Nonparametric Methods for Ascending Auctions with Unobserved Heterogeneity

Amit Gandhi and Daniel Quint

November 2, 2009

Preliminary and Incomplete

1 Introduction

For both theoretical and empirical work on auctions, it is standard to assume that bidders have symmetric and independently distributed private values for the good at auction, and that the set of bidders participating in an auction is independent of the characteristics of the auction itself. These are powerful assumptions. Theoretically, they establish the optimality of any of the standard auction formats with a suitably chosen reserve price (Myerson (1981)). Empirically, they ensure that observations of transaction prices in ascending auctions with a fixed number of bidders nonparametrically identify the entire model, allowing explicit calculation of that profit-maximizing reserve price (see Athey and Haile (2002), Paarsch (1992), Paarsch (1997)); if auctions with different numbers of bidders are observed, the model is overidentified, and therefore testable (Athey and Haile (2002)).

However, both the theoretical prescription and the empirical approach lean heavily on the assumptions of symmetric independent private values and a fixed set of bidders. Even ignoring the question of common versus private values, introducing any small degree of correlation in valuations among bidders (Cremer and McLean (1988), McAfee and Reny (1992)), or endogenizing bidders' participation decisions (Samuelson (1985), McAfee and McMillan (1987), Levin and Smith (1994)), significantly alters the design of the optimal mechanism. And once bidder valuations are allowed to be correlated, the model is not identified from bid data in ascending auctions (Athey and Haile (2002)).

If the objects in different auctions are identical, independence seems like a reasonable assumption: differences in valuations are based on idiosyncratic differences in tastes, which need not be correlated. If the objects are heterogeneous, independence is unlikely to hold in its literal form; standard practice is to control for as much auction-to-auction variation as possible, and hope that whatever residual variation is left is similarly idiosyncratic taste shocks and therefore independent across bidders. (**Cite examples.**) This technique has clear limitations. Controlling for multi-dimensional heterogeneity nonparametrically leads to a severe curse of dimensionality (see, e.g., Silverman (1986)). Controlling for heterogeneity parametrically leaves the concern that any deviation of the parametric model from the true data-generating process will leave residual correlation between bidder valuations; and any sources of heterogeneity which are not observed by the econometrician will similarly lead to correlation.

We propose a different approach, requiring the econometrician to observe only the number of bidders n and transaction prices of past auctions. We show that if the variation in n is exogenous and unbounded, a symmetric private values model is "identified enough" to allow calculation of expected revenue, expected bidder surplus, and the result of imposing a reserve price in future auctions. If (unobserved) heterogeneity across auctions is the only source of correlation in bidder values, we introduce a nonparametric test for whether variation in size is exogenous, and argue that the test should have power against the most likely sources of endogeneity. (We do not require the econometrician to observe or model the sources of heterogeneity, but controlling for some of them will strengthen the test.) In a companion paper, we discuss three standard models of participation in auctions, two of them consistent with exogenous n; and show how, given *any* exogenous variation in reserve prices (or an instrument for reserve price), we can distinguish between them.

We introduce the results under the standard assumption that the second-place bidder bids up to his valuation; we later show how to extend our results to an incomplete model of an English (ascending) auction (as in Haile and Tamer (2003)) if we can observe the highest two bids in each auction.

2 Testing – Theory

In this section, we introduce our model – symmetric, conditionally independent private values – and a nonparametric test of our identifying assumption.

2.1 Model

We use standard notation for private value auctions. Each bidder i in an auction with n bidders has a valuation V^i which is his or her private information. Bidder i gets payoff $V^i - P$ from winning the auction, where P is the price paid, and 0 from losing.

Bidders' valuations will be symmetrically distributed and conditionally independent. Each auction is characterized by a set of characteristics $\theta \in \Theta$.[1] Bidder valuations for the object for sale in an auction with characteristics θ are *i.i.d.* draws from a probability distribution $H(\,\cdot\,|\theta)$. Thus, variation in θ induces correlation in bidders' valuations, while any variation not caused by θ is idiosyncratic and independent across bidders.[2]

[1] These can include both characteristics of the object for sale, and of the auction more generally (reliability of a particular seller, shipping costs being charged, etc.)

[2] De Finetti's theorem, as formalized by Hewitt and Savage (1955), says that if bidders

Let $V_{j:k}$ be the j^{th} lowest bidder valuation in auctions with k bidders, and $F_{j:k}$ its probability distribution. Let $B_{j:k}$ be the j^{th} lowest bid in auctions with k bidders.

To present our results simply, we make a strong assumption about bidding behavior; we will later discuss how to relax it:

Assumption 1. *The highest losing bidder bids up to his valuation, or, equivalently,* $B_{n-1:n} = V_{n-1:n}$.

This assumption would be exactly true in a second-price sealed-bid auction, or in a so-called "button auction," where bidders hold down a button to remain active and release it to drop out. It is approximately true (up to the minimum bid increment) in an ascending auction with no "jump bids," and is a common assumption when modeling ascending auctions. In a later section, we show how to extend our analysis to a more general model of bidding behavior.

Finally, we define our key assumption, which we will be testing and using for identification:

Definition. *There is* **exogenous variation in n** *if the number of bidders in an auction is independent of the joint distribution of the valuations of the bidders in the auction.*

Under this assumption, any randomly-chosen subset of m bidders taken from an auction of size n is indistinguishable from a randomly-chosen subset of m bidders taken from an auction of size $n-1$. Putting it another way, exogenous variation in n requires n to be statistically independent of θ, as well as statistically independent of the realizations of the draws of bidder values from the distribution $H(\,\cdot\,|\theta)$.

are drawn at random from an *infinite* set potential bidders with an ex-ante joint distribution of valuations which is exchangeable, then this formulation is without loss of generality. Without appealing to infinite bidders, since we place no restrictions on the dimensionality of θ or how it affects bidder preferences (other than that it affects them symmetrically), we still feel this is a very general formulation.

2.2 Testing under IPV

Next, we introduce a nonparametric test of the joint hypothesis of symmetric, conditionally independent private values and exogenous variation in n. For expositional clarity, we begin by considering a test of symmetric, independent private values and exogenous variation in n, and then show how that test changes when conditional independence is allowed.

First, suppose there is no heterogeneity in the objects for sale, so bidders have symmetric, independent private values. For $n \geq 2$, define two functions $\psi_{n:n}, \psi_{n-1:n} : [0,1] \rightarrow [0,1]$ by

$$\begin{aligned} \psi_{n:n}(s) &= s^n \\ \psi_{n-1:n}(s) &= ns^{n-1} - (n-1)s^n \end{aligned} \tag{1}$$

These functions have the property that given n independent draws from a probability distribution $H(\,\cdot\,)$, the CDFs of the highest and second-highest of the draws are $\psi_{n:n}\left(H(\,\cdot\,)\right)$ and $\psi_{n-1:n}\left(H(\,\cdot\,)\right)$, respectively.

Under the assumption of symmetric IPV with exogenous n, valuations of participating bidders are all independent draws from some probability distribution H, which is the same regardless of n. So for any n,

$$F_{n-1:n}(v) = \psi_{n-1:n}\left(H(v)\right) \tag{2}$$

or[3]

$$\psi^{-1}_{n-1:n}\left(F_{n-1:n}(v)\right) = H(v) \tag{3}$$

Since the right-hand side does not depend on n, a test of the symmetric IPV model with exogenous n is whether

$$\psi^{-1}_{n'-1:n'}\left(F_{n'-1:n'}(v)\right) = \psi^{-1}_{n-1:n}\left(F_{n-1:n}(v)\right) \tag{4}$$

[3]From (1), it is straightforward to show that $\psi_{n:n}$ and $\psi_{n-1:n}$ are both strictly increasing and onto, and therefore invertible.

for each pair (n', n). (This is the test introduced in Theorem 1 of Athey and Haile (2002).)

2.3 Testing under CIPV

Returning to our model of conditionally independent private values, the assumption of exogenous n would imply that the valuations of bidders participating in an auction of size n are independent draws from a distribution $H(\,\cdot\,|\theta)$, where θ are the characteristics of the auction (unobserved by the econometrician) and H and θ are independent of n.

To see why the test above is not still valid, consider the extreme case of perfectly correlated values, where θ is one-dimensional and each bidder's valuation in an auction with characteristics θ is simply θ. In this case, $F_{n-1:n}(v)$ for $n \geq 2$ is simply the ex-ante distribution of θ, and therefore does not vary with n; $\psi^{-1}_{n-1:n}\left(F_{n-1:n}(v)\right)$, then, is strictly increasing in n, and (4) fails.

However, under CIPV, the way in which the test fails is predictable, and leads us to a similarly-motivated inequality test. Under our model, $F_{n-1:n}(v)$ must still be weakly decreasing in n; but we show below that under CIPV with exogenous n, it decreases more slowly than it would under IPV, and so $\psi^{-1}_{n-1:n}\left(F_{n-1:n}(v)\right)$ must be increasing in n. We prove this result, then argue why this test should have power against the most likely source of endogeneity of n.

Theorem 1. *Under symmetric, conditionally independent private values with exogenous n, for any v, $F_{n-1:n}(v)$ is decreasing in n and $\psi^{-1}_{n-1:n}\left(F_{n-1:n}(v)\right)$ is increasing in n.*

Proof. For the first result, note that $\psi_{n-1:n}(s)$ is decreasing in n.[4] At a

[4] For $n \geq 2$,

$$\begin{aligned}
\psi_{n-1:n}(s) - \psi_{n-2:n-1}(s) &= ns^{n-1} - (n-1)s^n - \left((n-1)s^{n-2} - (n-2)s^{n-1}\right) \\
&= -(n-1)s^{n-2} + (2n-2)s^{n-1} - (n-1)s^n \\
&= -(n-1)s^{n-2}(1-s)^2 \quad \leq \quad 0
\end{aligned} \tag{5}$$

given realization of θ, the distribution of $V_{n-1:n}$ is $\psi_{n-1:n}(H(\cdot|\theta))$; so the unconditional distribution of $V_{n-1:n}$,

$$F_{n-1:n}(v) \quad = \quad E_\theta\left\{\psi_{n-1:n}\left(H(v|\theta)\right)\right\} \tag{6}$$

is decreasing in n as well.

For the second, it suffices to show $\psi^{-1}_{n:n+1}\left(F_{n:n+1}(s)\right) \geq \psi^{-1}_{n-1:n}\left(F_{n-1:n}(s)\right)$, or, equivalently, $\left(\psi_{n-1:n} \circ \psi^{-1}_{n:n+1}\right)\left(F_{n:n+1}(s)\right) \geq F_{n-1:n}(s)$. The key step of the proof is the observation that $\psi_{n-1:n} \circ \psi^{-1}_{n:n+1} : [0,1] \to [0,1]$ is concave. For any two differentiable, invertible functions f and g,

$$\tfrac{d}{ds}\left(f \circ g^{-1}\right)(s) \quad = \quad f'\left(g^{-1}(s)\right) \cdot \left(g^{-1}\right)'(s) \quad = \quad \tfrac{f'\left(g^{-1}(s)\right)}{g'\left(g^{-1}(s)\right)} \tag{7}$$

Since $\psi_{n-1:n}(t) = nt^{n-1} - (n-1)t^n$, differentiating yields

$$\tfrac{d}{ds}\left(\psi_{n-1:n} \circ \psi^{-1}_{n:n+1}\right)(s) \quad = \quad \tfrac{\psi'_{n-1:n}(t)}{\psi'_{n:n+1}(t)} \quad = \quad \tfrac{n(n-1)t^{n-2}(1-t)}{(n+1)nt^{n-1}(1-t)} \quad = \quad \tfrac{n-1}{n+1} \cdot \tfrac{1}{t} \tag{8}$$

where $t = \psi^{-1}_{n:n+1}(s)$. So $\left(\psi_{n-1:n} \circ \psi^{-1}_{n:n+1}\right)'(s) = \frac{n-1}{n+1} / \psi^{-1}_{n:n+1}(s)$, which is decreasing in s, making $\psi_{n-1:n} \circ \psi^{-1}_{n:n+1}$ concave. Applying Jensen's inequality then yields

$$\begin{aligned}
\left(\psi_{n-1:n} \circ \psi^{-1}_{n:n+1}\right)\left(F_{n:n+1}(v)\right) &= \left(\psi_{n-1:n} \circ \psi^{-1}_{n:n+1}\right) E_\theta\left\{\psi_{n:n+1}(H(v|\theta))\right\} \\
&\geq E_\theta\left\{\left(\psi_{n-1:n} \circ \psi^{-1}_{n:n+1}\right)\left(\psi_{n:n+1}(H(v|\theta))\right)\right\} \\
&= E_\theta\left\{\psi_{n-1:n}\left(\left(\psi^{-1}_{n:n+1} \circ \psi_{n:n+1}\right)(H(v|\theta))\right)\right\} \\
&= E_\theta\left\{\psi_{n-1:n}(H(v|\theta))\right\} \\
&= F_{n-1:n}(v)
\end{aligned} \tag{9}$$

so $\psi^{-1}_{n:n+1}\left(F_{n:n+1}(v)\right) \geq \psi^{-1}_{n-1:n}\left(F_{n-1:n}(v)\right)$. □

Since we are assuming $F_{n-1:n}$ is known for each n, this gives us a test of exogenous n under the assumption of symmetric, conditionally independent private values: specifically, that

$$F_{n-1:n}(v) \quad \le \quad F_{n'-1:n'}(v) \tag{10}$$

and

$$\psi^{-1}_{n-1:n}\left(F_{n-1:n}(v)\right) \quad \ge \quad \psi_{n'-1:n'}\left(F_{n'-1:n'}(v)\right) \tag{11}$$

for any $n > n'$ and any v.

2.4 Power of the Test

It remains to argue why this test has power, that is, why endogenous n would likely lead to a violation of either (10) or (11). The logic is as follows.

Under IPV with exogenous n, as n increases, the distribution of $V_{n-1:n}$ shifts to the right, so $F_{n-1:n}(v)$ decreases; but the function $\psi^{-1}_{n-1:n}(\,\cdot\,)$ increases, and the two effects exactly balance each other, so $\psi^{-1}_{n-1:n}\left(F_{n-1:n}(v)\right)$ is unchanged. Under CIPV with exogenous n, the change in $\psi^{-1}_{n-1:n}(\,\cdot\,)$ is the same as before, but $\psi^{-1}_{n-1:n}\left(F_{n-1:n}(v)\right)$ is now increasing rather than constant; thus, the second part of Theorem 1 says that $F_{n-1:n}(v)$ must decrease in n more slowly under CIPV than it would under IPV. Anything that "speeds up" the dependence of $F_{n-1:n}(v)$ on n, beyond the rate of change under IPV, would lead to a violation of (11), and therefore a failure of the test.

The most plausible source of endogeneity of n would be if bidders were more prone to enter auctions for more valuable objects – either auctions with characteristics which make them likely to be more valuable (if the bidders knew θ but not their actual valuation at the time the entry decision is made), or auctions for objects that the bidder already knows he values more highly. Such "positive selection" would mean that auctions with larger n were more likely to have high prices, beyond the natural effect of drawing from a larger pool of bidders; this is *exactly* the behavior that the test (11) will detect. (However, weak positive selection combined with correlation could be undetectable – selection increases the dependence of $F_{n-1:n}$ on n, but correlation weakens it, so the two effects could cancel if selection were weak enough.)

Negative selection – bidders being more prone to participate in auctions for less-valuable objects – would not lead to a violation of (11); but if it were a strong enough effect, it would lead to a violation of (10). So strong negative selection should also be detected. Weak negative selection, however, could be indistinguishable from correlation in values, since either one would slow the dependence of $F_{n-1:n}$ without reversing its direction.[5,6]

(While counterintuitive, negative selection could occur naturally if more valuable objects also tend to be valued more consistently by different buyers. This could be the case, for example, if the most valuable objects tended to be purchased by professional dealers, who were fairly unanimous in their assessments, while lower-value objects appealed to individual collectors, whose tastes were more idiosyncratic. Still, we feel intuitively that positive selection is the more likely source of endogeneity in n; thus, we expect (11) to be the more meaningful test.)

The two tests together imply two strings of inequalities as n varies:

$$\begin{array}{ccccccc} F_{1:2}(v) & \geq & F_{2:3}(v) & \geq & F_{3:4}(v) & \geq & \cdots \\ \psi_{1:2}^{-1}\left(F_{1:2}(v)\right) & \leq & \psi_{2:3}^{-1}\left(F_{2:3}(v)\right) & \leq & \psi_{3:4}^{-1}\left(F_{3:4}(v)\right) & \leq & \cdots \end{array} \tag{13}$$

While we will of course present a formal econometric test of this joint hypoth-

[5]In theory, we can replace $F_{n-1:n}(v) \leq F_{n-2:n-1}(v)$ with the stronger result that

$$F_{n-1:n}(v) \quad \leq \quad F_{n-2:n-1}(v) - \frac{(n-1)(n-3)}{(n-2)^2} \frac{\left(F_{n-3:n-2}(v) - F_{n-2:n-1}(v)\right)^2}{F_{n-4:n-3}(v) - F_{n-3:n-2}(v)} \tag{12}$$

(shown in the appendix) to give a stronger test against negative selection. In practice, however, due to the difference term in the denominator, this latter test is highly unstable when applied to a moderately-sized dataset.

[6]In settings where exogenous n is taken for granted, a violation of (10) (a left-shift in the distribution of winning bids as n increases) is interpreted as an indication of common values, since bidders shade their bids more as n increases due to a stronger winner's curse. **(CITE EXAMPLES.)** Here, we take private values as given, and argue that such a left-shift must therefore indicate negative selection.

esis, simply plotting $F_{n-1:n}(v)$ and $\psi^{-1}_{n-1:n}\left(F_{n-1:n}(v)\right)$ against v for various n should give some visual intuition for whether exogenous n seems plausible, or whether either positive or negative selection seems a likely problem.

If we were to apply these tests under the weaker bidding assumptions of Haile and Tamer (2003) rather than Assumption 1, we would begin with pointwise upper and lower bounds on $F_{n-1:n}$ for each n, rather than exact identification. The tests would then become

$$\begin{array}{rcl} \underline{F}_{n-1:n}(v) & \leq & \overline{F}_{n'-1:n'}(v) \\ \psi^{-1}_{n-1:n}\left(\overline{F}_{n-1:n}(v)\right) & \geq & \psi_{n'-1:n'}\left(\underline{F}_{n'-1:n'}(v)\right) \end{array} \tag{14}$$

for $n > n'$ and any v. Of course, the wider are the bounds on each distribution $F_{n-1:n}$, the more difficult it will be to violate (14), and therefore the less power the test will have.

3 Testing – Empirics

4 Identification – Theory

We have already defined exogenous variation in n, which will allow us to use variation in $F_{n-1:n}$ as n changes to identify the unobserved distribution $F_{n:n}$. However, for point identification, we will need the additional (unrealistically strong) assumption of unbounded variation in n:

Definition. *There is* **unbounded variation in n** *if for any m, the number of auctions in the data which have exactly m bidders grows without bound as the sample size grows.*

We will give theoretical identification results using this assumption, then show how we can identify upper and lower bounds on objects of interest when it is violated.

4.1 Identification under Unbounded Variation in n

Theorem 2. *In the symmetric private values model, the marginal distribution of every order statistic $V_{j:k}$ is identified from the size and highest losing bid in past ascending auctions if there is unbounded exogenous variation in the number of bidders n.*

Proof. A complete proof is given in the appendix. Exogenous variation in n leads to the relation discussed in Athey and Haile (2002): for any $j < k$,

$$F_{j:k-1}(v) \quad = \quad \frac{k-j}{k} F_{j:k}(v) \quad + \quad \frac{j}{k} F_{j+1:k}(v) \tag{15}$$

(To see where this comes from, begin with an auction with k bidders, and then choose one at random to remove. With probability $\frac{j}{k}$, the bidder removed was one of those with the j lowest values, in which case the j^{th} lowest of the remaining $k-1$ was the $j+1^{st}$ lowest originally. With probability $\frac{k-j}{k}$, the removed bidder is not one of the j lowest, and so the j^{th} lowest of the remaining $k-1$ was also the j^{th} lowest out of the original k.) Letting $j = m$ and $k = m+1$ and applying (15) recursively M times gives

$$F_{m:m}(v) \quad = \quad \frac{1}{m-1} \sum_{n=m+1}^{M} \left(\prod_{i=m}^{n-1} \frac{i-1}{i+1} \right) F_{n-1:n}(v) \quad + \quad \frac{m}{M} F_{M:M}(v) \tag{16}$$

Under Assumption 1, n and $V_{n-1:n}$ are observed in each auction; under unbounded variation, then, $F_{n-1:n}(v)$ is identified for every n. Taking the limit $M \to \infty$, and noting that $\frac{m}{M} F_{M:M}(v) \leq \frac{m}{M} \to 0$, we conclude that $F_{m:m}(v) \;=\; \lim_{M\to\infty} \frac{1}{m-1} \sum_{n=m+1}^{M} \left(\prod_{i=m}^{n-1} \frac{i-1}{i+1} \right) F_{n-1:n}(v)$.[7]

[7] Under weaker assumptions on bidding behavior, $F_{n-1:n}$ would be bounded above and below rather than identified exactly. With bounded data, M would be limited to the largest auctions for which there was data. Both of these would lead to upper and lower bounds on $F_{m:m}(v)$ rather than exact identification; this is discussed further below.

Rearranging (15) gives

$$F_{k-i:k}(v) \;=\; \frac{k}{i}F_{k-i:k-1}(v) \;-\; \frac{k-i}{i}F_{k-i+1:k}(v) \tag{17}$$

so for $i > 1$, $F_{k-i:k}(v)$ is pinned down by induction on i. $\square$

Theorem 2 does not require conditional independence, only symmetric private values. The additional structure of conditional independence, however, yields nice intuition for the identification result. (It will also pin down tighter bounds when variation in n is not unbounded.) Consider the problem of identifying $F_{3:3}$ from $\{F_{n-1:n}\}_{n>3}$. Recalling that $F_{n-1:n}(v) = E_\theta\left\{\psi_{n-1:n}\left(H(v|\theta)\right)\right\}$, and letting H denote $H(v|\theta)$,

$$\begin{array}{rclcccccccccl}
F_{3:4}(v) & = & E_\theta\{ & 4H^3 & - & 3H^4 & & & & & & & \} \\
F_{4:5}(v) & = & E_\theta\{ & & & 5H^4 & - & 4H^5 & & & & & \} \\
F_{5:6}(v) & = & E_\theta\{ & & & & & 6H^5 & - & 5H^6 & & & \} \\
F_{6:7}(v) & = & E_\theta\{ & & & & & & & 7H^6 & - & 6H^7 & \}
\end{array} \tag{18}$$

Identification simply requires constructing linear combinations to cancel middle terms:

$$\begin{array}{lcl}
\frac{1}{4}F_{3:4}(v) & = & E_\theta\left\{H^3 - \frac{3}{4}H^4\right\} \\
\frac{1}{4}F_{3:4}(v) + \frac{3}{20}F_{4:5}(v) & = & E_\theta\left\{H^3 - \frac{3}{5}H^5\right\} \\
\frac{1}{4}F_{3:4}(v) + \frac{3}{20}F_{4:5}(v) + \frac{1}{10}F_{5:6}(v) & = & E_\theta\left\{H^3 - \frac{3}{6}H^6\right\} \\
\frac{1}{4}F_{3:4}(v) + \frac{3}{20}F_{4:5}(v) + \frac{1}{10}F_{5:6}(v) + \frac{5}{70}F_{6:7}(v) & = & E_\theta\left\{H^3 - \frac{3}{7}H^7\right\}
\end{array} \tag{19}$$

With each added term, the left-hand side is identified from the highest losing bid, and the right-hand side moves toward $E_\theta H^3 = F_{3:3}(v)$; taking the limit as we keep adding terms gives the result.

In addition to the highest losing bid, we require the econometrician to observe the number of bidders n in each past auction. This may be impossible in some ascending auctions – particularly in a setting such as eBay, where we may know how many people actually submitted bids on a particular item, but may not be able to observe how many considered bidding and chose not to based on the existing bids. We hope to address imperfect observation on n in future work; for now, we focus on applications where n is cleanly observed.

Corollary 1. *If there is unbounded exogenous variation in n, then the seller's expected profit, and each bidder's ex-ante expected surplus, in a second-price sealed-bid auction with n bidders and reserve price r, is identified from the size and highest losing bid in past ascending auctions.*

Proof. Again, a full proof is in the appendix. For a second-price sealed-bid auction, both these quantities are expressible as functions of only the marginal distributions of $V_{n-1:n}$ and $V_{n:n}$:[8]

$$\begin{aligned} \pi_n(r) &= \left(F_{n-1:n}(r) - F_{n:n}(r)\right)(r - v_0) + \int_r^{+\infty} (v - v_0) dF_{n-1:n}(v) \\ u_n(r) &= \tfrac{1}{n}\left(E\left\{\max\{V_{n:n}, r\}\right\} - E\left\{\max\{V_{n-1:n}, r\}\right\}\right) \end{aligned} \tag{20}$$

where v_0 is the seller's valuation for the unsold good (and $V_{0:1}$ is understood to be 0). □

If Assumption 1 were to hold in future ascending auctions as well as for those in the data set, then the price paid in each future auction would be $\max\{r, V_{n-1:n} + \delta\}$, where δ is the minimum bid increment; (20) could be modified to apply to future ascending auctions. Without Assumption 1, if bidders do not make "jump bids" and only outbid each other by δ at a time, the price paid would be within $V_{n-1:n} \pm \delta$, and so (20) would hold to within $\pm\delta$. Finally, under conditionally independent private values, if we assume that each bidder observes θ as well as his own valuation, then conditional on a particular realization of θ (which is common knowledge among the bidders), we are in a symmetric IPV model, and therefore revenue equivalence holds;

[8] In a different but analogous setting, Athey and Haile (2007) point out that identification of the *joint* distribution of $(V_{n-1:n}, V_{n:n})$ is sufficient for "evaluation of rent extraction by the seller, the effects of introducing a reserve price, and the outcomes under a number of alternative selling mechanisms." Here, however, we do not rely on the joint distribution of $V_{n-1:n}$ and $V_{n:n}$, only the *marginal* distribution of each.

taking expectations over θ, (20) would then hold exactly under equilibrium behavior in *any* mechanism which always allocates the object to the bidder with the highest valuation provided it is above r.

Of course, once $\pi_n(r)$ is known, we can solve $\max_r \pi_n(r)$ to calculate the optimal reserve price for an auction with a fixed number n bidders. (Under independent private values, the optimal r does not depend on n; but with correlated values, it does.)

Corollary 2. *If there is exogenous variation in n, the second-price auction is testable if the size, highest losing bid, and another losing bid are observed in each auction.*

Proof. Suppose for example that the third-highest bid is observed in each auction. From Theorem 2, $F_{m-2:m}(v)$ is identified from the highest losing bid; this distribution can then be tested against the observed distribution of third-highest bids in auctions of size m. Identification of $F_{m-2:m}(v)$ only requires identification of $F_{m-2:m-1}$ and $F_{m-1:m}$, so exact identification of $F_{m:m}$ (and therefore unbounded variation in n) is not required. □

4.2 Bounded n, Finite Data, Weaker Bidding Assumptions

While Theorem 2 is nice theoretically, we will obviously never do empirical work on a dataset with unbounded variation in the size of auctions, or with perfect identification of $F_{n-1:n}$ for a given n. However, the identification results extend easily to bounds under more realistic assumptions on data.

Relax Assumption 1 (that $V_{n-1:n}$ is directly observed in each auction), and instead impose the bidding restrictions from Haile and Tamer (2003): that each bidder bids no more than his private value, and that the winning bid plus the minimum bid increment is no less than any losing bidder's valuation. (These are roughly equivalent to the assumption that bidders do not play weakly dominated strategies.) Under these assumptions, $F_{n-1:n}(v)$ is not exactly identified by bid data, but it is bounded above and below: specifically,

$$\begin{array}{rcccl} F_{n-1:n}(v) & \leq & \overline{F}_{n-1:n}(v) & \equiv & G_{n-1:n}(v) \\ F_{n-1:n}(v) & \geq & \underline{F}_{n-1:n}(v) & \equiv & G_{n:n}(v-\delta) \end{array} \tag{21}$$

where $G_{j:k}$ is the distribution of the j^{th} lowest bid in auctions with k bidders and δ the minimum bid increment. (When the top two bids are close together, this pins down $F_{n-1:n}$ to a narrow interval.) Similarly, even under Assumption 1, with finite data we would begin with confidence intervals for $F_{n-1:n}(v)$ rather than certain knowledge of it. With bounded variation in n, and with pointwise bounds instead of exact identification of $F_{n-1:n}$, we can still identify bounds on the objects of interest:

Theorem 3. *Assume symmetric, conditionally independent private values and exogenous variation in n. Suppose we have pointwise bounds*

$$\underline{F}_{n-1:n}(v) \quad \leq \quad F_{n-1:n}(v) \quad \leq \quad \overline{F}_{n-1:n}(v) \tag{22}$$

for every v and each $n \in \{m, m+1, \dots, M\}$. Then $\underline{F}_{m:m}(v) \leq F_{m:m}(v) \leq \overline{F}_{m:m}(v)$, where

$$\begin{array}{rcl} \overline{F}_{m:m}(v) & = & \sum_{n=m+1}^{M} \alpha_{m,n} \overline{F}_{n-1:n}(v) + \frac{m}{M} \overline{F}_{M-1:M}(v) \\ \underline{F}_{m:m}(v) & = & \sum_{n=m+1}^{M} \alpha_{m,n} \underline{F}_{n-1:n}(v) + \frac{m}{M} \psi_{M:M} \circ \psi_{M-1:M}^{-1} \left(\underline{F}_{M-1:M}(v) \right) \end{array} \tag{23}$$

and $\alpha_{m,n} = \frac{1}{m-1} \prod_{i=m}^{n-1} \frac{i-1}{i+1}$. Further, in a second-price sealed-bid auction with n bidders and reserve price r, expected revenue $\pi_n(r) \in [\underline{\pi}_n(r), \overline{\pi}_n(r)]$, and ex-ante expected bidder surplus $u_n(r) \in [\underline{u}_n(r), \overline{u}_n(r)]$, where

$$\begin{array}{rcl} \underline{\pi}_n(r) & = & \left(\overline{F}_{n-1:n}(r) - \overline{F}_{n:n}(r) \right) (r - v_0) + \int_r^\infty (v - v_0) d\overline{F}_{n-1:n}(v) \\ \overline{\pi}_n(r) & = & \left(\underline{F}_{n-1:n}(r) - \underline{F}_{n:n}(r) \right) (r - v_0) + \int_r^\infty (v - v_0) d\underline{F}_{n-1:n}(v) \end{array} \tag{24}$$

and

$$\begin{array}{rcl} \underline{u}_n(r) & = & \frac{1}{n}\left(\int_0^\infty \max\{r,v\}d\overline{F}_{n:n}(v) - \int_0^\infty \max\{r,v\}d\underline{F}_{n-1:n}(v)\right) \\ \overline{u}_n(r) & = & \frac{1}{n}\left(\int_0^\infty \max\{r,v\}d\underline{F}_{n:n}(v) - \int_0^\infty \max\{r,v\}d\overline{F}_{n-1:n}(v)\right) \end{array} \tag{25}$$

Proof. Since the right-hand side of (16) is increasing in $F_{n-1:n}(v)$ (for each n) and $F_{M:M}(v)$, we plug in upper bounds to get $\overline{F}_{m:m}$ and lower bounds to get $\underline{F}_{m:m}$. For the upper bound on $F_{M:M}(v)$, we use the fact that $V_{M:M} \geq V_{M-1:M}$, and so

$$F_{M:M}(v) \;\leq\; F_{M-1:M}(v) \;\leq\; \overline{F}_{M-1:M}(v) \tag{26}$$

For the lower bound, first note that

$$\left(\psi_{M:M} \circ \psi_{M-1:M}^{-1}\right)'(s) \;=\; \frac{\psi'_{M:M}(t)}{\psi'_{M-1:M}(t)} \;=\; \frac{Mt^{M-1}}{M(M-1)t^{M-2}(1-t)} \;=\; \frac{t}{(M-1)(1-t)} \tag{27}$$

where $t = \psi_{M-1:M}^{-1}(s)$; (27) is increasing in t and therefore s, so $\psi_{M:M} \circ \psi_{M-1:M}^{-1}$ is convex. Letting H denote $H(v|\theta)$, applying Jensen's Inequality gives

$$\begin{array}{rcl} F_{M:M}(v) & = & E_\theta\left\{\psi_{M:M}(H)\right\} \\ & = & E_\theta\left\{\left(\psi_{M:M} \circ \psi_{M-1:M}^{-1}\right)\left(\psi_{M-1:M}(H)\right)\right\} \\ & \geq & \left(\psi_{M:M} \circ \psi_{M-1:M}^{-1}\right)\left(E_\theta\left\{\psi_{M-1:M}(H)\right\}\right) \\ & = & \psi_{M:M} \circ \psi_{M-1:M}^{-1}\left(F_{M-1:M}(v)\right) \\ & \geq & \psi_{M:M} \circ \psi_{M-1:M}^{-1}\left(\underline{F}_{M-1:M}(v)\right) \end{array} \tag{28}$$

since $\psi_{M:M}$ and $\psi_{M-1:M}^{-1}$ are both increasing.[9]

For the bounds on $\pi_n(r)$, Quint (2008) shows that the expression for expected profit (20) is stochastically increasing in both $V_{n-1:n}$ and $V_{n:n}$ (in the first-order stochastic dominance sense); so plugging the upper bounds $\overline{F}_{n-1:n}$ and $\overline{F}_{n:n}$ into (20) gives the lower bound $\underline{\pi}_n(r)$, and the lower bounds

[9] Quint (2008) shows that the lower bound $F_{n:n}(v) \geq \psi_{n:n} \circ \psi_{n-1:n}^{-1}\left(F_{n-1:n}(v)\right)$ holds for symmetric affiliated private values which are not conditionally independent as well.

$\underline{F}_{n-1:n}$ and $\underline{F}_{n:n}$ give the upper bound $\overline{\pi}_n(r)$. Similarly, the expression for expected bidder surplus is stochastically increasing in $V_{n:n}$ and decreasing in $V_{n-1:n}$; so plugging $\overline{F}_{n:n}$ and $\underline{F}_{n-1:n}$ into (20) gives a lower bound on $u_n(r)$, and $\underline{F}_{n:n}$ and $\overline{F}_{n-1:n}$ yield an upper bound. $\square$

Once $\overline{\pi}_n(r)$ and $\underline{\pi}_n(r)$ are known for each r, as in Haile and Tamer (2003), we can use them to bound the profit-maximizing reserve price for an auction of a given size. Specifically, letting $r_* = \arg\max_{r'} \underline{\pi}_n(r')$, the true optimal reserve price r_n^* for an auction with n bidders must satisfy

$$\overline{\pi}_n(r_n^*) \quad \geq \quad \pi_n(r_n^*) \quad \geq \quad \pi_n(r_*) \quad \geq \quad \underline{\pi}_n(r_*) \quad = \quad \max_{r'} \underline{\pi}_n(r') \tag{29}$$

and so $r_n^* \in \{r : \overline{\pi}_n(r) \geq \max_{r'} \underline{\pi}_n(r')\}$.

It is also worth noting that the bounds on $F_{m:m}$, and therefore $\pi_m(r)$ (and therefore r_m^*), will be tightest when m is lowest – exactly the cases where setting the correct reserve price is most important.[10] To see this, consider the case where the data set is huge but bounded, so that $F_{n-1:n}$ is learned very precisely for n up to an upper bound M. By (16), $F_{m:m}(v) = \sum_{n=m+1}^{M} \alpha_{m,n} F_{n-1:n}(v) + \frac{m}{M} F_{M:M}(v)$, so all uncertainty about $F_{m:m}(v)$ comes from uncertainty about $F_{M:M}(v)$; but the lower is m, the lower is the coefficient $\frac{m}{M}$ on $F_{M:M}(v)$, and therefore the lower is the uncertainty in $F_{m:m}$.

5 Identification – Empirics

6 Conclusion

[10] As m grows, the probabilities that a given reserve price r either binds or precludes a sale both go to 0.

Appendix

A.1 Proof of Theorem 2

(Equations (15) and (16) are numbered below to match their numbering in the text.)

The proof is based on the relation discussed in Athey and Haile (2002) and in the text above,

$$F_{j:k-1}(v) \quad = \quad \frac{k-j}{k} F_{j:k}(v) \quad + \quad \frac{j}{k} F_{j+1:k}(v) \tag{15}$$

To identify $F_{m:m}$, we fix $m > 1$ and use induction on M to show that

$$F_{m:m}(v) \quad = \quad \frac{1}{m-1} \sum_{n=m+1}^{M} \left(\prod_{i=m}^{n-1} \frac{i-1}{i+1} \right) F_{n-1:n}(v) \quad + \quad \frac{m}{M} F_{M:M}(v) \tag{16}$$

for any $M > m$. First, let $M = m + 1$. Then the right-hand side is

$$\frac{1}{m+1} F_{m:m+1}(v) + \frac{m}{m+1} F_{m+1:m+1}(v)$$

which is equal to $F_{m:m}(v)$ by (15) (letting $j = m$ and $k = m + 1$).

For the inductive step, if (16) holds for $M = K$, then

$$\begin{aligned}
&\tfrac{1}{m-1} \textstyle\sum_{n=m+1}^{K+1} \left(\prod_{i=m}^{n-1} \tfrac{i-1}{i+1} \right) F_{n-1:n}(v) + \tfrac{m}{K+1} F_{K+1:K+1}(v) \\
&\quad = \tfrac{1}{m-1} \textstyle\sum_{n=m+1}^{K} \left(\prod_{i=m}^{n-1} \tfrac{i-1}{i+1} \right) F_{n-1:n}(v) + \tfrac{1}{m-1} \left(\prod_{i=m}^{K} \tfrac{i-1}{i+1} \right) F_{K:K+1}(v) \\
&\qquad\quad + \tfrac{m}{K+1} F_{K+1:K+1}(v) \\
&\quad = F_{m:m}(v) - \tfrac{m}{K} F_{K:K}(v) + \tfrac{1}{m-1} \tfrac{(m-1)m}{K(K+1)} F_{K:K+1}(v) + \tfrac{m}{K+1} F_{K+1:K+1}(v) \\
&\quad = F_{m:m}(v) + \tfrac{m}{K} \left(-F_{K:K}(v) + \tfrac{1}{K+1} F_{K:K+1}(v) + \tfrac{K}{K+1} F_{K+1:K+1}(v) \right)
\end{aligned}$$

which is $F_{m:m}(v)$ by (15) (letting $j = K$ and $k = K + 1$), so (16) holds for $M = K + 1$.

So $F_{k:k}(v)$ and $F_{k-1:k}(v)$ are identified. For $i > 1$, we pin down $F_{k-i:k}$ by induction on i: rearranging (15) gives

$$F_{k-i:k}(v) \quad = \quad \frac{k}{i} F_{(k-1)-(i-1):k-1}(v) \quad - \quad \frac{k-i}{i} F_{k-(i-1):k}(v)$$

so once $F_{k'-(i-1):k'}(v)$ is known for every k', $F_{k-i:k}(v)$ can be calculated directly. (So $\{F_{n-2:n}\}$ are pinned down from $\{F_{n-1:n}\}$, $\{F_{n-3:n}\}$ from $\{F_{n-2:n}\}$, and so on.)

A.2 Proof of Corollary 1

Seller profits are 0 if $V_{n:n} < r$, $r - v_0$ if $V_{n:n} \geq r \geq V_{n-1:n}$, and $V_{n-1:n} - v_0$ if $V_{n:n} \geq V_{n-1:n} > r$ (where v_0 is the seller's cost and $V_{0:1}$ is understood to be 0). We can therefore write expected profits as

$$\pi_n(r) \quad = \quad E_{V_{n-1:n},V_{n:n}} \left\{ \mathbf{1}_{V_{n:n} \geq r \geq V_{n-1:n}} (r - v_0) + \mathbf{1}_{V_{n:n} \geq V_{n-1:n} > r} (V_{n-1:n} - v_0) \right\}$$

Since the first event happens with probability $F_{n-1:n}(r) - F_{n:n}(r)$, and the second happens whenever $V_{n-1:n} > r$, we can rewrite this as

$$\pi_n(r) \quad = \quad (F_{n-1:n}(r) - F_{n:n}(r)) (r - v_0) + \int_r^{+\infty} (v - v_0) dF_{n-1:n}(v)$$

As for bidder surplus, each bidder has ex-ante probability $\frac{1}{n}$ of having the highest value, which earns a surplus of 0 when $V_{n:n} \leq r$ and $V_{n:n} - \max\{V_{n-1:n}, r\}$ when $V_{n:n} > r$ (where $V_{0:1}$ is under stood to be 0). So

$$\begin{aligned}
u_n(r) \quad &= \quad \tfrac{1}{n} E_{V_{n-1:n},V_{n:n}} \left\{ \mathbf{1}_{V_{n:n} > r} (V_{n:n} - \max\{V_{n-1:n}, r\}) \right\} \\
&= \quad \tfrac{1}{n} E_{V_{n-1:n},V_{n:n}} \left\{ \mathbf{1}_{r \geq V_{n:n} \geq V_{n-1:n}} (r - r) \right. \\
&\qquad + \mathbf{1}_{V_{n:n} > r \geq V_{n-1:n}} (V_{n:n} - r) \\
&\qquad \left. + \mathbf{1}_{V_{n:n} \geq V_{n-1:n} > r} (V_{n:n} - V_{n-1:n}) \right\} \\
&= \quad \tfrac{1}{n} E_{V_{n-1:n},V_{n:n}} \left\{ \mathbf{1}_{V_{n:n} \leq r} r + \mathbf{1}_{V_{n:n} > r} V_{n:n} \right. \\
&\qquad \left. + \mathbf{1}_{V_{n-1:n} \leq r} r + \mathbf{1}_{V_{n-1:n} > r} V_{n-1:n} \right\} \\
&= \quad \tfrac{1}{n} \left(E \left\{ \max\{V_{n:n}, r\} \right\} - E \left\{ \max\{V_{n-1:n}, r\} \right\} \right)
\end{aligned}$$

A.3 Proof of Equation (12)

The exact claim is that under symmetric, conditionally independent private values with exogenous n,

$$F_{n-1:n}(v) \quad \leq \quad F_{n-2:n-1}(v) - \frac{(n-1)(n-3)}{(n-2)^2} \frac{\left(F_{n-3:n-2}(v) - F_{n-2:n-1}(v)\right)^2}{F_{n-4:n-3}(v) - F_{n-3:n-2}(v)}$$

Let X_n denote the statement "$v_2 > r$, $v_3 > r$, and $v_i \leq r$ for $i \in \{4, 5, \ldots, n\}$". We first show that $\Pr(v_1 \leq r | X_n)$ is increasing in n. To see this, fix r, and let G^n denote the probability distribution of $H(r|\theta)$ when the distribution of θ is conditioned on the information X_n. (That is, begin with the prior distribution of θ, apply Bayes' Law to determine the posterior distribution of θ conditional on X_n; G^n is then the posterior distribution of $H(r|\theta)$.) Then $\Pr(v_1 \leq r | X_n) = E_{\theta|X_n} H(r|\theta) = \int p \; dG^n(p)$.

Since X_{n+1} is just X_n, plus the information that $v_{n+1} \leq r$, we can generate dG^{n+1} from dG^n through Bayes' Law: abusing notation slightly,

$$dG^{n+1}(p) = \frac{\Pr(H(r|\theta) = p | X_n) \; \Pr(v_1 \leq r | H(r|\theta) = p)}{\Pr(v_1 \leq r | X_n)} = \frac{p \; dG^n(p)}{\int p' \; dG^n(p')}$$

and so, taking the expectation,

$$\begin{aligned} \Pr(v_1 \leq r | X_{n+1}) &= \int p \; dG^{n+1}(p) &= \frac{\int p^2 \; dG^n(p)}{\int p' \; dG^n(p')} \\ &\geq \int p' \; dG^n(p') &= \Pr(v_1 \leq r | X_n) \end{aligned}$$

where the inequality is simply $E(p^2) \geq (E(p))^2$ when both expectations are taken with respect to the distribution G^n.

Once we know $\Pr(v_1 \leq r | X_n)$ is increasing in n, we know that

$$\frac{\Pr(X_n)}{\Pr(X_n)} \frac{\Pr(v_1 > r | X_n)}{\Pr(v_1 \leq r | X_n)}$$

is decreasing in n. By symmetry,

$$\begin{aligned}
F_{n-2:n}(r) - F_{n-1:n}(r) &= {}_nC_2 \Pr(v_1 \le r, v_2 > r, v_3 > r, v_4 \le 4, \ldots, v_n \le r) \\
&= {}_nC_2 \Pr(X_n) \Pr(v_1 \le r | X_n)
\end{aligned}$$

$$\begin{aligned}
F_{n-3:n}(r) - F_{n-2:n}(r) &= {}_nC_3 \Pr(v_1 > r, v_2 > r, v_3 > r, v_4 \le 4, \ldots, v_n \le r) \\
&= {}_nC_3 \Pr(X_n) \Pr(v_1 > r | X_n)
\end{aligned}$$

so

$$\frac{\frac{1}{{}_nC_3}(F_{n-3:n}(r) - F_{n-2:n}(r))}{\frac{1}{{}_nC_2}(F_{n-2:n}(r) - F_{n-1:n}(r))}$$

is decreasing in n.

Solving (15) for $F_{j:k}$ gives $F_{j:k} = \frac{k}{k-j} F_{j:k-1} - \frac{j}{k-j} F_{j+1:k}$ (suppressing the dependence on r), leading to

$$\begin{aligned}
F_{n-2:n} - F_{n-1:n} &= \tfrac{n}{2} F_{n-2:n-1} - \tfrac{n-2}{2} F_{n-1:n} - F_{n-1:n} \\
&= \tfrac{n}{2} \left(F_{n-2:n-1} - F_{n-1:n}\right)
\end{aligned}$$

$$\begin{aligned}
F_{n-3:n} - F_{n-2:n} &= \tfrac{n}{3} F_{n-3:n-1} - \tfrac{n-3}{3} F_{n-2:n} - F_{n-2:n} \\
&= \tfrac{n}{3} \left(F_{n-3:n-1} - F_{n-2:n}\right) \\
&= \tfrac{n}{3} \left(\tfrac{n-1}{2} F_{n-3:n-2} - \tfrac{n-3}{2} F_{n-2:n-1} - \left(\tfrac{n}{2} F_{n-2:n-1} - \tfrac{n-2}{2} F_{n-1:n}\right)\right) \\
&= \tfrac{n}{6} \left((n-1) F_{n-3:n-2} - (2n-3) F_{n-2:n-1} + (n-2) F_{n-1:n}\right) \\
&= \tfrac{n}{6} \left((n-1)\left(F_{n-3:n-2} - F_{n-2:n-1}\right) - (n-2)\left(F_{n-2:n-1} - F_{n-1:n}\right)\right)
\end{aligned}$$

and so

$$\begin{aligned}
\frac{\frac{1}{{}_nC_3}(F_{n-3:n} - F_{n-2:n})}{\frac{1}{{}_nC_2}(F_{n-2:n} - F_{n-1:n})} &= \frac{{}_nC_2}{{}_nC_3} \frac{\frac{n}{6}((n-1)(F_{n-3:n-2} - F_{n-2:n-1}) - (n-2)(F_{n-2:n-1} - F_{n-1:n}))}{\frac{n}{2}(F_{n-2:n-1} - F_{n-1:n})} \\
&= \frac{1}{n-2} \left(\frac{(n-1)(F_{n-3:n-2} - F_{n-2:n-1})}{(F_{n-2:n-1} - F_{n-1:n})} - (n-2) \right) \\
&= \frac{(n-1)(F_{n-3:n-2} - F_{n-2:n-1})}{(n-2)(F_{n-2:n-1} - F_{n-1:n})} - 1
\end{aligned}$$

must be decreasing in n. Solving

$$\frac{(n-1)\left(F_{n-3:n-2}-F_{n-2:n-1}\right)}{(n-2)\left(F_{n-2:n-1}-F_{n-1:n}\right)} \leq \frac{(n-2)\left(F_{n-4:n-3}-F_{n-3:n-2}\right)}{(n-3)\left(F_{n-3:n-2}-F_{n-2:n-1}\right)}$$

for $F_{n-1:n}$ gives the upper bound.

References

Athey, S. and P. Haile (2002). Identification of standard auction models. *Econometrica 70*(6), 2107–2140.

Athey, S. and P. Haile (2007). Nonparametric approaches to auctions. In J. Heckman and E. Leamer (Eds.), *Handbook of Econometrics, vol. 6A*, Chapter 60, pp. 3847–3965. Elsevier.

Cremer, J. and R. McLean (1988). Full extraction of the surplus in Bayesian and dominant strategy auctions. *Econometrica 56*(6), 1247–1257.

Haile, P. and E. Tamer (2003). Inference with an incomplete model of English auctions. *Journal of Political Economy 111*(1), 1–51.

Hewitt, E. and L. Savage (1955). Symmetric measures on Cartesian products. *Transactions of the American Mathematical Society 80*(2), 470–501.

Levin, D. and J. Smith (1994). Equilibrium in auctions with entry. *The American Economic Review 84*(3), 585–599.

McAfee, R. and J. McMillan (1987). Auctions with entry. *Economics Letters 23*(4), 343–347.

McAfee, R. and P. Reny (1992). Correlated Information and Mechanism Design. *Econometrica 60*(2), 395–421.

Myerson, R. (1981). Optimal auction design. *Mathematics of Operations Research 6*(1), 58–73.

Paarsch, H. (1997). Deriving an estimate of the optimal reserve price: an application to British Columbian timber sales. *Journal of Econometrics 78*(2), 333–357.

Paarsch, H. J. (1992). Deciding between the common and private value paradigms in empirical models of auctions. *Journal of Econometrics 51*(1-2), 191–215.

Quint, D. (2008). Unobserved correlation in private-value ascending auctions. *Economics Letters 100*(3), 432–434.

Samuelson, W. (1985). Competitive bidding with entry costs. *Economics Letters 17*(1-2), 53–57.

Silverman, B. (1986). *Density estimation for statistics and data analysis.* Chapman & Hall/CRC.