $\rho$-concavity is equivalent to log-concavity.

generally. And even in the model without wealth effects, log-supermodularity does not hold generally. In the online-only appendix, I show a simple example – two products and two attributes, with preferences over attributes independently and uniformly distributed – in which log-demand can be locally either supermodular or submodular, depending on the level of prices.

### 4.4. Commonly-used closed-form demand functions

Vives [31] considers a number of commonly-used demand systems which give closed-form functions for $Q_k(P_1, \dots, P_K)$. The first two models considered are the linear formulations of Shapley and Shubik [28] and Bowley [7]. The third is the location model of Salop [27] with uniformly-distributed customers, which near equilibrium prices is linear as well. The first three models, then, can all be thought of as special cases of the demand equation

$$Q_k = a_k - b_k P_k + \sum_{k' \neq k} c_{k,k'} P_{k'} \tag{6}$$

with $\{b_k\}$ and $\{c_{k,k'}\}$ all positive, and with each model imposing some further restrictions on the values of these parameters. Differentiating gives $\frac{\partial \log Q_k}{\partial P_k} = -\frac{b_k}{Q_k}$, which is decreasing in $P_k$ and increasing in $P'_k$ ($k' \neq k$). Thus, any linear demand formulation gives demand which is log-concave and log-supermodular, as in the model in this paper. In addition, under linear demand, the ratio $-\frac{\partial Q_k}{\partial P_k} / \frac{\partial Q_k}{\partial P_{k'}} = \frac{b_k}{c_{k,k'}}$ remains constant as $P_k$ increases, and as a result Assumption 3 can be shown to hold as well.

The final four models considered by Vives are the constant elasticity of substitution (CES) model, two formulations of a constant-expenditure model, and the logit model:

$$\begin{aligned}
&\text{CES}: && Q_k = S(\beta\theta)^{\frac{1}{1-\beta\theta}} \frac{P_k^{-1/(1-\beta)}}{(\sum_j P_j^{-\beta/(1-\beta)})^{(1-\theta)/(1-\beta\theta)}}, \quad \beta, \theta \in (0,1) \\
&\text{CE-exponential}: && Q_k = S \frac{e^{-\beta P_k}}{P_k \sum_j e^{-\beta P_j}}, \quad \beta > 0 \\
&\text{CE-constant elasticity}: && Q_k = S \frac{P_k^{-\sigma}}{\sum_j P_j^{1-\sigma}}, \quad \sigma > 1 \\
&\text{Logit}: && Q_k = S \frac{e^{-P_k/\mu}}{\sum_j e^{-P_j/\mu}}, \quad \mu > 0
\end{aligned} \tag{7}$$

The logit model, in fact, is a discrete-choice model with a particular choice of distribution $F_k$ satisfying Assumption 1(c); thus, by Theorem 1, it yields log-concave and log-supermodular demand. In addition, under logit, $-\frac{\partial Q_k}{\partial P_k} / \frac{\partial Q_k}{\partial P_{k'}}$ turns out to be constant in $P_k$, so Assumption 3 holds.

For the other three models, we can explicitly calculate derivatives and find that in all three cases, demand is log-supermodular but not necessarily log-concave. However, while demand can be locally either log-concave or log-convex depending on parameter values and prices, the own-price elasticity is always increasing, so nearly all of our results still hold. For all three of

Table 3
Summary of results comparing my results to other demand models.

| Demand system | Curvature of demand | Strategic interaction | Assumption 3 or 3′? |
|---|---|---|---|
| Discrete choice, independent preferences over products, $F_k$ and $1 - F_k$ log-concave | log-concave (Theorem 1) | log-supermodular (Theorem 1) | ?[a] |
| Discrete choice, independent preferences over products, $f_k$ log-convex | may be log-convex[b] | may be log-submodular[b] | |
| Discrete choice, prefs over characteristics | either[c] | either | |
| Linear (Shapley-Shubik/Bowley/Salop) | log-concave | log-supermodular | 3 holds |
| Logit | log-concave | log-supermodular | 3 holds |
| Constant elasticity of substitution | either[d] | log-supermodular | no |
| Constant expenditure – exponential | either[d] | log-supermodular | 3′ but not 3 |
| Constant expenditure – CES | either[d] | log-supermodular | 3′ but not 3 |

[a] Assumption 3 holds for logit demand, which is a special case of this model; it's not clear whether it holds generally.
[b] Specific examples (solved numerically) using the Weibull, Gamma, and Pareto distributions yielded log-convex, log-submodular demand, but no general results are available.
[c] In a linear model without heterogeneous wealth effects, demand is log-concave if the joint density of preference parameters is log-concave; with wealth effects, this need not hold.
[d] Own-price elasticity is strictly increasing, so whether demand is log-convex or log-concave, Lemma 1, and therefore most of Theorems 2 and 3, still hold.

these models, Assumption 3 is often violated[15]; but the two constant-expenditure models do satisfy the weaker Assumption 3′.

### 4.5. *Summing up*

Table 3 summarizes the results on other demand specifications. The linear and logit models satisfy all the assumptions of this paper; the CES model and the two constant-expenditure models considered by Vives do not satisfy these assumptions exactly, but still yield most of the same results. On the other hand, discrete choice with log-convex distributions, or BLP-style preferences over product characteristics, have substantially different properties, and the results of this paper do not apply to those demand systems.

## 5. Extensions to the baseline model

### 5.1. *Wholesale pricing through bilateral negotiation*

A different, but closely related, model of firm competition would be if each product $k \in \mathcal{K}$ was sold by a single retailer, who purchased the necessary components from the remaining (upstream) firms in $\mathcal{T}_k$. In such a setting, a more natural assumption than simultaneous price-setting might be that upstream prices were determined through a series of bilateral negotiations between the retailer and each component manufacturer prior to downstream competition. In a separate online-only appendix, I analyze such a model. I assume that the wholesale price of each component is set via Nash bargaining between the component manufacturer and the corresponding retailer.

[15] Assumption 3 holds under the CE-exponential model when there are many products or product prices are close to the same, but not necessarily otherwise. Under the CES and CE-CES models, it is generally violated.

Bargaining with correct beliefs about each wholesale price's impact on downstream competition is too complicated a model to work with, so a simpler assumption, in the spirit of conjectural variations, is used: each Nash bargain is struck under the assumption that wholesale prices are passed through to downstream prices at a constant rate, although this rate can again be different for each firm. Under these assumptions, I show that a simple transformation maps this bargaining model to the baseline model above, and that identical results therefore hold.

### 5.2. *Essential components*

Next, I extend the model to allow for components which are common to *all* products. That is, instead of requiring no overlap in the components of the various products ($\mathcal{T}_k \cap \mathcal{T}_{k'} = \emptyset$), I now allow for a common, nonempty overlap, $\mathcal{T}_k \cap \mathcal{T}_{k'} = \mathcal{T}_1 \cap \mathcal{T}_2 \cap \ldots \cap \mathcal{T}_K \equiv \mathcal{T}^E$. I refer to the components required for every product as *essential* – think of these as monopolists in a supply chain (a single national railroad that transports coal for all coal companies), or necessary components with no substitutes (Microsoft Windows for the mainstream PC market). To differentiate the other components from the essential ones, let $\mathcal{T}_k^N = \mathcal{T}_k - \mathcal{T}^E$ denote the set of components required only for product $k$.

I assume that the essential suppliers do not price-discriminate, that is, they set a single price for the entire market. Let $P^E = \sum_{i \in \mathcal{T}^E} p_i$ denote the combined price of all the essential components, and $P_k^N$ the combined price of the nonessential components of product $k$. A consumer wishing to consume product $k$ must therefore buy each of the components in $\mathcal{T}^E \cup \mathcal{T}_k^N$, at a price $P_k = P^E + P_k^N$.

Similar to Lemma 1, the equilibrium of the pricing game here can be linked to the equilibrium of a different game, now with $K + 1$ players, which is a supermodular game in $(P_1^1, \ldots, P_K^N, -P^E)$. This requires $\log Q_k$ to be concave in $P_k^N$ and have increasing differences in $(P_k^N, P_{k'}^N)$ and $(P_k^N, P^E)$; and $\log Q_A = \log \Sigma_{k \in \mathcal{K}} Q_k$ to be concave in $P^E$ with increasing differences in $(P^E, P_k^N)$. The first three properties follow directly from Assumption 1. The last two, however, do not. I have not found a simple condition on model primitives which guarantees these conditions; instead, I explicitly assume what I need, and then argue that it's a reasonable assumption.

**Assumption 4.** $\frac{\partial \log Q_A}{\partial P^E}$ is decreasing in $P^E$ and in $P_k^N$.

In the online-only appendix, I discuss why this is a reasonable assumption. Note that it holds under linear and logit demand, but not under the other closed-form demand systems considered by Vives [31]. As with the baseline model, Assumptions 1, 2 and 4 together guarantee log-concave profit functions and supermodularity of a transformed game characterizing equilibrium prices, which leads to comparative statics on price. The equilibrium characterization is given in the online-only appendix. This then leads to results analogous to Theorems 2 and 3. Results for essential firms are similar to those for firms in a pure-complements model; results for non-essential firms are similar to those in the baseline model:

**Theorem 5.** *Under Assumptions 1, 2 and 4,*

1. *A decrease in costs $c_i$ for $i \in \mathcal{T}_k^N$, or a merger between two firms $i, j \in \mathcal{T}_k^N$, lead to...*
   - *a lower price $P_k$ and greater demand $Q_k$ for product $k$*

- *higher markups for (nonmerging) firms* $i' \in \mathcal{T}_k^N$
- *lower prices for firms* $i' \in \mathcal{T}_{k'}^N$ *for* $k' \neq k$, *and higher prices for firms* $i' \in \mathcal{T}^E$
- *an ambiguous effect on the prices* $P_{k'}$ *of other products*

2. *A decrease in costs* $c_i$ *for* $i \in \mathcal{T}^E$, *or a merger between two firms* $i, j \in \mathcal{T}^E$, *both lead to lower prices for every product and higher markups for every (nonmerging) firm.*

As in the baseline model, an additional assumption (an analog to Assumption 3) would allow for comparative statics on the effect of mergers and price changes on other firms' equilibrium demand and profit level. For brevity, I omit these further results, as the conditions required are not intuitive. (The added conditions do, however, hold under logit demand. Quint [25] analyzes the model with essential components under logit demand, focusing on the application of the model to patent pools.)

## 6. Related literature

The discrete-choice demand framework I use is closely related to work done by others. Deneckere and Davidson [14] consider mergers of firms producing substitutes in a linear demand system, and show that such mergers are profitable for both the merging firms and for outsiders. Perloff and Salop [24] consider the symmetric case ($F_k = F_{k'}$) of the discrete choice model I use, and show existence and uniqueness of a "single-price" equilibrium. Chen and Riordan [13] consider the two-firm case, with a more general (symmetric) joint distribution of valuations. Gabaix et al. [17] characterize equilibrium prices in the limit as the number of (identical) firms in the market goes to infinity. Anderson, de Palma and Thisse [1] and many subsequent papers explore the logit demand model, which is frequently used in empirical work (and satisfies the assumptions of this paper). The most general model of this sort that I'm aware of is that of Caplin and Nalebuff [9]. They situate each product in an $m$-dimensional space of product attributes over which consumers have preferences. They allow preferences over these different product dimensions to be correlated; they show that a condition similar to log-concavity of the density function (but slightly weaker) is sufficient to guarantee existence of an equilibrium, although they show uniqueness and log-supermodularity only for special cases. Berry, Levinsohn and Pakes [5] use a similar model as the framework for empirical estimation. I focus on competition in prices (à la Bertrand) as opposed to quantities (à la Cournot)[16]; Vives [30], among others, explores sufficient conditions for equilibrium existence, uniqueness, and other properties under both models, and compares insights across the two models.

A number of other recent papers examine settings with both complements and substitutes. Tan and Yuan [29] use a model similar to mine, but with two products, to study the incentives of an integrated firm to divest. Horn and Wolinsky [19] consider the duopoly case as well, assuming each downstream firm bargains over wholesale prices with a single supplier. Casadesus-Masanell, Nalebuff and Yoffie [11] consider a downstream monopolist (Microsoft) and perfect competition among two upstream suppliers (Intel and AMD); Chen and Nalebuff [12] consider a monopolist in one market (Microsoft as the supplier of an operating system) who also competes in a

[16] Jaffe and Weyl [21] point out that what differs across these two models (among others) is not so much the name attached to each firm's strategic variable, but what assumptions they make about other firms' responses to their own actions – that is, what they hold fixed when they optimize behavior – and focus on empirical merger evaluation in a framework that nests both models.

complementary market (Microsoft and Netscape in the browser market). The recent literature on two-sided markets (such as newspapers, which must attract both advertisers and readers) and competition among platforms (such as XBox and PlayStation, which may be substitutes for consumers but are each accompanied by a collection of complementary products) also considers both complementarities and substitutes, although the focus is different – see, for example, Carrillo and Tan [10], Rochet and Tirole [26], Armstrong [2], and Weyl [32]. Coexistence of substitutes and complements is also explicitly allowed in the recent extension of the two-sided matching literature to supply chains and other settings: see Ostrovsky [23] and Hatfield and Kominers [18].

## 7. Conclusion

Certain received wisdom about price competition – for example, that mergers lead to higher prices when firms produce substitutes, but lower prices when firms produce complements – comes from simple models where only one type of competition is considered. I show that these effects persist in a setting where a given good has both complements and substitutes. In particular, when competing supply chains do not overlap, vertical mergers are consumer-friendly, while horizontal mergers between levels of competing supply chains are not.

The Department of Justice ruling on 3G patent licensing was fully in line with these insights. The DOJ rejected a proposal to create a single Patent Platform (similar to a traditional patent pool, but with more flexibility) which would handle licensing of all 3G-related patents; instead, five separate Patent Platforms were formed, one for each competing radio interface technology. In other words, full *vertical* integration was allowed – for pricing purposes, each set of complementary firms was replaced by a single entity – while the proposed horizontal merger was not. This is consistent with the DOJ's mandate to promote competition, as this policy would be expected to lead to the lowest possible licensing costs.

Many other applications might fit this model reasonably well. The market for Windows-compatible personal computers involves many competing retailers (Acer, Dell, Gateway, HP, Lenovo, etc.); some essential components (Intel and Microsoft); and lots of nonessential manufacturers (sources for hard drives, optical drives, and other components). Similarly, to the extent that competing car manufacturers have non-overlapping supply chains, the model presented above could apply. The market for delivered coal, discussed in Busse and Keohane [8], does not fit the model perfectly, as many of the power plants purchasing coal are serviced by only one railroad, and therefore do not face the full menu of available "products", and railroads each deliver coal from many mines; but the model could potentially be adapted to this type of market. A similar modification might apply the model to the cell phone market, where phone manufacturers (Nokia, Samsung, LG, Motorola, Sony Ericsson, Apple) partner with service providers (AT&T, Verizon, Sprint).

One significant limitation of the model considered in this paper is the exclusive focus on single-product firms. When each firm produces multiple components, or a single retailer offers several different products, the supermodular equilibrium structure demonstrated above does not hold, as a single firm's log-profit function need not have increasing differences in any two of its own prices. Finding conditions under which comparable results can be achieved for multi-product firms is a significant challenge left for future work.

## Appendix A. Omitted proofs

### A.1. Proof of Theorem 1

As noted in the text (footnote 13), Barlow and Proschan [4, p. 100] use techniques from Karlin [22] to show that if two independent random variables have distributions with increasing hazard rates, so does their sum: that is, if $X_1$ and $X_2$ are independent, with $X_1 \sim G_1$ and $X_2 \sim G_2$, and $G$ is the distribution of $X_1 + X_2$, then if $1 - G_1$ and $1 - G_2$ are log-concave, so is $1 - G$. So fix $k \in \mathcal{K}$, and apply this result with $X_1 = v_k$ and $X_2 = -\max_{j \in \mathcal{K} - \{k\} \cup \{0\}}\{v_j^l - P_l\}$. Under Assumption 1, $1 - G_1 = 1 - F_k$ is log-concave; as for $G_2$,

$$\begin{aligned} 1 - G_2(t) &= \Pr\bigl(-\max\{v_j^l - P_j\} > t\bigr) \\ &= \Pr\bigl(\max\{v_j^l - P_j\} < -t\bigr) \\ &= \prod_{j \in \mathcal{K} - \{k\} \cup \{0\}} F_j(-t + P_j) \\ \log\bigl(1 - G_2(t)\bigr) &= \sum_{j \in \mathcal{K} - \{k\} \cup \{0\}} \log F_j(-t + P_j) \end{aligned}$$

Since each $F_j$ is log-concave (Assumption 1), $\log(1 - G_2)$ is the sum of concave functions, and therefore concave. Given these definitions for $X_1$ and $X_2$, demand $Q_k$ is

$$\begin{aligned} Q_k &= \Pr\Bigl(v_k^l - P_k > \max_{j \in \mathcal{K} - \{k\} \cup \{0\}}\{v_j^l - P_j\}\Bigr) \\ &= \Pr(X_1 - P_k > -X_2) = \Pr(X_1 + X_2 > P_k) = 1 - G(P_k) \end{aligned}$$

so $Q_k$ is log-concave.

To prove the second part of the theorem, we follow a similar outline to the proof in Barlow and Proschan, but with several changes. This time, we want to show that $\log Q_k$ has increasing differences in $P_k$ and $P_{k'}$. We will therefore fix $k \in \mathcal{K}$ and $k' \in \mathcal{K} - \{k\}$, let $X_1 = v_k^l$, and this time let $X_2 = \max_{j \in \mathcal{K} - \{k\} \cup \{0\}}\{v_j^l - P_j\}$, and this time let $G$ denote the distribution of the difference $X_1 - X_2$, so that once again $Q_k = 1 - G(P_k)$. We will think of $G$ as a function of two arguments – the point at which it is evaluated, and the value of $P_{k'}$. As in Barlow and Proschan, we will let $\overline{G}$ denote $1 - G$ (and likewise for other distributions).

Letting $t_1 > t_2$ and $a_1 > a_2$, establishing that $\log Q_k$ has increasing differences in $P_k$ and $P_{k'}$ is equivalent to showing that the determinant

$$D = \begin{vmatrix} \overline{G}(t_1, a_1) & \overline{G}(t_1, a_2) \\ \overline{G}(t_2, a_1) & \overline{G}(t_2, a_2) \end{vmatrix}$$

is positive. Since this part of the argument is not based directly on Barlow and Proschan [4], we give the argument in full detail. First, since $G$ is now the distribution of a difference rather than a sum,

$$\overline{G}(t, a) = E_{X_2 | P_{k'} = a} \Pr(X_1 > t + X_2) = \int_{-\infty}^{\infty} \overline{G}_1(t + s) g_2(s, a) ds$$

(Note that $G_1 = F_k$ does not depend on $P_{k'}$, so only $g_2$ is written with $a$ as an argument.) We can rewrite $D$ as

$$D = \begin{vmatrix} \int_{-\infty}^{\infty} \overline{G}_1(t_1+s)g_2(s,a_1)ds & \int_{-\infty}^{\infty} \overline{G}_1(t_1+s)g_2(s,a_2)ds \\ \int_{-\infty}^{\infty} \overline{G}_1(t_2+s)g_2(s,a_1)ds & \int_{-\infty}^{\infty} \overline{G}_1(t_2+s)g_2(s,a_2)ds \end{vmatrix}$$

which, after a little bit of manipulation, can be written as

$$\begin{aligned} D = &\int_{-\infty}^{\infty}\int_{-\infty}^{s_1} \overline{G}_1(t_1+s_1)\overline{G}_1(t_2+s_2)\big(g_2(s_1,a_1)g_2(s_2,a_2)-g_2(s_2,a_1)g_2(s_1,a_2)\big)ds_2ds_1 \\ &- \int_{-\infty}^{\infty}\int_{s_1}^{\infty} \overline{G}_1(t_1+s_1)\overline{G}_1(t_2+s_2)\big(g_2(s_2,a_1)g_2(s_1,a_2)-g_2(s_1,a_1)g_2(s_2,a_2)\big)ds_2ds_1 \end{aligned}$$

Switching the order of integration in the second integral, and then exchanging the names of the variables $s_1$ and $s_2$ in the second integral, gives

$$\begin{aligned} D = &\int_{-\infty}^{\infty}\int_{-\infty}^{s_1} \overline{G}_1(t_1+s_1)\overline{G}_1(t_2+s_2)\big(g_2(s_1,a_1)g_2(s_2,a_2)-g_2(s_2,a_1)g_2(s_1,a_2)\big)ds_2ds_1 \\ &- \int_{-\infty}^{\infty}\int_{-\infty}^{s_1} \overline{G}_1(t_1+s_2)\overline{G}_1(t_2+s_1)\big(g_2(s_1,a_1)g_2(s_2,a_2)-g_2(s_2,a_1)g_2(s_1,a_2)\big)ds_2ds_1 \end{aligned}$$

Recalling that $d\overline{G}_1 = -g_1$ and $dG_2 = g_2$, evaluate both inner ($ds_2$) integrals by parts, giving

$$\begin{aligned} D = &\int_{-\infty}^{\infty} \big[\overline{G}_1(t_1+s_1)\overline{G}_1(t_2+s_2)\big(g_2(s_1,a_1)G_2(s_2,a_2)-G_2(s_2,a_1)g_2(s_1,a_2)\big)\big|_{s_2=-\infty}^{s_2=s_1} \\ &+ \int_{-\infty}^{s_1} \overline{G}_1(t_1+s_1)g_1(t_2+s_2)\big(g_2(s_1,a_1)G_2(s_2,a_2)-G_2(s_2,a_1)g_2(s_1,a_2)\big)ds_2\big]ds_1 \\ &- \int_{-\infty}^{\infty} \big[\overline{G}_1(t_1+s_2)\overline{G}_1(t_2+s_1)\big(g_2(s_1,a_1)G_2(s_2,a_2)-G_2(s_2,a_1)g_2(s_1,a_2)\big)\big|_{s_2=-\infty}^{s_2=s_1} \\ &+ \int_{-\infty}^{s_1} g_1(t_1+s_2)\overline{G}_1(t_2+s_1)\big(g_2(s_1,a_1)G_2(s_2,a_2)-G_2(s_2,a_1)g_2(s_1,a_2)\big)ds_2\big]ds_1 \end{aligned}$$

Plugging $s_2 = -\infty$ into $G_2(s_2, a)$ in the first and third lines gives 0; plugging in $s_2 = s_1$ in the first and third gives the four terms

$$\begin{aligned} &\int_{-\infty}^{\infty} \overline{G}_1(t_1+s_1)\overline{G}_1(t_2+s_1)\big(g_2(s_1,a_1)G_2(s_1,a_2)-G_2(s_1,a_1)g_2(s_1,a_2)\big)ds_1 \\ &\quad - \int_{-\infty}^{\infty} \overline{G}_1(t_1+s_1)\overline{G}_1(t_2+s_1)\big(g_2(s_1,a_1)G_2(s_1,a_2)-G_2(s_1,a_1)g_2(s_1,a_2)\big)ds_1 \end{aligned}$$

which conveniently cancel, leaving

$$D = \int_{-\infty}^{\infty}\int_{-\infty}^{s_1} \overline{G}_1(t_1+s_1)g_1(t_2+s_2)\big(g_2(s_1,a_1)G_2(s_2,a_2) - G_2(s_2,a_1)g_2(s_1,a_2)\big)ds_2 ds_1$$

$$- \int_{-\infty}^{\infty}\int_{-\infty}^{s_1} g_1(t_1+s_2)\overline{G}_1(t_2+s_1)\big(g_2(s_1,a_1)G_2(s_2,a_2) - G_2(s_2,a_1)g_2(s_1,a_2)\big)ds_2 ds_1$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{s_1} \begin{vmatrix} \overline{G}_1(t_1+s_1) & g_1(t_1+s_2) \\ \overline{G}_1(t_2+s_1) & g_1(t_2+s_2) \end{vmatrix} \begin{vmatrix} g_2(s_1,a_1) & g_2(s_1,a_2) \\ G_2(s_2,a_1) & G_2(s_2,a_2) \end{vmatrix} ds_2 ds_1$$

The first determinant has the same sign as

$$\frac{g_1(t_2+s_2)}{\overline{G}_1(t_2+s_1)} - \frac{g_1(t_1+s_2)}{\overline{G}_1(t_1+s_1)} = \frac{g_1(t_2+s_2)}{\overline{G}_1(t_2+s_2)}\frac{\overline{G}_1(t_2+s_2)}{\overline{G}_1(t_2+s_1)} - \frac{g_1(t_1+s_2)}{\overline{G}_1(t_1+s_2)}\frac{\overline{G}_1(t_1+s_2)}{\overline{G}_1(t_1+s_1)}$$

By assumption, $\frac{g_1}{\overline{G}_1} = \frac{f_k}{1-F_k}$ is increasing in its argument and $t_1 > t_2$, so $\frac{g_1(t_2+s_2)}{\overline{G}_1(t_2+s_2)} < \frac{g_1(t_1+s_2)}{\overline{G}_1(t_1+s_2)}$; and since $\overline{G}_1$ is log-concave, $\overline{G}_1(t+s)$ is log-submodular in $(t,s)$, so $\overline{G}_1(t_1+s_1)\overline{G}_1(t_2+s_2) < \overline{G}_1(t_1+s_2)\overline{G}_1(t_2+s_1)$, or $\frac{\overline{G}_1(t_2+s_2)}{\overline{G}_1(t_2+s_1)} < \frac{\overline{G}_1(t_1+s_2)}{\overline{G}_1(t_1+s_1)}$. This means the first determinant is negative.

The second determinant has the same sign as

$$\frac{g_2(s_1,a_1)}{G_2(s_2,a_1)} - \frac{g_2(s_1,a_2)}{G_2(s_2,a_2)} = \frac{g_2(s_1,a_1)}{G_2(s_1,a_1)}\frac{G_2(s_1,a_1)}{G_2(s_2,a_1)} - \frac{g_2(s_1,a_2)}{G_2(s_1,a_2)}\frac{G_2(s_1,a_2)}{G_2(s_2,a_2)}$$

This time,

$$\begin{aligned} G_2(t) &= \Pr\big(\max\{v_j^l - P_j\} < t\big) \\ &= \Pr\big(v_j^l < P_j + t \ \forall j\big) \\ &= \prod_{j\in\mathcal{K}-\{k\}\cup\{0\}} F_j(P_j+t) \\ \log G_2(t) &= \sum_{j\in\mathcal{K}-\{k\}\cup\{0\}} \log F_j(P_j+t) \\ \big(\log G_2(t)\big)' &= \sum_{j\in\mathcal{K}-\{k\}\cup\{0\}} \frac{f_j(P_j+t)}{F_j(P_j+t)} \end{aligned}$$

which, since each $F_j$ is log-concave by assumption, is decreasing in $P_j$, so $G_2$ is log-submodular in $t$ and $P_{k'} = a$. This means that $\frac{g_2(s_1,a_1)}{G_2(s_1,a_1)} < \frac{g_2(s_1,a_2)}{G_2(s_1,a_2)}$ and $\frac{G_2(s_1,a_1)}{G_2(s_2,a_1)} < \frac{G_2(s_1,a_2)}{G_2(s_2,a_2)}$, so the second determinant is negative.

Thus, both determinants are negative, making the integrand everywhere positive and therefore $D > 0$, so $Q_k = \overline{F}_k(P_k)$ is log-supermodular in $(P_k, P_{k'})$. □

### A.2. *Proof of Lemma 1*

Fix $k \in \mathcal{K}$ and $i \in \mathcal{T}_k$, and let $p_{-i} = \sum_{j\in\mathcal{T}_k-\{i\}} p_j$. Given other firms' prices, firm $i$ solves

$$\max_{p_i}\big\{\log(p_i - c_i) + \log Q_k(p_i + p_{-i}, P_{-k})\big\} \tag{A.1}$$

By Theorem 1, $Q_k$ is log-concave in $P_k$, so the maximand in (A.1) is strictly concave in $p_i$; so $p_i$ is a solution if and only if it satisfies the first-order condition

$$\frac{1}{p_i - c_i} = -\frac{\partial}{\partial p_i} \log Q_k(p_i + p_{-i}, P_{-k}) = -\frac{\partial \log Q_k}{\partial P_k}(P_k, P_{-k}) \tag{A.2}$$

The right-hand side of (A.2) depends on $k$ but not $i$, so in equilibrium, every firm $i \in \mathcal{T}_k$ sets the same markup $p_i - c_i$; so $P_k - C_k = n_k(p_i - c_i)$, and (A.2) becomes

$$\frac{n_k}{P_k - C_k} = -\frac{\partial \log Q_k}{\partial P_k}(P_k, P_{-k}) \tag{A.3}$$

This is the first-order condition to the maximization problem

$$\max_{P_k}\left\{n_k \log(P_k - C_k) + \log Q_k(P_k, P_{-k})\right\} \tag{A.4}$$

which is strictly concave, so (A.3) is satisfied if and only if $P_k$ solves (A.4). So all firms $i \in \mathcal{T}_k$ are simultaneously best-responding to $P_{-k}$ if and only if their markups are equal to each other and their combined price solves (A.4). Since this holds for every $k$, the aggregate prices in the equilibrium of the actual pricing game simultaneously maximize (A.4) for each $k \in \{1, 2, \dots, K\}$, and therefore correspond to the equilibrium of the $K$-player game with payoffs given by (A.4); and the equilibrium prices of this new game, along with each firm $i \in \mathcal{T}_k$ setting markup $p_i - c_i = \frac{1}{n_k}(P_k - C_k)$, satisfy (A.2) and therefore solve (A.1), and are therefore an equilibrium of the full game. By Theorem 1, $\frac{\partial u_k}{\partial P_k} = \frac{n_k}{P_k - C_k} + \frac{\partial \log Q_k}{\partial P_k}$ is increasing in $P_{k'}$ for every $k' \neq k$, so the new $K$-player game is a supermodular game.

Next, we establish existence of an equilibrium of this new game. Since the game is supermodular, player $k$'s best-response is bounded above by the limit of his best-responses as $P_{k'} \to +\infty$ for $P_{k'}$. This is finite, since even if all other products had infinite prices, product $k$ would still be competing against the (free) outside option; even without any other products available, $Q_k$ is still strictly log-concave under Assumption 1, so $-\frac{\partial \log Q_k}{\partial P_k}$ is strictly positive and increasing, and therefore $\frac{n_k}{P_k - C_k} = -\frac{\partial \log Q_k}{\partial P_k}$ has a finite solution. Letting $P_k^*$ denote this solution, it's easy to show that prices above $P_k^*$ are strictly dominated by $P_k^*$, and can therefore be eliminated without loss. Given supermodularity, best-responses are similarly bounded below by the best-response to zero prices by all competitors, which will be strictly above $C_k$. Thus, we can eliminate strategies for player $k$ outside of some range $[\underline{P}_k, P_k^*]$ with $\underline{P}_k > C_k$ and $P_k^* < \infty$. (The lower bound is needed because we focus on log-profits, and $\log(P_k - C_k)$ is not continuous at $P_k = C_k$.) Continuous supermodular games on bounded strategy spaces are guaranteed to have an equilibrium, and one can be found by iterating best-responses from either the "lower-left" or "upper-right" corner of the strategy space.

Next, we establish uniqueness of this equilibrium. Suppose instead that there were two distinct equilibria, with aggregate prices $(\overline{P}_0, \overline{P}_1, \overline{P}_2, \dots, \overline{P}_K)$ and $(\overline{P}'_0, \overline{P}'_1, \overline{P}'_2, \dots, \overline{P}'_K)$. Note that we include imaginary prices for the outside option (buying nothing), but $\overline{P}_0 = \overline{P}'_0 = 0$. (Since the outisde option is treated symmetrically to any other option, $\frac{\partial \log Q_k}{\partial P_k}$ would be increasing in $P_0$; the proof is identical to the proof of the last part of Theorem 1.) Fix $k \in \arg\max_{j \in \mathcal{K}} |\overline{P}'_j - \overline{P}_j|$, assume without loss of generality that $\overline{P}'_k > \overline{P}_k$, and let $\epsilon = \overline{P}'_k - \overline{P}_k > 0$. Let $P_{-k} + \epsilon$ denote adding $\epsilon$ to every price in $P_{-k}$. Each consumer's problem, and therefore demand and price elasticity, are unaffected when the same constant is added to every price including the price of the outside good; so

$$\frac{\partial \log Q_k}{\partial P_k}(\overline{P}_k, \overline{P}_0, \overline{P}_{-k}) = \frac{\partial \log Q_k}{\partial P_k}(\overline{P}_k + \epsilon, \overline{P}_0 + \epsilon, \overline{P}_{-k} + \epsilon) > \frac{\partial \log Q_k}{\partial P_k}\left(\overline{P}'_k, \overline{P}'_0, \overline{P}'_{-k}\right)$$

The latter inequality holds because $\frac{\partial \log Q_k}{\partial P_k}$ is increasing in $P_{k'}$ for every $k' \in \mathcal{K} \cup \{0\} - \{k\}$ (by Theorem 1), $\overline{P}'_0 = 0 < \epsilon = \overline{P} + \epsilon$, and $\overline{P}'_{k'} \leq \overline{P}_{k'} + \epsilon$ for every $k' \in \mathcal{K} - \{k\}$. But since $\overline{P}_k < \overline{P}'_k$, $\frac{n_k}{\overline{P}_k - C_k} > \frac{n_k}{\overline{P}'_k - C_k}$ as well, so the first-order condition (A.3) cannot hold at both equilibria, giving a contradiction.

Finally, since $\frac{\partial}{\partial P_k}(n_k \log(P_k - C_k) + \log Q_k) = \frac{n_k}{P_k - C_k} + \frac{\partial \log Q_k}{\partial P_k}$ is increasing in $n_k$ and $C_k$, the game is indexed by $n_k$ and $C_k$ for every $k$. □

### A.3. Proof of Theorem 2

*Part 1.* Since the supermodular game described in Lemma 1 has a unique equilibrium and is indexed by $C_k$, an increase in $C_k$ leads to higher prices for all products. By the same logic as the uniqueness proof in Lemma 1, if $k' \in \arg\max_{k''} \Delta P_{k''}$ and $k' \neq k$, then both $\frac{n_{k'}}{P_{k'} - C_{k'}}$ and $\frac{\partial \log Q_{k'}}{\partial P_{k'}}$ are lower after the change, and the first-order condition $\frac{n_{k'}}{P_{k'} - C_{k'}} = -\frac{\partial \log Q_{k'}}{\partial P_{k'}}$ could therefore not hold both before and after the change; so $\Delta P_k > \Delta P_{k'}$. But since $\Delta P_k > \Delta P_{k'}$ for every $k'$, $\frac{\partial \log Q_k}{\partial P_k}$ is lower after the change, so $\frac{n_k}{P_k - C_k}$ must be higher, meaning $P_k - C_k$ is lower or $\Delta P_k < \Delta C_k$. Since $P_k$ rose, and rose by more than the price of any other product, $Q_k$ must fall; in addition, $p_{i'} - c_{i'} = \frac{1}{n_k}(P_k - C_k)$ fell for any $i' \in \mathcal{T}_k$, giving lower profits $(p_{i'} - c_{i'})Q_k$.

*Part 2.* Let $t \in \{0, 1\}$ indicate whether or not the right-shift in $F_k$ has occurred, and let $\mathbf{Q}^t(P_1, \ldots, P_K) = (Q_1^t(\cdot), \ldots, Q_K^t(\cdot))$ denote the demand system before and after the shift. Given quasilinear consumer preferences, a price reduction would have the same effect on demand as a quality increase, so $\mathbf{Q}^1(P_k, P_{-k}) = \mathbf{Q}^0(P_k - \Delta v, P_{-k})$; or, letting $P_k^t \equiv P_k - t\Delta v$ denote the "quality-adjusted price" of good $k$, $\mathbf{Q}^t(P_k, P_{-k}) = \mathbf{Q}^0(P_k^t, P_{-k})$. Thus, rather than the demand system changing, we can think of the demand system remaining constant, but as a function of $P_k^t$. Firms $i \in \mathcal{T} - \mathcal{T}_k$ have the same payoff functions as before, but as a function of $P_k^t$, and so their aggregate prices solve $\max_{P_{k'}}\{n_{k'} \log(P_{k'} - C_{k'}) + \log Q_{k'}^0(P_{k'}, P_k^t, P_{-k,k'})\}$; the aggregate price of product $k$ now solves

$$\max_{P_k^t}\left\{n_k \log\left(P_k^t + t\Delta v - C_k\right) + \log Q_k^0\left(P_k^t, P_{-k}\right)\right\}$$

This relabeled game is as before a supermodular game in $(P_k^t, P_{-k})$, and is indexed by $-t$; so opposite to Part 1, the increase in $t$ (from 0 to 1) causes $P_k^t$ and $P_{k'}$ to fall, with $P_k^t$ falling by more (thus $\Delta P_k^t = \Delta P_k - \Delta v < \Delta P_{k'} < 0$). Since $P_k^t$ falls by more than each $P_{k'}$, $\frac{\partial \log Q_k}{\partial P_k^t}$ is higher than before, so $\frac{1}{P_k^t + t\Delta v - C_k}$ must be lower than before if the first-order condition is to hold; thus $P_k - C_k = P_k^t + t\Delta v - C_k$ has risen, or $\Delta P_k > 0$. Since $P_k^t$ falls by more than each $P_{k'}$, $Q_k$ rises, and since $p_i - c_i = \frac{1}{n_k}(P_k - C_k)$ also rises for each $i \in \mathcal{T}_k$, profits are higher.

*Part 3.* The introduction of a new product can be thought of as the limit, as $M \to +\infty$, of a reduction in product $k$'s costs from $C_k + M$ to $C_k$; by the reverse of part 1, this implies lower prices for all products. □

### A.4. Proof of Theorem 3

*Part 1.* Since the supermodular game in Lemma 1 is indexed by $n_k$, a reduction in $n_k$ while holding $C_k$ constant (a merger between complements with no cost synergies) lowers all prices.

By the same logic as in the proof of uniqueness in Lemma 1, $P_k$ must fall more than any of the other prices, since otherwise, whichever product's price fell the most could not have satisfied its first-order condition (A.3) both before and after the merger. By the same logic, since $\Delta P_k < \Delta P_{k'} < 0$, $\frac{\partial \log Q_k}{\partial P_k}$ is higher than before, so for any $i' \in \mathcal{T}_k - \{i, j\}$, $p_{i'}$ must rise in order for $\frac{1}{p_{i'} - c_{i'}} = -\frac{\partial \log Q_k}{\partial P_k}$ to hold before and after. Since $P_k$ fell by more than any other price, $Q_k$ must rise, so these firms $i'$ sell more at a higher markup and therefore earn higher profits. (However, since $P_k$ fell and $p_{i'}$ rose for all $i' \in \mathcal{T}_k - \{i, j\}$, the merged firm must be setting a lower combined price than before, and such a merger may not profitable.)

*Part 2.* Note that a merger between substitutes destroys the supermodular structure of the game, so the uniqueness proof from earlier no longer holds; however, the second result holds for *any* post-merger equilibrium, provided one exists. Suppose the merger is between firms $i \in \mathcal{T}_1$ and $j \in \mathcal{T}_2$. The merged firm maximizes $(p_i - c_i)Q_1 + (p_j - c_j)Q_2$; the first-order condition with respect to $p_i$ is $Q_1 + (p_i - c_i)\frac{\partial Q_1}{\partial P_1} + (p_j - c_j)\frac{\partial Q_2}{\partial P_1} = 0$, or rearranging,

$$\frac{1}{p_i - c_i}\left(1 + \frac{p_j - c_j}{Q_1}\frac{\partial Q_2}{\partial P_1}\right) = -\frac{\partial \log Q_1}{\partial P_1}$$

Let $x$ be the equilibrium value of $\frac{p_j - c_j}{Q_1}\frac{\partial Q_2}{\partial P_1}$, so this becomes

$$p_i - c_i = \frac{1 + x}{-\partial \log Q_1 / \partial P_1}$$

Along with the usual first-order condition $p_{i'} - c_{i'} = \frac{1}{-\partial \log Q_1 / \partial P_1}$ of the firms $i' \in \mathcal{T}_1 - \{i\}$, this establishes

$$P_1 - C_1 = \frac{n_1 + x}{-\partial \log Q_1 / \partial P_1}$$

or

$$P_1 = \arg\max_{P_1} \left\{(n_1 + x)\log(P_1 - C_1) + \log Q_1\right\}$$

By identical arguments,

$$P_2 = \arg\max_{P_2} \left\{(n_2 + y)\log(P_2 - C_2) + \log Q_2\right\}$$

where $y$ is the equilibrium value of $\frac{p_i - c_i}{Q_2}\frac{\partial Q_1}{\partial P_2}$. Since components $i$ and $j$ are substitutes, nonpositive markups for either product are strictly dominated for the merged firm, and so $x, y > 0$; so the merger corresponds to increases in $(n_1, n_2)$ from their old values to $(n_1 + x, n_2 + y)$. Since the $K$-player game described in Lemma 1 is supermodular and indexed by $n_1$ and $n_2$, this means all prices are higher post-merger. □

### A.5. Proof of Lemma 2

What we actually need is for

$$-\frac{\partial \log Q_k}{\partial \log(P_k - C_k)} \cdot \frac{1}{\varepsilon_{k,k'}}$$

to be increasing in $P_k$, which is weaker than Assumption 3 because

$$-\frac{\partial \log Q_k}{\partial \log(P_k - C_k)} \cdot \frac{1}{\varepsilon_{k,k'}} = -(P_k - C_k)\frac{\partial \log Q_k}{\partial P_k} \cdot \frac{1}{\varepsilon_{k,k'}} = \frac{P_k - C_k}{P_k}\frac{\varepsilon_{k,k}}{\varepsilon_{k,k'}}$$

and $\frac{P_k - C_k}{P_k}$ is increasing in $P_k$. Letting $\bar{\varepsilon}_{k,k} \equiv -\frac{\partial \log Q_k}{\partial \log(P_k - C_k)}$, rearranging (A.3) shows that the mutual best-responses of the $n_k$ firms $i \in \mathcal{T}_k$ are the unique solution to $\bar{\varepsilon}_{k,k} = n_k$. Consider an incremental increase in $\log P_{k'}$ of $d \log P_{k'}$, followed by the resulting change in $\log(P_k - C_k)$ of $d \log(P_k - C_k)$. Since $\bar{\varepsilon}_{k,k} = n_k$ holds both before and after,

$$\frac{\partial \bar{\varepsilon}_{k,k}}{\partial \log P_{k'}} d \log P_{k'} + \frac{\partial \bar{\varepsilon}_{k,k}}{\partial \log(P_k - C_k)} d \log(P_k - C_k) = 0$$

Defining $\Delta = d \log P_{k'} / \frac{\partial \bar{\varepsilon}_{k,k}}{\partial \log(P_k - C_k)}$, which we know is positive, we get

$$d \log P_{k'} = \frac{\partial \bar{\varepsilon}_{k,k}}{\partial \log(P_k - C_k)} \Delta \quad \text{and} \quad d \log(P_k - C_k) = -\frac{\partial \bar{\varepsilon}_{k,k}}{\partial \log P_{k'}} \Delta$$

The net effect on $\log Q_k$, then, is

$$\begin{aligned} d \log Q_k &= \frac{\partial \log Q_k}{\partial \log P_{k'}} d \log P_{k'} + \frac{\partial \log Q_k}{\partial \log(P_k - C_k)} d \log(P_k - C_k) \\ &= \frac{\partial \log Q_k}{\partial \log P_{k'}} \frac{\partial \bar{\varepsilon}_{k,k}}{\partial \log(P_k - C_k)} \Delta - \frac{\partial \log Q_k}{\partial \log(P_k - C_k)} \frac{\partial \bar{\varepsilon}_{k,k}}{\partial \log P_{k'}} \Delta \\ &= \varepsilon_{k,k'} \frac{\partial \bar{\varepsilon}_{k,k}}{\partial \log(P_k - C_k)} \Delta - (-\bar{\varepsilon}_{k,k}) \frac{\partial^2(-\log Q_k)}{\partial \log P_{k'} \partial \log(P_k - C_k)} \Delta \end{aligned}$$

Switching the order of the two partial derivatives in the last term gives

$$\begin{aligned} d \log Q_k &= \varepsilon_{k,k'} \frac{\partial \bar{\varepsilon}_{k,k}}{\partial \log(P_k - C_k)} \Delta - \bar{\varepsilon}_{k,k} \frac{\partial}{\partial \log(P_k - C_k)} \left( \frac{\partial \log Q_k}{\partial \log P_{k'}} \right) \Delta \\ &= \varepsilon_{k,k'} \bar{\varepsilon}_{k,k} \Delta \left[ \frac{1}{\bar{\varepsilon}_{k,k}} \frac{\partial \bar{\varepsilon}_{k,k}}{\partial \log(P_k - C_k)} - \frac{1}{\varepsilon_{k,k'}} \frac{\partial \varepsilon_{k,k'}}{\partial \log(P_k - C_k)} \right] \\ &= \varepsilon_{k,k'} \bar{\varepsilon}_{k,k} \Delta \left[ \frac{\partial \log \bar{\varepsilon}_{k,k}}{\partial \log(P_k - C_k)} - \frac{\partial \log \varepsilon_{k,k'}}{\partial \log(P_k - C_k)} \right] \\ &= \varepsilon_{k,k'} \bar{\varepsilon}_{k,k} \Delta \frac{\partial}{\partial \log(P_k - C_k)} \log \frac{\bar{\varepsilon}_{k,k}}{\varepsilon_{k,k'}} \end{aligned}$$

Under Assumption 3, $\bar{\varepsilon}_{k,k}/\varepsilon_{k,k'}$ is increasing in $P_k$, so $\log(\bar{\varepsilon}_{k,k}/\varepsilon_{k,k'})$ is increasing in $\log(P_k - C_k)$, so $d \log Q_k > 0$. For a "large" change in $P_{k'}^N$, $\Delta Q_k = \int dQ_k > 0$, so $Q_k$ ends up higher than it started. □

We can similarly calculate the change in log-profits $d(\log(P_k - C_k) + \log Q_k)$; this turns out to be positive if $(\bar{\varepsilon}_{k,k} - 1)/\varepsilon_{k,k'}$ is increasing in $P_k$. So under this weaker condition, the changes considered in Lemma 2 leave the profits of the firms in $\mathcal{T}_k$ higher, though not necessarily the demand for product $k$; and thus the results on profit, though not demand, in Theorem 4 would still hold under this weaker condition.

### A.6. Proof of Theorem 4

Following any of these changes, decompose the move from old equilibrium prices to new prices into $K-1$ steps: in each step, for one $k'' \in \mathcal{K} - \{k\}$, the firms in $\mathcal{T}_{k''}$ change from their old to their new equilibrium prices, and the firms in $\mathcal{T}_k$ move to their new simultaneous best-responses. By Theorems 2 and 3, each step involves a reduction in $P_{k''}$ (or, in the case of firm $k'$ following a right-shift in $F_{k'}$, a reduction in the quality-adjusted price $P_{k'} - \Delta v$); by Lemma 2, then, each step leaves $Q_k$ lower than before. Since $Q_k$ and $P_k$ both end up lower than before, for $j \in \mathcal{T}_k$, firm $j$'s profits $\pi_j = \frac{1}{n_k}(P_k - C_k)Q_k$ are lower than before. For the merger between firms $i \in \mathcal{T}_{k'}$ and $j \in \mathcal{T}_{k''}$, the same logic holds in reverse. $\square$

## Appendix B. Supplementary material

Supplementary material related to this article can be found online at http://dx.doi.org/10.1016/j.jet.2014.05.004.

## References

[1] Simon Anderson, André de Palma, Jacques-Francois Thisse, Discrete Choice Theory of Product Differentiation, MIT Press, Cambridge, MA, 1992.
[2] Mark Armstrong, Competition in two-sided markets, RAND J. Econ. 37 (3) (2006) 668–691.
[3] Mark Bagnoli, Ted Bergstrom, Log-concave probability and its applications, Econ. Theory 26 (2) (2005) 445–469.
[4] Richard Barlow, Frank Proschan, Statistical Theory of Reliability and Life Testing, To Begin With, Silver Spring, MD, 1975.
[5] Steven Berry, James Levinsohn, Ariel Pakes, Automobile prices in market equilibrium, Econometrica 63 (4) (1995) 841–890.
[6] Henry Block, Thomas Savits, Harshinder Singh, The reversed hazard rate function, Probab. Eng. Inf. Sci. 12 (1) (1998) 69–90.
[7] Arthur Bowley, The Mathematical Groundwork of Economics, Oxford University Press, 1924.
[8] Meghan Busse, Nathaniel Keohane, Market effects of environmental regulation: coal, railroads, and the 1990 clean air act, RAND J. Econ. 38 (4) (2007) 1159–1179.
[9] Andrew Caplin, Barry Nalebuff, Aggregation and imperfect competition: on the existence of equilibrium, Econometrica 59 (1) (1991) 25–29.
[10] Juan Carrillo, Guofu Tan, Platform competition: the role of multi-homing and complementors, 2006, working paper.
[11] Ramon Casadesus-Masanell, Barry Nalebuff, David Yoffie, Competing complements, 2007, working paper.
[12] Keith Chen, Barry Nalebuff, One-way essential complements, 2006, working paper.
[13] Yongmin Chen, Michael Riordan, Price-increasing competition, RAND J. Econ. 39 (4) (2008) 1042–1058.
[14] Raymond Deneckere, Carl Davidson, Incentives to form coalitions with Bertrand competition, RAND J. Econ. 16 (4) (1985) 473–486.
[15] Liran Einav, Theresa Kuchler, Jonathan Levin, Neel Sundaresan, Learning from seller experiments in online markets, 2012, working paper.
[16] Michal Fabinger, Glen Weyl, Pass-through and demand forms, 2012, working paper.
[17] Xavier Gabaix, David Laibson, Deyuan Li, Hongyi Li, Sidney Resnick, Casper G. de Vries, The impact of competition on prices with numerous firms, 2013, working paper.
[18] John Hatfield, Scott Kominers, Matching in networks with bilateral contracts, Amer. Econ. J. Microecon. 4 (1) (2012) 176–208.
[19] Henrick Horn, Asher Wolinsky, Bilateral monopolies and incentives for merger, RAND J. Econ. 19 (3) (1988) 408–419.
[20] Charles James, Department of Justice Business Review Letter to K.P. Ewing, available at http://www.justice.gov/atr/public/busreview/200455.htm, 12 November, 2002.
[21] Sonia Jaffe, Glen Weyl, The first-order approach to merger analysis, Amer. Econ. J. Microecon. 5 (4) (2013) 188–218.
[22] Samuel Karlin, Total Positivity, Volume I, Stanford University Press, 1968.

[23] Michael Ostrovsky, Stability in supply chain networks, Amer. Econ. Rev. 98 (3) (2008) 897–923.
[24] Jeffrey Perloff, Steven Salop, Equilibrium with product differentiation, Rev. Econ. Stud. 52 (1) (1985) 107–120.
[25] Daniel Quint, Pooling with essential and nonessential patents, Amer. Econ. J. Microecon. 6 (1) (2014) 23–57.
[26] Jean-Charles Rochet, Jean Tirole, Two-sided markets: a progress report, RAND J. Econ. 37 (3) (2006) 645–667.
[27] Steven Salop, Monopolistic competition with outside goods, Bell J. Econ. 10 (1979) 141–146.
[28] Lloyd Shapley, Martin Shubik, Price strategy oligopoly with product differentiation, Kyklos 22 (1969) 30–44.
[29] Guofu Tan, Lasheng Yuan, Strategic incentives of divestitures of competing conglomerates, Int. J. Ind. Organ. 21 (2003) 673–697.
[30] Xavier Vives, Oligopoly Pricing: Old Ideas and New Tools, MIT Press, Cambridge, MA, 1999.
[31] Xavier Vives, Innovation and competitive pressure, J. Ind. Econ. 56 (3) (2008) 419–469.
[32] Glen Weyl, A price theory of multi-sided platforms, Amer. Econ. Rev. 100 (4) (2010) 1642–1672.
[33] Glen Weyl, Michal Fabinger, Pass-through as an economic tool: principles of incidence under imperfect competition, J. Polit. Economy 121 (3) (2013) 528–583.