ECONOMICS OF PATENT POOLS WHEN SOME (BUT NOT ALL) PATENTS ARE ESSENTIAL

TECHNICAL APPENDIX

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1 Pools of Essential Patents (Theorem 1)

1.1 Marginal Effect of n_E on Prices

Under logit demand,

$$Q_k = \frac{\exp(v_k - P_k^N - P^E)}{1 + \sum_{k' \in \mathcal{K}} \exp(v_{k'} - P_{k'}^N - P^E)}$$

Let $Q_0 = 1 - \sum_{k \in \mathcal{K}} Q_k$. Differentiation establishes that for $k, k' \in \mathcal{K}$, $k' \neq k$,

$$\frac{\partial Q_k}{\partial P_k^N} = -Q_k(1-Q_k), \quad \frac{\partial Q_k}{\partial P_{k'}^N} = Q_k Q_{k'}, \quad \frac{\partial Q_k}{\partial P^E} = -Q_k Q_0, \quad \frac{\partial Q_0}{\partial P_k^N} = Q_0 Q_k, \quad \frac{\partial Q_0}{\partial P^E} = Q_0(1-Q_0)$$

Logit demand satisfies the conditions for Lemmas 1 and 2, so we know that P^E is increasing in n_E and P_k^N decreasing in n_E. To calculate the exact effects, we can treat n_E as a continuous variable, affecting equilibrium prices through the simultaneous equations $P_k^N(1-Q_k) = n_k$ and $P^E Q_0 = n_E$, and differentiate these with respect to n_E:

$$\begin{aligned}
P_k^N(1-Q_k) &= n_k \\
(1-Q_k)\tfrac{\partial P_k^N}{\partial n_E} - P_k^N \left(\sum_{k' \in \mathcal{K}} \tfrac{\partial Q_k}{\partial P_{k'}^N} \tfrac{\partial P_{k'}^N}{\partial n_E} + \tfrac{\partial Q_k}{\partial P^E} \tfrac{\partial P^E}{\partial n_E} \right) &= 0 \\
(1-Q_k)\tfrac{\partial P_k^N}{\partial n_E} - P_k^N \left(-Q_k(1-Q_k)\tfrac{\partial P_k^N}{\partial n_E} + \sum_{k' \neq k} Q_k Q_{k'} \tfrac{\partial P_{k'}^N}{\partial n_E} - Q_k Q_0 \tfrac{\partial P^E}{\partial n_E} \right) &= 0 \\
(1-Q_k+P_k^N Q_k)\tfrac{\partial P_k^N}{\partial n_E} - P_k^N Q_k \left(\sum_{k' \in \mathcal{K}} Q_{k'} \tfrac{\partial P_{k'}^N}{\partial n_E} - Q_0 \tfrac{\partial P^E}{\partial n_E} \right) &= 0
\end{aligned}$$

and similarly

$$\begin{aligned}
P^E Q_0 &= n_E \\
Q_0 \tfrac{\partial P^E}{\partial n_E} + P^E \left(\tfrac{\partial Q_0}{\partial P^E} \tfrac{\partial P^E}{\partial n_E} + \sum_{k' \in \mathcal{K}} \tfrac{\partial Q_0}{\partial P_{k'}^N} \tfrac{\partial P_{k'}^N}{\partial n_E} \right) &= 1 \\
Q_0 \tfrac{\partial P^E}{\partial n_E} + P^E \left(Q_0(1-Q_0) \tfrac{\partial P^E}{\partial n_E} + \sum_{k' \in \mathcal{K}} Q_0 Q_{k'} \tfrac{\partial P_{k'}^N}{\partial n_E} \right) &= 1 \\
\left(Q_0 + P^E Q_0 \right) \tfrac{\partial P^E}{\partial n_E} - P^E Q_0 \left(Q_0 \tfrac{\partial P^E}{\partial n_E} - \sum_{k' \in \mathcal{K}} Q_{k'} \tfrac{\partial P_{k'}^N}{\partial n_E} \right) &= 1
\end{aligned}$$

Letting $\Delta = Q_0 \frac{\partial P^E}{\partial n_E} - \sum_{k' \in \mathcal{K}} Q_{k'} \frac{\partial P_{k'}^N}{\partial n_E}$, which we know from supermodularity is positive, gives

$$\begin{aligned}
\tfrac{\partial P_k^N}{\partial n_E} &= -\tfrac{P_k^N Q_k}{1-Q_k+P_k^N Q_k} \Delta \\
\tfrac{\partial P^E}{\partial n_E} &= \tfrac{1}{Q_0 + P^E Q_0} + \tfrac{P^E}{1+P^E} \Delta \\
\Delta &= Q_0 \left(\tfrac{1}{Q_0+P^E Q_0} + \tfrac{P^E}{1+P^E} \Delta \right) - \sum_{k' \in \mathcal{K}} Q_{k'} \left(-\tfrac{P_{k'}^N Q_{k'}}{1-Q_{k'}+P_{k'}^N Q_{k'}} \Delta \right) \\
\tfrac{1}{1+P^E} &= \Delta \left(1 - \tfrac{P^E Q_0}{1+P^E} - \sum_{k' \in \mathcal{K}} Q_{k'} \tfrac{P_{k'}^N Q_{k'}}{1-Q_{k'}+P_{k'}^N Q_{k'}} \right) \\
\tfrac{1}{1+P^E} &= \Delta \left(Q_0 + \sum_{k' \in \mathcal{K}} Q_{k'} - \tfrac{P^E Q_0}{1+P^E} - \sum_{k' \in \mathcal{K}} Q_{k'} \tfrac{P_{k'}^N Q_{k'}}{1-Q_{k'}+P_{k'}^N Q_{k'}} \right) \\
\tfrac{1}{1+P^E} &= \Delta \left(\tfrac{Q_0}{1+P^E} + \sum_{k' \in \mathcal{K}} Q_{k'} \tfrac{1-Q_{k'}}{1-Q_{k'}+P_{k'}^N Q_{k'}} \right)
\end{aligned}$$

and so

$$\frac{\partial P_k^N}{\partial n_E} = -\frac{P_k^N Q_k}{1-Q_k+P_k^N Q_k}\frac{\delta}{1+P^E}$$

$$\frac{\partial P^E}{\partial n_E} = \frac{1}{Q_0+P^E Q_0}+\frac{P^E}{1+P^E}\frac{\delta}{1+P^E}$$

where $\frac{1}{\delta}=\frac{Q_0}{1+P^E}+\sum_{k'\in\mathcal{K}}Q_{k'}\frac{1-Q_{k'}}{1-Q_{k'}+P_{k'}^N Q_{k'}}$. We can also write $\frac{\partial P^E}{\partial n_E}$ as

$$\begin{aligned}
\frac{\partial P^E}{\partial n_E} &= \frac{1}{Q_0+P^E Q_0}\frac{1+P^E}{\delta}\frac{\delta}{1+P^E}+\frac{P^E}{1+P^E}\frac{\delta}{1+P^E} \\
&= \left(\frac{1}{Q_0}\left(\frac{Q_0}{1+P^E}+\sum_{k'\in\mathcal{K}}Q_{k'}\frac{1-Q_{k'}}{1-Q_{k'}+P_{k'}^N Q_{k'}}\right)+\frac{P^E}{1+P^E}\right)\frac{\delta}{1+P^E} \\
&= \left(1+\frac{1}{Q_0}\sum_{k'\in\mathcal{K}}Q_{k'}\frac{1-Q_{k'}}{1-Q_{k'}+P_{k'}^N Q_{k'}}\right)\frac{\delta}{1+P^E}
\end{aligned}$$

which makes it easier to show that

$$\frac{\partial\left(P_k^N+P^E\right)}{\partial n_E} = \frac{\delta}{1+P^E}\left(-\frac{P_k^N Q_k}{1-Q_k+P_k^N Q_k}+1+\frac{1}{Q_0}\sum_{k'\in\mathcal{K}}\frac{Q_{k'}(1-Q_{k'})}{1-Q_{k'}+P_{k'}^N Q_{k'}}\right) > 0$$

Finally, since $P_k^N(1-Q_k)=n_k$ or $\log P_k^N+\log(1-Q_k)=\log n_k$,

$$0 = \frac{\partial\left(\log P_k^N+\log(1-Q_k)\right)}{\partial n_E} = \frac{1}{P_k^N}\frac{\partial P_k^N}{\partial n_E}-\frac{1}{1-Q_k}\frac{\partial Q_k}{\partial n_E} \rightarrow \frac{\partial Q_k}{\partial n_E} = \frac{1-Q_k}{P_k^N}\frac{\partial P_k^N}{\partial n_E} = -\frac{(1-Q_k)Q_k}{1-Q_k+P_k^N Q_k}\frac{\delta}{1+P^E}$$

1.2 Welfare

First, note that consumer surplus can be written as

$$CS=\mathbb{E}_{\epsilon_0^l,\epsilon_1^l,\epsilon_2^l,\ldots,\epsilon_K^l}\max\left\{\epsilon_0^l,\max_{k'\in\mathcal{K}}\left\{v_{k'}-P_{k'}+\epsilon_{k'}^l\right\}\right\}=\log\left(1+\sum_{k'\in\mathcal{K}}e^{v_{k'}-P_{k'}}\right)$$

and differentiating, $\frac{\partial CS}{\partial P_k}=-Q_k$. Total patentholder profits are $P^E\left(\sum_{k'\in\mathcal{K}}Q_{k'}\right)+\sum_{k'\in\mathcal{K}}P_{k'}^N Q_{k'}=\sum_{k'\in\mathcal{K}}P_{k'}Q_{k'}$, so the effect of one price on total welfare is

$$\begin{aligned}
\frac{\partial Welfare}{\partial P_k} &= \frac{\partial CS}{\partial P_k}+\frac{\partial}{\partial P_k}\sum_{k'\in\mathcal{K}}P_{k'}Q_{k'} \\
&= -Q_k-P_kQ_k(1-Q_k)+Q_k+\sum_{k'\neq k}P_{k'}Q_{k'}Q_k \\
&= -P_kQ_k+\sum_{k'\in\mathcal{K}}P_{k'}Q_{k'}Q_k \\
&= Q_k\left(\overline{P}-P_k\right)
\end{aligned}$$

where $\overline{P}\equiv\sum_{k'\in\mathcal{K}}Q_{k'}P_{k'}$. Summing over k,

$$\frac{\partial Welfare}{\partial P^E} = \sum_{k\in\mathcal{K}}\frac{\partial Welfare}{\partial P_k} = \sum_{k\in\mathcal{K}}Q_k\left(\overline{P}-P_k\right) = \overline{P}\sum_{k\in\mathcal{K}}Q_k-\overline{P} = -Q_0\overline{P}$$

Using these,

$$\begin{aligned}
\frac{\partial Welfare}{\partial n_E} &= \frac{\partial Welfare}{\partial P^E}\frac{\partial P^E}{\partial n_E}+\sum_{k'\in\mathcal{K}}\frac{\partial Welfare}{\partial P_{k'}^N}\frac{\partial P_{k'}^N}{\partial n_E} \\
&= -Q_0\overline{P}\frac{\partial P^E}{\partial n_E}+\sum_{k'\in\mathcal{K}}Q_{k'}\left(\overline{P}-P_{k'}\right)\frac{\partial P_{k'}^N}{\partial n_E} \\
&= -Q_0\overline{P}\left(1+\frac{1}{Q_0}\sum_{k'\in\mathcal{K}}Q_{k'}\frac{1-Q_{k'}}{1-Q_{k'}+P_{k'}^N Q_{k'}}\right)\frac{\delta}{1+P^E}-\sum_{k'\in\mathcal{K}}Q_{k'}\left(\overline{P}-P_{k'}\right)\frac{P_{k'}^N Q_{k'}}{1-Q_{k'}+P_{k'}^N Q_{k'}}\frac{\delta}{1+P^E} \\
&= \frac{\delta}{1+P^E}\left(-Q_0\overline{P}-\overline{P}\sum_{k'\in\mathcal{K}}Q_{k'}\frac{1-Q_{k'}}{1-Q_{k'}+P_{k'}^N Q_{k'}}-\overline{P}\sum_{k'\in\mathcal{K}}Q_{k'}\frac{P_{k'}^N Q_{k'}}{1-Q_{k'}+P_{k'}^N Q_{k'}}+\sum_{k'\in\mathcal{K}}Q_{k'}P_{k'}\frac{P_{k'}^N Q_{k'}}{1-Q_{k'}+P_{k'}^N Q_{k'}}\right) \\
&= \frac{\delta}{1+P^E}\left(-Q_0\overline{P}-\overline{P}\sum_{k'\in\mathcal{K}}Q_{k'}+\sum_{k'\in\mathcal{K}}Q_{k'}P_{k'}\frac{P_{k'}^N Q_{k'}}{1-Q_{k'}+P_{k'}^N Q_{k'}}\right) \\
&= \frac{\delta}{1+P^E}\left(-\overline{P}+\sum_{k'\in\mathcal{K}}Q_{k'}P_{k'}\frac{P_{k'}^N Q_{k'}}{1-Q_{k'}+P_{k'}^N Q_{k'}}\right) \\
&= \frac{\delta}{1+P^E}\sum_{k'\in\mathcal{K}}Q_{k'}P_{k'}\left(\frac{P_{k'}^N Q_{k'}}{1-Q_{k'}+P_{k'}^N Q_{k'}}-1\right) \\
&< 0
\end{aligned}$$

Pools of essential patents have the same impact on equilibrium prices as a decrease in n_E; we can integrate the marginal effect to calculate the effect of a pool.

We showed above that each price $P_k = P_k^N + P^E$ is increasing in n_E, so each price P_k is lower following the formation of a pool; so total consumer surplus (as well as each individual consumer's payoff) is increased by a pool.

We showed that for each k, P_k^N and Q_k are both decreasing in n_E, and therefore higher after a pool. For $j \in \mathcal{J}_k^N$, $\pi_j = \frac{1}{n_k} P_k^N Q_k$, so π_j is higher after a pool for every $j \in \mathcal{J} - \mathcal{J}^E$. As for essential patentholders who remain outside of a pool, they earn profits of $\frac{1}{n_E} P^E (1 - Q_0)$ both before and after the pool forms. Since $P^E Q_0 = n_E$, we can rewrite this as $\frac{1}{Q_0}(1-Q_0) = \frac{1}{Q_0} - 1$. Since each Q_k is decreasing in n_E, Q_0 is increasing in n_E, so $\frac{1}{n_E} P^E (1-Q_0) = \frac{1}{Q_0} - 1$ is decreasing in n_E and therefore higher after a pool.

Finally, we showed that total welfare is decreasing in n_E, and therefore higher after a pool.

2 Pools of Nonessential Complements (Theorem 2)

2.1 Marginal Effect of n_1 on Equilibrium Prices

Again, we begin by calculating the marginal effect, this time of n_1, on equilibrium prices through the simultaneous equations characterizing them. Differentiating, we find

$$\begin{aligned}
P_1^N(1-Q_1) &= n_1 \\
(1-Q_1)\frac{\partial P_1^N}{\partial n_1} - P_1^N \left(\sum_{k' \in \mathcal{K}} \frac{\partial Q_1}{\partial P_{k'}^N} \frac{\partial P_{k'}^N}{\partial n_1} + \frac{\partial Q_1}{\partial P^E} \frac{\partial P^E}{\partial n_1} \right) &= 1 \\
(1-Q_1)\frac{\partial P_1^N}{\partial n_1} - P_1^N \left(-Q_1(1-Q_1) \frac{\partial P_1^N}{\partial n_1} + \sum_{k' \neq 1} Q_1 Q_{k'} \frac{\partial P_{k'}^N}{\partial n_1} - Q_1 Q_0 \frac{\partial P^E}{\partial n_1} \right) &= 1 \\
(1 - Q_1 + P_1^N Q_1)\frac{\partial P_1^N}{\partial n_1} - P_1^N Q_1 \left(\sum_{k' \in \mathcal{K}} Q_{k'} \frac{\partial P_{k'}^N}{\partial n_1} - Q_0 \frac{\partial P^E}{\partial n_1} \right) &= 1
\end{aligned}$$

$$\begin{aligned}
P_k^N(1-Q_k) &= n_k \\
(1-Q_k)\frac{\partial P_k^N}{\partial n_1} - P_k^N \left(\sum_{k' \in \mathcal{K}} \frac{\partial Q_k}{\partial P_{k'}^N} \frac{\partial P_{k'}^N}{\partial n_1} + \frac{\partial Q_k}{\partial P^E} \frac{\partial P^E}{\partial n_1} \right) &= 0 \\
(1-Q_k)\frac{\partial P_k^N}{\partial n_1} - P_k^N \left(-Q_k(1-Q_k) \frac{\partial P_k^N}{\partial n_1} + \sum_{k' \neq k} Q_k Q_{k'} \frac{\partial P_{k'}^N}{\partial n_1} - Q_k Q_0 \frac{\partial P^E}{\partial n_1} \right) &= 0 \\
(1 - Q_k + P_k^N Q_k)\frac{\partial P_k^N}{\partial n_1} - P_k^N Q_k \left(\sum_{k' \in \mathcal{K}} Q_{k'} \frac{\partial P_{k'}^N}{\partial n_1} - Q_0 \frac{\partial P^E}{\partial n_1} \right) &= 0
\end{aligned}$$

$$\begin{aligned}
P^E Q_0 &= n_E \\
Q_0 \frac{\partial P^E}{\partial n_1} + P^E \left(\sum_{k' \in \mathcal{K}} \frac{\partial Q_0}{\partial P_{k'}^N} \frac{\partial P_{k'}^N}{\partial n_1} + \frac{\partial Q_0}{\partial P^E} \frac{\partial P^E}{\partial n_1} \right) &= 0 \\
Q_0 \frac{\partial P^E}{\partial n_1} + P^E \left(\sum_{k' \in \mathcal{K}} Q_0 Q_{k'} \frac{\partial P_{k'}^N}{\partial n_1} + Q_0(1-Q_0) \frac{\partial P^E}{\partial n_1} \right) &= 0 \\
(Q_0 + P^E Q_0) \frac{\partial P^E}{\partial n_1} + P^E Q_0 \left(\sum_{k' \in \mathcal{K}} Q_{k'} \frac{\partial P_{k'}^N}{\partial n_1} - Q_0 \frac{\partial P^E}{\partial n_1} \right) &= 0
\end{aligned}$$

Let $\Lambda = \sum_{k' \in \mathcal{K}} Q_{k'} \frac{\partial P_{k'}^N}{\partial n_1} - Q_0 \frac{\partial P^E}{\partial n_1}$, which (as before) we know from supermodularity is positive; then

$$\begin{aligned}
\frac{\partial P_1^N}{\partial n_1} &= \frac{1}{1-Q_1+P_1^N Q_1} + \frac{P_1^N Q_1}{1-Q_1+P_1^N Q_1} \Lambda \\
\frac{\partial P_k^N}{\partial n_1} &= \frac{P_k^N Q_k}{1-Q_k+P_k^N Q_k} \Lambda \\
\frac{\partial P^E}{\partial n_1} &= -\frac{P^E}{1+P^E} \Lambda
\end{aligned}$$

and

$$\begin{aligned}
\Lambda &= \sum_{k' \in \mathcal{K}} Q_{k'} \frac{\partial P_{k'}^N}{\partial n_1} - Q_0 \frac{\partial P^E}{\partial n_1} \\
\Lambda &= \frac{Q_1}{1-Q_1+P_1^N Q_1} + Q_1 \frac{P_1^N Q_1}{1-Q_1+P_1^N Q_1} \Lambda + \sum_{k' \neq 1} Q_{k'} \frac{P_{k'}^N Q_{k'}}{1-Q_{k'}+P_{k'}^N Q_{k'}} \Lambda + Q_0 \frac{P^E}{1+P^E} \Lambda \\
\frac{Q_1}{1-Q_1+P_1^N Q_1} &= \Lambda \left(1 - \sum_{k' \in \mathcal{K}} Q_{k'} \frac{P_{k'}^N Q_{k'}}{1-Q_{k'}+P_{k'}^N Q_{k'}} - Q_0 \frac{P^E}{1+P^E} \right) \\
\frac{Q_1}{1-Q_1+P_1^N Q_1} &= \Lambda \left(\sum_{k' \in \mathcal{K}} Q_{k'} + Q_0 - \sum_{k' \in \mathcal{K}} Q_{k'} \frac{P_{k'}^N Q_{k'}}{1-Q_{k'}+P_{k'}^N Q_{k'}} - Q_0 \frac{P^E}{1+P^E} \right) \\
\frac{Q_1}{1-Q_1+P_1^N Q_1} &= \Lambda \left(\sum_{k' \in \mathcal{K}} \frac{Q_{k'}(1-Q_{k'})}{1-Q_{k'}+P_{k'}^N Q_{k'}} + \frac{Q_0}{1+P^E} \right)
\end{aligned}$$

so $\Lambda = \frac{Q_1}{1-Q_1+P_1^N Q_1}\delta$, with (as before) $\frac{1}{\delta} = \sum_{k'\in\mathcal{K}} \frac{Q_{k'}(1-Q_{k'})}{1-Q_{k'}+P_{k'}^N Q_{k'}} + \frac{Q_0}{1+P^E}$. So we can write

$$
\begin{aligned}
\frac{\partial P_k^N}{\partial n_1} &= \frac{P_k^N Q_k}{1-Q_k+P_k^N Q_k}\Lambda \\
\frac{\partial P^E}{\partial n_1} &= -\frac{P^E}{1+P^E}\Lambda \\
\frac{\partial P_1^N}{\partial n_1} &= \frac{1}{1-Q_1+P_1^N Q_1} + \frac{P_1^N Q_1}{1-Q_1+P_1^N Q_1}\Lambda \\
&= \left(\frac{1}{Q_1}\left(\sum_{k'\in\mathcal{K}} \frac{Q_{k'}(1-Q_{k'})}{1-Q_{k'}+P_{k'}^N Q_{k'}} + \frac{Q_0}{1+P^E}\right) + \frac{P_1^N Q_1}{1-Q_1+P_1^N Q_1}\right)\Lambda \\
&= \left(1 + \frac{1}{Q_1}\left(\sum_{k'\neq 1} \frac{Q_{k'}(1-Q_{k'})}{1-Q_{k'}+P_{k'}^N Q_{k'}} + \frac{Q_0}{1+P^E}\right)\right)\Lambda \\
\Lambda &= \frac{Q_1}{1-Q_1+P_1^N Q_1}\delta = \frac{Q_1}{1-Q_1+P_1^N Q_1} \Bigg/ \left(\sum_{k'\in\mathcal{K}} \frac{Q_{k'}(1-Q_{k'})}{1-Q_{k'}+P_{k'}^N Q_{k'}} + \frac{Q_0}{1+P^E}\right)
\end{aligned}
$$

Note also that since $\frac{\partial P_1^N}{\partial n_1} > \Lambda$, $P_1 = P_1^N + P^E$ is increasing in n_1. For $k \neq 1$,

$$
\frac{\partial(\log P_k^N + \log(1-Q_k))}{\partial n_1} = \frac{\partial \log n_k}{\partial n_1} = 0 \quad \rightarrow \quad \frac{\partial Q_k}{\partial n_1} = \frac{1-Q_k}{P_k^N}\frac{\partial P_k^N}{\partial n_1} = \frac{Q_k(1-Q_k)}{1-Q_k+P_k^N Q_k}\Lambda
$$

As for Q_1,

$$
\begin{aligned}
\frac{\partial Q_1}{\partial n_1} &= \frac{\partial Q_1}{\partial P_1^N}\frac{\partial P_1^N}{\partial n_1} + \sum_{k'\neq 1} \frac{\partial Q_1}{\partial P_{k'}^N}\frac{\partial P_{k'}^N}{\partial n_1} + \frac{\partial Q_1}{\partial P^E}\frac{\partial P^E}{\partial n_1} \\
&= -Q_1(1-Q_1)\frac{\partial P_1^N}{\partial n_1} + \sum_{k'\neq 1} Q_1 Q_{k'}\frac{\partial P_{k'}^N}{\partial n_1} - Q_1 Q_0 \frac{\partial P^E}{\partial n_1} \\
&= -Q_1 \frac{\partial P_1^N}{\partial n_1} + \sum_{k'\in\mathcal{K}} Q_1 Q_{k'}\frac{\partial P_{k'}^N}{\partial n_1} - Q_1 Q_0 \frac{\partial P^E}{\partial n_1} \\
&= -Q_1 \frac{\partial P_1^N}{\partial n_1} + Q_1\Lambda \\
&= -Q_1\Lambda - \left(\sum_{k'\neq 1} \frac{Q_{k'}(1-Q_{k'})}{1-Q_{k'}+P_{k'}^N Q_{k'}} + \frac{Q_0}{1+P^E}\right)\Lambda + Q_1\Lambda \\
&= -\left(\sum_{k'\neq 1} \frac{Q_{k'}(1-Q_{k'})}{1-Q_{k'}+P_{k'}^N Q_{k'}} + \frac{Q_0}{1+P^E}\right)\Lambda
\end{aligned}
$$

2.2 Consumer Surplus

Since $\frac{\partial CS}{\partial P_k} = -Q_k$,

$$
\begin{aligned}
\frac{\partial CS}{\partial n_1} &= -Q_1 \frac{\partial P_1^N}{\partial n_1} - \sum_{k'\neq 1} Q_{k'}\frac{\partial P_{k'}^N}{\partial n_1} - \sum_{k'\in\mathcal{K}} Q_{k'}\frac{\partial P^E}{\partial n_1} \\
&= \Lambda\left(-Q_1 - \sum_{k'\neq 1}\frac{Q_{k'}(1-Q_{k'})}{1-Q_{k'}+P_{k'}^N Q_{k'}} - \frac{Q_0}{1+P^E} - \sum_{k'\neq 1} Q_{k'}\frac{P_{k'}^N Q_{k'}}{1-Q_{k'}+P_{k'}^N Q_{k'}} + \sum_{k'\in\mathcal{K}} Q_{k'}\frac{P^E}{1+P^E}\right) \\
&= \Lambda\left(-Q_1 - \sum_{k'\neq 1} Q_{k'} - \frac{Q_0}{1+P^E} + \sum_{k'\in\mathcal{K}} Q_{k'}\frac{P^E}{1+P^E}\right) \\
&= \Lambda\left(-\frac{Q_0}{1+P^E} - \sum_{k'\in\mathcal{K}} Q_{k'}\frac{1}{1+P^E}\right) \\
&= -\frac{1}{1+P^E}\Lambda
\end{aligned}
$$

2.3 Welfare

Since $\frac{\partial Welfare}{\partial P_k} = Q_k(\overline{P} - P_k)$ and $\frac{\partial Welfare}{\partial P^E} = -Q_0\overline{P}$,

$$\begin{aligned}
\frac{\partial Welfare}{\partial n_1} &= Q_1(\overline{P} - P_1)\frac{\partial P_1^N}{\partial n_1} + \sum_{k'\neq 1} Q_{k'}(\overline{P} - P_{k'})\frac{\partial P_{k'}^N}{\partial n_1} - Q_0\overline{P}\frac{\partial P^E}{\partial n_1} \\
&= \Bigg(Q_1(\overline{P} - P_1)\left(1 + \frac{1}{Q_1}\left(\sum_{k'\neq 1}\frac{Q_{k'}(1-Q_{k'})}{1-Q_{k'}+P_{k'}^N Q_{k'}} + \frac{Q_0}{1+P^E}\right)\right) \\
&\quad + \sum_{k'\neq 1} Q_{k'}(\overline{P} - P_{k'})\frac{P_{k'}^N Q_{k'}}{1-Q_{k'}+P_{k'}^N Q_{k'}} + Q_0\overline{P}\frac{P^E}{1+P^E}\Bigg)\Lambda \\
&= \Bigg(Q_1\overline{P} - Q_1P_1 + \overline{P}\sum_{k'\neq 1}\frac{Q_{k'}(1-Q_{k'})}{1-Q_{k'}+P_{k'}^N Q_{k'}} + \overline{P}\frac{Q_0}{1+P^E} - P_1\left(\sum_{k'\neq 1}\frac{Q_{k'}(1-Q_{k'})}{1-Q_{k'}+P_{k'}^N Q_{k'}} + \frac{Q_0}{1+P^E}\right) \\
&\quad + \overline{P}\sum_{k'\neq 1} Q_{k'}\frac{P_{k'}^N Q_{k'}}{1-Q_{k'}+P_{k'}^N Q_{k'}} - \sum_{k'\neq 1} Q_{k'}P_{k'}\frac{P_{k'}^N Q_{k'}}{1-Q_{k'}+P_{k'}^N Q_{k'}} + Q_0\overline{P}\frac{P^E}{1+P^E}\Bigg)\Lambda \\
&= \Bigg(Q_1\overline{P} - Q_1P_1 + \overline{P}\sum_{k'\neq 1} Q_{k'} + \overline{P}Q_0 - P_1\left(\sum_{k'\neq 1}\frac{Q_{k'}(1-Q_{k'})}{1-Q_{k'}+P_{k'}^N Q_{k'}} + \frac{Q_0}{1+P^E}\right) \\
&\quad - \sum_{k'\neq 1} Q_{k'}P_{k'}\frac{P_{k'}^N Q_{k'}}{1-Q_{k'}+P_{k'}^N Q_{k'}}\Bigg)\Lambda \\
&= \left(\overline{P} - Q_1P_1 - P_1\left(\sum_{k'\neq 1}\frac{Q_{k'}(1-Q_{k'})}{1-Q_{k'}+P_{k'}^N Q_{k'}} + \frac{Q_0}{1+P^E}\right) - \sum_{k'\neq 1} Q_{k'}P_{k'}\frac{P_{k'}^N Q_{k'}}{1-Q_{k'}+P_{k'}^N Q_{k'}}\right)\Lambda \\
&= \left(\sum_{k'\neq 1} Q_{k'}P_{k'} - P_1\left(\sum_{k'\neq 1}\frac{Q_{k'}(1-Q_{k'})}{1-Q_{k'}+P_{k'}^N Q_{k'}} + \frac{Q_0}{1+P^E}\right) - \sum_{k'\neq 1} Q_{k'}P_{k'}\frac{P_{k'}^N Q_{k'}}{1-Q_{k'}+P_{k'}^N Q_{k'}}\right)\Lambda \\
&= \left(\sum_{k'\neq 1} P_{k'}\frac{Q_{k'}(1-Q_{k'})}{1-Q_{k'}+P_{k'}^N Q_{k'}} - \sum_{k'\neq 1} P_1\frac{Q_{k'}(1-Q_{k'})}{1-Q_{k'}+P_{k'}^N Q_{k'}} - P_1\frac{Q_0}{1+P^E}\right)\Lambda
\end{aligned}$$

2.4 Theorem 2

A pool of patentholders in $\mathcal{J}_1^N$ has the same effect on equilibrium prices as a decrease in n_1. We showed above that P_k^N is increasing in n_1 for every k (including 1); P^E is decreasing in n_1; Q_1 is decreasing in n_1; Q_k ($k \neq 1$) is increasing in n_1; and P_1 is increasing in n_1. Examples in the text show that P_k can increase or decrease in n_1. We showed above that total consumer surplus is decreasing in n_1; but a pool which increases P_k ($k \neq 1$) will harm some individual consumers (at a minimum, those who still purchase product k after the pool).

For $k \neq 1$, P_k^N and Q_k are both increasing in n_1, so $\frac{1}{n_k}P_k^N Q_k$ is increasing in n_1, so profits of patentholders in $\mathcal{J}_k^N$ are lower following the pool. Since $P^E Q_0 = n_E$ and P^E is decreasing in n_1, Q_0 must be increasing in n_1, so $\frac{1}{n_E}P^E(1-Q_0)$ is decreasing in n_1, so profits of patentholders in $\mathcal{J}^E$ are higher after the pool. As for outsiders within $\mathcal{J}_1^N$, they earn $\frac{1}{n_1}P_1^N Q_1$; since $P_1^N(1-Q_1) = n_1$, we can rewrite this as $\frac{Q_1}{1-Q_1}$; since Q_1 is higher after the pool, these outsiders earn more. Examples in the text show both welfare-positive and welfare-negative pools.

3 Welfare and Profitability When $K = 2$ (Theorem 3)

Using the above, we can calculate (for general K)

$$\begin{aligned}
\frac{\partial}{\partial n_1}\left(Welfare - P_1^N Q_1\right) &= \Big(\textstyle\sum_{k'\neq 1} P_{k'} \frac{Q_{k'}(1-Q_{k'})}{1-Q_{k'}+P_{k'}^N Q_{k'}} - \sum_{k'\neq 1} P_1 \frac{Q_{k'}(1-Q_{k'})}{1-Q_{k'}+P_{k'}^N Q_{k'}} - P_1 \frac{Q_0}{1+P^E} \\
&\quad -P_1^N \Big(-\textstyle\sum_{k'\neq 1} \frac{Q_{k'}(1-Q_{k'})}{1-Q_{k'}+P_{k'}^N Q_{k'}} - \frac{Q_0}{1+P^E}\Big) \\
&\quad -Q_1 \Big(1 + \frac{1}{Q_1}\Big(\textstyle\sum_{k'\neq 1} \frac{Q_{k'}(1-Q_{k'})}{1-Q_{k'}+P_{k'}^N Q_{k'}} + \frac{Q_0}{1+P^E}\Big)\Big)\Big)\Lambda \\
&= \Big(\textstyle\sum_{k'\neq 1}(P_{k'}^N - P_1^N)\frac{Q_{k'}(1-Q_{k'})}{1-Q_{k'}+P_{k'}^N Q_{k'}} - (P_1^N + P^E)\frac{Q_0}{1+P^E} \\
&\quad +(P_1^N - 1)\textstyle\sum_{k'\neq 1} \frac{Q_{k'}(1-Q_{k'})}{1-Q_{k'}+P_{k'}^N Q_{k'}} + (P_1^N - 1)\frac{Q_0}{1+P^E} - Q_1\Big)\Lambda \\
&= \Big(\textstyle\sum_{k'\neq 1}(P_{k'}^N - 1)\frac{Q_{k'}(1-Q_{k'})}{1-Q_{k'}+P_{k'}^N Q_{k'}} - (1+P^E)\frac{Q_0}{1+P^E} - Q_1\Big)\Lambda \\
&= \Big(\textstyle\sum_{k'\neq 1} \frac{(P_{k'}^N-1)Q_{k'}(1-Q_{k'})}{1-Q_{k'}+P_{k'}^N Q_{k'}} - Q_0 - Q_1\Big)\Lambda \\
&= \Big(\textstyle\sum_{k'\neq 1}\Big(\frac{P_{k'}^N Q_k}{1-Q_{k'}+P_{k'}^N Q_{k'}} - Q_{k'}\Big) - Q_0 - Q_1\Big)\Lambda \\
&= \Big(\textstyle\sum_{k'\neq 1} \frac{P_{k'}^N Q_k}{1-Q_{k'}+P_{k'}^N Q_{k'}} - 1\Big)\Lambda
\end{aligned}$$

When $K = 2$, the summation has only one term, which is less than 1, so the expression is negative; so a decrease in n_1 (a pool of patentholders in $\mathcal{J}_1^N$) has a net positive externality on everyone outside of $\mathcal{J}_1^N$. Since we already showed that it has a positive externality on outsiders within $\mathcal{J}_1^N$, if it increases the profits of its members, the overall welfare impact must be positive.

4 Profitability of Small Pools of Essential Patents (Claim 2)

Claim. *Fix $m \geq 1$. If n_E is sufficiently large, the formation of a pool of $m+1$ essential patents increases the profits of its participants.*

I will treat n_E as the name of the variable, $\tilde{n}_E$ as its starting value, and $\tilde{n}_E - m$ as its ending value (after pool formation). Let $\Pi_E(\cdot)$ be the equilibrium value of $P^E \sum_{k'\in\mathcal{K}} Q_{k'}$, as a function of n_E. Combined profits of the patentholders forming the pool are $\frac{m+1}{\tilde{n}_E}\Pi^E(\tilde{n}_E)$ before pool formation, and $\frac{1}{\tilde{n}_E - m}\Pi^E(\tilde{n}_E - m)$ after. We can therefore write the gain in profits as

$$d\pi = (m+1)^r \frac{\Pi^E(\tilde{n}_E - (1-r)m)}{\tilde{n}_E - (1-r)m}\bigg|_{r=1}^{r=0} = -\int_0^1 \frac{d}{dr}\left((m+1)^r \frac{\Pi^E(\tilde{n}_E - (1-r)m)}{\tilde{n}_E - (1-r)m}\right) dr$$

Thus, it will suffice to show that for n_E sufficiently large, $(m+1)^r \frac{\Pi^E(\tilde{n}_E-(1-r)m)}{\tilde{n}_E-(1-r)m}$ is decreasing in r for all $r \in (0,1)$. Letting $n_E = \tilde{n}_E - (1-r)m$, it will suffice to show that

$$\begin{aligned}
0 &> \frac{\partial}{\partial r}\left((m+1)^r \frac{\Pi^E(\tilde{n}_E-(1-r)m)}{\tilde{n}_E-(1-r)m}\right) \\
&= \frac{d}{dr}\left(e^{r\log(m+1)} \frac{\Pi^E(\tilde{n}_E-m+rm)}{\tilde{n}_E-m+rm}\right) \\
&= \log(m+1)e^{r\log(m+1)} \frac{\Pi^E(n_E)}{n_E} + e^{r\log(m+1)} m \frac{\partial}{\partial n_E}\frac{\Pi^E(n_E)}{n_E} \\
&\propto \frac{\log(m+1)}{m}\frac{\Pi^E(n_E)}{n_E} + \frac{\partial}{\partial n_E}\frac{\Pi^E(n_E)}{n_E}
\end{aligned}$$

Recall from before that since $P^E Q_0 = n_E$, $\frac{\Pi^E}{n_E} = \frac{P^E(1-Q_0)}{n_E} = \frac{1-Q_0}{Q_0} = \frac{1}{Q_0} - 1$. We calculated earlier that

$$\frac{\partial Q_k}{\partial n_E} = -\frac{1}{1+P^E}\frac{(1-Q_k)Q_k}{1-Q_k+P_k^N Q_k} \Bigg/ \left(\sum_{k'\in\mathcal{K}} \frac{Q_{k'}(1-Q_{k'})}{1-Q_{k'}+P_{k'}^N Q_{k'}} + \frac{Q_0}{1+P^E}\right)$$

so since $Q_0 = 1 - \sum_{k' \in \mathcal{K}} Q_{k'}$,

$$\frac{\partial Q_0}{\partial n_E} = \frac{1}{1+P^E} \sum_{k' \in \mathcal{K}} \frac{(1-Q_{k'})Q_{k'}}{1-Q_{k'}+P^N_{k'}Q_{k'}} \Bigg/ \left(\sum_{k' \in \mathcal{K}} \frac{Q_{k'}(1-Q_{k'})}{1-Q_{k'}+P^N_{k'}Q_{k'}} + \frac{Q_0}{1+P^E} \right)$$

and

$$\frac{\partial}{\partial n_E}\left(\frac{\Pi^E}{n_E}\right) = \frac{\partial}{\partial n_E}\left(\frac{1}{Q_0} - 1\right) = -\frac{1}{Q_0^2}\frac{\partial Q_0}{\partial n_E}$$

so we want to show that

$$\begin{aligned}
0 &> \tfrac{\log(m+1)}{m}\tfrac{\Pi^E(n_E)}{n_E} + \tfrac{\partial}{\partial n_E}\tfrac{\Pi^E(n_E)}{n_E} \\
&= \tfrac{\log(m+1)}{m}\tfrac{1-Q_0}{Q_0} - \tfrac{1}{Q_0^2}\tfrac{1}{1+P^E}\textstyle\sum_{k'\in\mathcal{K}} \tfrac{(1-Q_{k'})Q_{k'}}{1-Q_{k'}+P^N_{k'}Q_{k'}} \Big/ \left(\textstyle\sum_{k'\in\mathcal{K}} \tfrac{Q_{k'}(1-Q_{k'})}{1-Q_{k'}+P^N_{k'}Q_{k'}} + \tfrac{Q_0}{1+P^E}\right) \\
&\propto \tfrac{\log(m+1)}{m}Q_0(1-Q_0)(1+P^E) - \textstyle\sum_{k'\in\mathcal{K}} \tfrac{(1-Q_{k'})Q_{k'}}{1-Q_{k'}+P^N_{k'}Q_{k'}} \Big/ \left(\textstyle\sum_{k'\in\mathcal{K}} \tfrac{Q_{k'}(1-Q_{k'})}{1-Q_{k'}+P^N_{k'}Q_{k'}} + \tfrac{Q_0}{1+P^E}\right) \\
&\propto \tfrac{\log(m+1)}{m}Q_0(1-Q_0)(1+P^E)\tfrac{Q_0}{1+P^E} + \left(\tfrac{\log(m+1)}{m}Q_0(1-Q_0)(1+P^E) - 1\right)\textstyle\sum_{k'\in\mathcal{K}} \tfrac{Q_{k'}(1-Q_{k'})}{1-Q_{k'}+P^N_{k'}Q_{k'}} \\
&= \tfrac{\log(m+1)}{m}Q_0^2(1-Q_0) + \left(\tfrac{\log(m+1)}{m}Q_0(1-Q_0)(1+P^E) - 1\right)\textstyle\sum_{k'\in\mathcal{K}} \tfrac{Q_{k'}(1-Q_{k'})}{1-Q_{k'}+P^N_{k'}Q_{k'}} \\
&\propto \tfrac{\log(m+1)}{m}Q_0^2 \frac{\sum_{k'\in\mathcal{K}} Q_{k'}}{\sum_{k'\in\mathcal{K}} \frac{Q_{k'}(1-Q_{k'})}{1-Q_{k'}+P^N_{k'}Q_{k'}}} + \tfrac{\log(m+1)}{m}Q_0(1-Q_0+P^E-P^EQ_0) - 1 \\
&= \tfrac{\log(m+1)}{m}Q_0^2 \left(\frac{\sum_{k'\in\mathcal{K}} Q_{k'}}{\sum_{k'\in\mathcal{K}} Q_{k'}\frac{1-Q_{k'}}{1-Q_{k'}+P^N_{k'}Q_{k'}}} - 1\right) + \tfrac{\log(m+1)}{m}Q_0(1+P^E(1-Q_0)) - 1 \\
&= \tfrac{\log(m+1)}{m}Q_0^2 \left[\frac{\sum_{k'\in\mathcal{K}} Q_{k'}\frac{P^N_{k'}Q_{k'}}{1-Q_{k'}+P^N_{k'}Q_{k'}}}{\sum_{k'\in\mathcal{K}} Q_{k'}\frac{1-Q_{k'}}{1-Q_{k'}+P^N_{k'}Q_{k'}}}\right] + \tfrac{\log(m+1)}{m}Q_0P^E(1-Q_0) - \left(1 - \tfrac{\log(m+1)}{m}Q_0\right)
\end{aligned}$$

We can write this last expression as

$$\frac{\log(m+1)}{m}Q_0^2\left[\frac{wtd\ avg\ P^N_{k'}Q_{k'}}{wtd\ avg\ 1-Q_{k'}}\right] + \frac{\log(m+1)}{m}Q_0P^E(1-Q_0) - \left(1 - \frac{\log(m+1)}{m}Q_0\right)$$

where the weighted averages are taken with respect to weights $\frac{Q_{k'}}{1-Q_{k'}+P^N_{k'}Q_{k'}}$, normalized to sum to 1.

Now, $Q_k = \frac{\exp(v_k-P_k)}{1+\sum_{k'\in\mathcal{K}}\exp(v_{k'}-P_{k'})} < e^{v_k}e^{-P^E}$ and $P^E > n_E$, so as n_E grows without bound, $P^N_kQ_k$ and $P^E(1-Q_0)$ go to 0 and $1-Q_k$ does not, so the first two terms vanish. For $m \geq 1$, $\frac{\log(m+1)}{m} < 0.7$, so the term in parentheses is at least 0.3. Pick $\tilde{n}_E$ sufficiently large so that the whole expression is negative for $n_E \geq \tilde{n}_E - m$ and the integrand is negative on $(0,1)$, making the pool profitable.

(Conversely, since the lead term is positive, the integrand will be positive whenever $\frac{\log(m+1)}{m}(Q_0+P^E(1-Q_0)Q_0) > 1$. For $m = 1$, this will hold for all $r \in (0,1)$ if prior to pool formation, $(\tilde{n}_E - 2)(1-Q_0) > \frac{1}{\log 2} - 1 \approx 0.442$.)

5 Large Pools of Nonessential Complements (Claim 3)

The claim is the same as the one proven in the text for essential patents: fixing m, if n_k is sufficiently large, a pool of all but m of the patentholders in $\mathcal{J}^N_k$ increases the profits of its participants.

This time, let π denote equilibrium profits to each player in $\mathcal{J}^N_k$ after the pool formed (that is, with $n_k = m+1$). Since there are $n_k - m$ pool members, each earns $\frac{1}{n_k-m}\pi$ given the pool. Without the pool, each earns $\frac{1}{n_k}P^N_kQ_k$. Pick n_k sufficiently large that $\pi e^{-v_k} > n_ke^{-n_k}$. Since $P^N_k > n_k$ and xe^x is decreasing above 1, $\pi e^{-v_k} > P^N_ke^{-P^N_k}$, so $\pi > P^N_ke^{v_k-P^N_k} > P^N_kQ_k$, so

$$\frac{1}{n_k-m}\pi > \frac{1}{n_k}\pi > \frac{1}{n_k}P^N_kQ_k$$

and the pool is profitable.

6 Small Pools of Nonessential Complements (Claim 4)

Mimicking the proof above of Claim 2, note that when $m+1$ nonessential patents blocking technology 1 form a pool, the change in their joint profits is

$$d\pi = (m+1)^r \frac{\Pi^1(\tilde{n}_1 - (1-r)m)}{\tilde{n}_1 - (1-r)m}\Bigg|_{r=1}^{r=0} = -\int_0^1 \frac{d}{dr}\left((m+1)^r \frac{\Pi^1(\tilde{n}_1-(1-r)m)}{\tilde{n}_1-(1-r)m}\right) dr$$

where Π^1 is the equilibrium value of $P_1^N Q_1$ as a function of n_1. Differentiating and letting $n_1 = \tilde{n}_1 - (1-r)m$, the integrand is

$$\log(m+1)e^{r\log(m+1)}\frac{P_1^N Q_1}{n_1} + e^{r\log(m+1)} m \frac{\partial}{\partial n_1}\left(\frac{P_1^N Q_1}{n_1}\right)$$

Again, $P_1^N(1-Q_1) = n_1$, so $\frac{1}{n_1}P_1^N Q_1 = \frac{Q_1}{1-Q_1} = \frac{1}{1-Q_1} - 1$; with $\frac{\partial Q_1}{\partial n_1}$ calculated earlier, the integrand is proportional to

$$\frac{\log(m+1)}{m}\frac{Q_1}{1-Q_1} - \frac{1}{(1-Q_1)^2}\frac{\sum_{k\in\mathcal{K}-\{1\}}\frac{Q_k(1-Q_k)}{1-Q_k+P_k^N Q_k} + \frac{Q_0}{1+P^E}}{\sum_{k\in\mathcal{K}}\frac{Q_k(1-Q_k)}{1-Q_k+P_k^N Q_k} + \frac{Q_0}{1+P^E}}\frac{Q_1}{1-Q_1+P_1^N Q_1}$$

$$\propto \frac{\log(m+1)}{m}(1-Q_1) - \frac{1}{1-Q_1+P_1^N Q_1}\left(1 - \frac{\frac{Q_1(1-Q_1)}{1-Q_1+P_1^N Q_1}}{\sum_{k\in\mathcal{K}}\frac{Q_k(1-Q_k)}{1-Q_k+P_k^N Q_k} + \frac{Q_0}{1+P^E}}\right)$$

As n_1 increases to infinity, P_1^N grows unboundedly and Q_1 and $P_1^N Q_1$ shrink to 0. The term $\frac{Q_1(1-Q_1)}{1-Q_1+P_1^N Q_1} < Q_1$ vanishes; since Q_0 increases in n_1 and P^E decreases, $\frac{Q_0}{1+P^E}$ does not vanish; so the messy fraction goes to 0 and the term within parentheses goes to 1. $\frac{1}{1-Q_1+P_1^N Q_1}$ goes to 1 as well, so the entire second expression goes to 1, while the first expression is bounded above by $\frac{\log(m+1)}{m} < 0.7$. So when n_1 is sufficiently large, the integrand is negative and the pool is profitable.

7 Pool of One Patent Per Technology (Theorem 4 part 2)

Such a pool has the same effect as an increase (by one) of n_E and decreases (by one) of each n_k. We calculated earlier that

$$\frac{\partial Welfare}{\partial n_E} = \frac{\delta}{1+P^E}\sum_{k'\in\mathcal{K}} Q_{k'}P_{k'}\left(\frac{P_{k'}^N Q_{k'}}{1-Q_{k'}+P_{k'}^N Q_{k'}} - 1\right) = -\frac{\delta}{1+P^E}\sum_{k'\in\mathcal{K}} Q_{k'}(P_{k'}^N + P^E)\left(\frac{1-Q_{k'}}{1-Q_{k'}+P_{k'}^N Q_{k'}}\right)$$

$$\frac{\partial Welfare}{\partial n_k} = \left(\sum_{k'\in\mathcal{K}}(P_{k'}^N - P_k^N)\frac{Q_{k'}(1-Q_{k'})}{1-Q_{k'}+P_{k'}^N Q_{k'}} - (P_k^N + P^E)\frac{Q_0}{1+P^E}\right)\frac{Q_k}{1-Q_k+P_k^N Q_k}\delta$$

First,

$$\begin{aligned}
\frac{1}{\delta}\sum_{k\in\mathcal{K}}\frac{\partial W}{\partial n_k} &= \sum_{k\in\mathcal{K}}\left(\sum_{k'\in\mathcal{K}}(P_{k'}^N - P_k^N)\frac{Q_{k'}(1-Q_{k'})}{1-Q_{k'}+P_{k'}^N Q_{k'}} - (P_k^N + P^E)\frac{Q_0}{1+P^E}\right)\frac{Q_k}{1-Q_k+P_k^N Q_k} \\
&= \sum_{k\in\mathcal{K}}\sum_{k'\in\mathcal{K}}(1-Q_{k'})(P_{k'}^N - P_k^N)\frac{Q_{k'}}{1-Q_{k'}+P_{k'}^N Q_{k'}}\frac{Q_k}{1-Q_k+P_k^N Q_k} - \sum_{k\in\mathcal{K}}(P_k^N + P^E)\frac{Q_0}{1+P^E}\frac{Q_k}{1-Q_k+P_k^N Q_k} \\
&= \sum\sum_{k'<k}(Q_k - Q_{k'})(P_{k'}^N - P_k^N)\frac{Q_{k'}}{1-Q_{k'}+P_{k'}^N Q_{k'}}\frac{Q_k}{1-Q_k+P_k^N Q_k} - \sum_{k\in\mathcal{K}}(P_k^N + P^E)\frac{Q_0}{1+P^E}\frac{Q_k}{1-Q_k+P_k^N Q_k}
\end{aligned}$$

and so

$$\begin{aligned}
\tfrac{1}{\delta}\left(\tfrac{\partial W}{\partial n_E} - \sum \tfrac{\partial W}{\partial n_k}\right) &= -\tfrac{1}{1+P^E}\sum_{k\in\mathcal{K}}(1-Q_k)(P_k^N+P^E)\tfrac{Q_k}{1-Q_k+P_k^N Q_k} + \tfrac{1}{1+P^E}\sum_{k\in\mathcal{K}} Q_0(P_k^N+P^E)\tfrac{Q_k}{1-Q_k+P_k^N Q_k} \\
&\quad -\sum\sum_{k'<k}(Q_k-Q_{k'})(P_{k'}^N-P_k^N)\tfrac{Q_{k'}}{1-Q_{k'}+P_{k'}^N Q_{k'}}\tfrac{Q_k}{1-Q_k+P_k^N Q_k} \\
&= -\tfrac{1}{1+P^E}\sum_{k\in\mathcal{K}}(1-Q_k-Q_0)(P_k^N+P^E)\tfrac{Q_k}{1-Q_k+P_k^N Q_k} \\
&\quad -\sum\sum_{k'<k}(Q_k-Q_{k'})(P_{k'}^N-P_k^N)\tfrac{Q_{k'}}{1-Q_{k'}+P_{k'}^N Q_{k'}}\tfrac{Q_k}{1-Q_k+P_k^N Q_k} \\
&= -\tfrac{1}{1+P^E}\sum_{k\in\mathcal{K}}\sum_{k'\neq k} Q_{k'}(P_k^N+P^E)\tfrac{Q_k}{1-Q_k+P_k^N Q_k} \\
&\quad -\sum\sum_{k'<k}(Q_k-Q_{k'})(P_{k'}^N-P_k^N)\tfrac{Q_{k'}}{1-Q_{k'}+P_{k'}^N Q_{k'}}\tfrac{Q_k}{1-Q_k+P_k^N Q_k}
\end{aligned}$$

Each unordered pair $\{k,k'\}$ contributes two terms to the first double-summation and one term to the second; these three terms are

$$\begin{aligned}
&-\tfrac{1}{1+P^E}Q_{k'}(P_k^N+P^E)\tfrac{Q_k}{1-Q_k+P_k^N Q_k} - \tfrac{1}{1+P^E}Q_k(P_{k'}^N+P^E)\tfrac{Q_{k'}}{1-Q_{k'}+P_{k'}^N Q_{k'}} \\
&-(Q_k-Q_{k'})(P_{k'}^N-P_k^N)\tfrac{Q_{k'}}{1-Q_{k'}+P_{k'}^N Q_{k'}}\tfrac{Q_k}{1-Q_k+P_k^N Q_k} \\
\propto\ & -(P_k^N+P^E)((1-Q_{k'})+P_{k'}^N Q_{k'}) - (P_{k'}^N+P^E)((1-Q_k)+P_k^N Q_k) \\
&-(Q_k-Q_{k'})(P_{k'}^N-P_k^N)(1+P^E) \\
=\ & -P_k^N(1-Q_{k'}) - P^E(1-Q_{k'}) - P_k^N P_{k'}^N Q_{k'} - P^E P_{k'}^N Q_{k'} \\
&-P_{k'}^N(1-Q_k) - P^E(1-Q_k) - P_{k'}^N P_k^N Q_k - P^E P_k^N Q_k \\
&+(1-Q_k)P_{k'}^N - (1-Q_{k'})P_{k'}^N - (1-Q_k)P_k^N + (1-Q_{k'})P_k^N \\
&-Q_k P_{k'}^N P^E + Q_{k'}P_{k'}^N P^E + Q_k P_k^N P^E - Q_{k'}P_k^N P^E \\
=\ & -P^E(1-Q_{k'}) - P_k^N P_{k'}^N Q_{k'} - P^E(1-Q_k) - P_{k'}^N P_k^N Q_k \\
&-(1-Q_{k'})P_{k'}^N - (1-Q_k)P_k^N - Q_k P_{k'}^N P^E - Q_{k'}P_k^N P^E \\
<\ & 0
\end{aligned}$$

so summing up over all $\{k,k'\}$ pairs gives $\frac{\partial W}{\partial n_E} - \sum \frac{\partial W}{\partial n_k} < 0$, making the pool welfare-destroying.

As for consumer surplus,

$$\begin{aligned}
\frac{\partial CS}{\partial n_E} &= -\frac{\delta}{1+P^E}\sum_{k\in\mathcal{K}}\frac{1-Q_k}{Q_0}\frac{Q_k}{1-Q_k+P_k^N Q_k} \\
\frac{\partial CS}{\partial n_k} &= -\frac{\delta}{1+P^E}\frac{Q_k}{1-Q_k+P_k^N Q_k}
\end{aligned}$$

Since $1-Q_k = Q_0 + \sum_{k'\neq k} Q_{k'} > Q_0$, $\frac{\partial CS}{\partial n_E} - \sum_{k'\in\mathcal{K}}\frac{\partial CS}{\partial n_{k'}} < 0$, so such a pool lowers consumer surplus.

8 Adding One Nonessential Patent to an Essential Pool

If a single essential patent (or a pool) joins with a single essential patent blocking technology 1, we can write the change in their joint profits as

$$d\pi = \int_0^1 \frac{d}{dr}\left(\frac{1}{n_E}\Pi^E(\tilde{n}_1 - r) + (1-r)\frac{\Pi^1(\tilde{n}_1 - r)}{\tilde{n}_1 - r}\right)dr$$

where $\tilde{n}_1$ is the starting value of n_1 and Π^E and Π^1 are the equilibrium values of $P^E(1-Q_0)$ and $P_1^N Q_1$, respectively, as a function of n_1. We find conditions for this to be negative. Differentiating, the integrand is

$$\frac{1}{n_E}\frac{P^E}{1+P^E}\frac{Q_1}{1-Q_1+P_1^N Q_1}\delta - \frac{\Pi^1(\tilde{n}_1 - r)}{\tilde{n}_1 - r} + (1-r)\frac{1}{(1-Q_1)^2}\left(\sum_{k\in\mathcal{K}-\{1\}}\frac{Q_k(1-Q_k)}{1-Q_k+P_k^N Q_k} + \frac{Q_0}{1+P^E}\right)\frac{Q_1}{1-Q_1+P_1^N Q_1}\delta$$

Letting $n_1 = \tilde{n}_1 - r$ and multiplying by $\frac{1}{\delta}$ gives

$$\frac{1}{Q_0}\frac{1}{1+P^E}\frac{Q_1}{1-Q_1+P_1^N Q_1} - \frac{P_1^N Q_1}{n_1}\left(\sum_{k\in\mathcal{K}}\frac{Q_k(1-Q_k)}{1-Q_k+P_k^N Q_k} + \frac{Q_0}{1+P^E}\right)$$

$$+(1-r)\frac{1}{(1-Q_1)^2}\left(\sum_{k\in\mathcal{K}-\{1\}}\frac{Q_k(1-Q_k)}{1-Q_k+P_k^N Q_k} + \frac{Q_0}{1+P^E}\right)\frac{Q_1}{1-Q_1+P_1^N Q_1}$$

or, multiplying by $\frac{1-Q_1}{Q_1}$,

$$\frac{1}{Q_0}\frac{1}{1+P^E}\frac{1-Q_1}{1-Q_1+P_1^N Q_1} - \left(\sum_{k\in\mathcal{K}}\frac{Q_k(1-Q_k)}{1-Q_k+P_k^N Q_k} + \frac{Q_0}{1+P^E}\right)$$

$$+(1-r)\frac{1}{1-Q_1}\frac{1}{1-Q_1+P_1^N Q_1}\left(\sum_{k\in\mathcal{K}-\{1\}}\frac{Q_k(1-Q_k)}{1-Q_k+P_k^N Q_k} + \frac{Q_0}{1+P^E}\right)$$

If $Q_1 \geq \frac{1}{Q_0}\frac{1}{1+P^E} = \frac{1}{Q_0+n_E}$ and $1 \geq (1-r)\frac{1}{1-Q_1}\frac{1}{1-Q_1+P_1^N Q_1}$, then the $k=1$ piece of the second term dominates the first term, and the remainder of the second term dominates the third term, making the whole expression negative. The latter requires that

$$1 \geq \frac{1-r}{(1-Q_1)^2 + P_1^N Q_1(1-Q_1)} = \frac{1-r}{1-2Q_1+Q_1^2+n_1 Q_1} = \frac{1-r}{1-2Q_1+Q_1^2+(\tilde{n}_1-r)Q_1}$$

If $\tilde{n}_1 \geq 2$, the condition

$$1 \geq \frac{1-r}{1+(\tilde{n}_1-2)Q_1+Q_1^2-rQ_1}$$

holds for all $r \in [0,1]$. Thus, $\tilde{n}_1 \geq 2$, along with the condition that $Q_1(Q_0+n_E) \geq 1$ at each $n_1 = \tilde{n}_1 - r$, guarantees all negative integrands. Since Q_1 is decreasing in n_1, if $Q_1 n_E \geq 1$ at $r=0$ (before the pool forms/grows), then $Q_1 n_E \geq 1$ at all r, which will suffice.