A Simple Example to Illustrate the Linkage Principle

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Abstract. I present a numerical example illustrating the revenue-superiority of an "open" over a "closed" auction format (the linkage principle), and use it to calculate the magnitude of the effect.

1 Introduction

One of the classic results of Milgrom and Weber's (1982) seminal paper is that in a symmetric, affiliated setting, an ascending auction will yield higher expected revenue if bidders are aware at each point during the auction of how many of their competitors are still bidding. This result – one of a set of related insights often collectively labeled the "linkage principle" – is well known: Loertscher, Marx and Wilkening (2015), for example, quote FCC economist Evan Kwerel (in the introduction to Milgrom (2004)) as citing it as a key consideration in the design of early U.S. spectrum auctions. However, the mechanism by which it operates is a bit subtle, and I can't recall ever seeing an explicit example illustrating it, or calculating its magnitude. Here, I offer one.

2 Model

2.1 Valuations

Let N be the number of bidders. Suppose there's an underlying state $\theta \in \{0, 1\}$, equally likely, with common values $v_i = \theta$. Each bidder gets a signal s_i which is *i.i.d.* conditional on θ, with density

$$f(s|\theta) \quad = \quad \begin{cases} \alpha s^{\alpha-1} & if \quad \theta = 1 \\ \alpha(1-s)^{\alpha-1} & if \quad \theta = 0 \end{cases}$$

and therefore CDF

$$F(s|\theta) \quad = \quad \begin{cases} s^{\alpha} & if \quad \theta = 1 \\ 1-(1-s)^{\alpha} & if \quad \theta = 0 \end{cases}$$

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conditional on the true state, with $\alpha > 1$. Note that an increase in α makes signals "more informative" about the realization of θ. While it's not immediately apparent, since $f(s|\theta)$ satisfies the monotone likelihood ratio property, $\{\theta, s_1, \ldots, s_N\}$ are affiliated.[1] Let $s^{(1)} > s^{(2)} > s^{(3)} > \cdots > s^{(N)}$ denote the order statistics of the bidders' signals.

2.2 Full-information versus no-information button auctions

As in Milgrom and Weber (1982), I model ascending auctions as button auctions, with either full information (every bidder knows who else is active and at what price the inactive bidders dropped out) or no information (bidders receive no feedback other than that the auction is still in progress until it is over). Milgrom and Weber refer to the former as an English auction, and the latter as a second-price auction. Symmetric, monotonic equilibria for both are established in Theorems 6 and 10 of their paper, and the revenue ranking result is Theorem 11.

2.3 Bidding in the no-information case

When there is no feedback during the auction, each bidder bids the expected value the asset would have if he was exactly tied for the highest signal, or

$$b(s) = \Pr(\theta = 1 | s^{(1)} = s^{(2)} = s)$$

By Bayes' Law, this is

$$\begin{aligned}
b(s) &= \frac{\frac{1}{2}\Pr(s^{(1)} = s^{(2)} = s|\theta = 1)}{\frac{1}{2}\Pr(s^{(1)} = s^{(2)} = s|\theta = 1) + \frac{1}{2}\Pr(s^{(1)} = s^{(2)} = s|\theta = 0)} \\
&= \frac{\frac{1}{2}N(N-1)\alpha s^{\alpha-1} \cdot \alpha s^{\alpha-1} \cdot (s^\alpha)^{N-2}}{\frac{1}{2}N(N-1)\alpha s^{\alpha-1} \cdot \alpha s^{\alpha-1} \cdot (s^\alpha)^{N-2} + \frac{1}{2}N(N-1)\alpha(1-s)^{\alpha-1} \cdot \alpha(1-s)^{\alpha-1} \cdot (1-(1-s)^\alpha)^{N-2}} \\
&= 1 \Bigg/ \left(1 + \frac{(1-s)^{\alpha-1}}{s^{\alpha-1}} \cdot \frac{(1-s)^{\alpha-1}}{s^{\alpha-1}} \cdot \left(\frac{1-(1-s)^\alpha}{s^\alpha}\right)^{N-2}\right)
\end{aligned}$$

2.4 Bidding in the full-information case

In the full-information case, each losing bidder's signal is revealed by the price at which he drops out. Revenue depends on the bid function when only two active bidders remain, which depends on the revealed signals of the $N-2$ bidders who have dropped out to that point. I write this bid function as $b(s, \mathbf{t})$, where $\mathbf{t} = (t_3, t_4, \ldots, t_N)$ are the revealed realizations of the order statistics $s^{(3)}$,

[1]Thus, if we think of bidder values as being $v_i = E(\theta|\{s_1, \ldots, s_n\})$ instead of $v_i = \theta$, this example fits within the general affiliated/interdependent model of Milgrom and Weber (1982).

$s^{(4)}$, etc. This bid function is

$$
\begin{aligned}
b(s,\mathbf{t}) &= \Pr(\theta=1|s^{(1)}=s^{(2)}=s, s^{(3-N)}=\mathbf{t}) \\
&= 1 \Big/ \left(1+\frac{(1-s)^{\alpha-1}}{s^{\alpha-1}}\cdot\frac{(1-s)^{\alpha-1}}{s^{\alpha-1}}\cdot\prod_{i=3}^{N}\frac{(1-t_i)^{\alpha-1}}{t_i^{\alpha-1}}\right)
\end{aligned}
$$

3 Results

3.1 Example: expected revenue when $N=3$, $\alpha=2$, and $s^{(2)}=\frac{1}{2}$

For an example we can solve analytically, let $N=3$, $\alpha=2$, and consider the realization $\frac{1}{2}$ of the second-highest signal. Note that $E(\theta|s^{(2)}=\frac{1}{2})=\frac{1}{2}$, as $s^{(2)}=\frac{1}{2}$ is equally likely under either state.

In the no-information case, when $s^{(2)}=\frac{1}{2}$, the second-highest bidder drops out at a bid of

$$
b(\tfrac{1}{2}) = 1 \Big/ \left(1+\frac{1}{1}\cdot\frac{1}{1}\cdot\frac{1-\frac{1}{4}}{\frac{1}{4}}\right) = 0.25
$$

so revenue is 0.25. In the full-information case, given the realized value of the lowest signal, the bidder with type $\frac{1}{2}$ bids up to

$$
b(\tfrac{1}{2}, s^{(3)}) = 1 \Big/ \left(1+\frac{1}{1}\cdot\frac{1}{1}\cdot\frac{1-s^{(3)}}{s^{(3)}}\right) = s^{(3)}
$$

so expected revenue when $s^{(2)}=\frac{1}{2}$ is $E(s^{(3)}|s^{(2)}=\frac{1}{2})$. Now, conditional on the realizations of θ and $s^{(2)}$, the distribution of $s^{(3)}$ is just the marginal (prior) distribution of a signal s_i, conditional on θ, truncated above at $s^{(2)}$; so we can calculate expected revenue as

$$
\begin{aligned}
E\left(b\left(\tfrac{1}{2}, s^{(3)}\right)|s^{(2)}=\tfrac{1}{2}\right) &= E\left(s^{(3)}|s^{(2)}=\tfrac{1}{2}\right) \\
&= \Pr(\theta=1|s^{(2)}=\tfrac{1}{2})\cdot E(s^{(3)}|s^{(2)}=\tfrac{1}{2},\theta=1) \\
&\quad +\Pr(\theta=0|s^{(2)}=\tfrac{1}{2})\cdot E(s^{(3)}|s^{(2)}=\tfrac{1}{2},\theta=0) \\
&= \frac{1}{2}\cdot\frac{\int_0^{\frac{1}{2}}(2s)s\,ds}{\int_0^{\frac{1}{2}}2s\,ds}+\frac{1}{2}\cdot\frac{\int_0^{\frac{1}{2}}2(1-s)s\,ds}{\int_0^{\frac{1}{2}}2(1-s)\,ds} \\
&= \frac{1}{2}\cdot\frac{1}{3}+\frac{1}{2}\cdot\frac{2}{9} = \frac{5}{18} \approx 0.2778
\end{aligned}
$$

So for this particular realization of $s^{(2)}$, expected revenue is 11% higher when the bidders learn the realization of $s^{(3)}$ from equilibrium bidding than when they do not.

3.2 What's going on

To avoid ex post regret, the second-to-last bidder must give up before he would regret winning the item if his opponent dropped out. This therefore stops him from bidding beyond the expected value of θ conditional on $s^{(1)} = s^{(2)} = \frac{1}{2}$, and whatever other information he has. If the signal of the lowest signal was observed, then its realization is also used to update the expected value of θ. If not, then it is not; by iterated expectations, this is the same, in expectation, as using the distribution of $s^{(3)}$ that would be consistent with $s^{(1)} = s^{(2)} = \frac{1}{2}$. But since this is pessimistic relative to the truth $s^{(1)} > s^{(2)} = \frac{1}{2}$, and since the signals are affiliated, this depresses the expected value of θ and therefore the second-highest bidder's bid.

In numbers, as noted above, expected revenue with full information when $s^{(2)} = \frac{1}{2}$ is

$$\begin{aligned} E(s^{(3)}|s^{(2)} = \tfrac{1}{2}) &= \Pr(\theta = 1|s^{(2)} = \tfrac{1}{2})E(s^{(3)}|\theta = 1, s^{(2)} = \tfrac{1}{2}) \\ &\quad + \Pr(\theta = 0|s^{(2)} = \tfrac{1}{2})E(s^{(3)}|\theta = 0, s^{(2)} = \tfrac{1}{2}) \\ &= \frac{\mathbf{1}}{\mathbf{2}} \cdot \frac{1}{3} + \frac{\mathbf{1}}{\mathbf{2}} \cdot \frac{2}{9} \quad = \quad \frac{5}{18}, \end{aligned}$$

while with no information, it's

$$\begin{aligned} E(\theta|s^{(1)} = s^{(2)} = \tfrac{1}{2}) &= E_{s^{(3)}|s^{(1)}=s^{(2)}=\frac{1}{2}} E(\theta|s^{(1)} = s^{(2)} = \tfrac{1}{2}, s^{(3)}) \\ &= E(s^{(3)}|s^{(1)} = s^{(2)} = \tfrac{1}{2}) \\ &= \Pr(\theta = 1|s^{(1)} = s^{(2)} = \tfrac{1}{2})E(s^{(3)}|\theta = 1, s^{(1)} = s^{(2)} = \tfrac{1}{2}) \\ &\quad + \Pr(\theta = 0|s^{(1)} = s^{(2)} = \tfrac{1}{2})E(s^{(3)}|\theta = 0, s^{(1)} = s^{(2)} = \tfrac{1}{2}) \\ &= \frac{\mathbf{1}}{\mathbf{4}} \cdot \frac{1}{3} + \frac{\mathbf{3}}{\mathbf{4}} \cdot \frac{2}{9} \quad = \quad \frac{1}{4}. \end{aligned}$$

3.3 Ex ante expected revenue

While that explains the mechanism through which this effect operates, in terms of magnitude, what we want is not revenue for a particular realization of signals, but expected revenue ex ante. For various values of α and N, I've calculated ex ante expected revenue for the two formats via numerical simulation, simulating 500,000 sets of signals and the resulting auction outcomes. Table 1 gives expected revenue (multiplied by $1,000$ for easier reading); keep in mind the expected value of the asset is exactly $\frac{1}{2}$, so 500 minus revenue gives bidder surplus.

Note that revenue goes to 500, regardless of N, as α approaches either 1 or ∞. As $\alpha \to 1$, the distributions $F(s_i|\theta = 1)$ and $F(s_i|\theta = 0)$ approach each other, so the signals become uninformative; bidders then share the common belief that $E(\theta) = \frac{1}{2}$ and earn no information rents. When α gets large, signals are very informative, but therefore also highly correlated, so information rents once again go to 0.

Table 1: Expected revenue ($\times 1000$), as a function of N, α, and information assumption

	No information					**Full information**				
α	$N=3$	$N=4$	$N=5$	$N=7$	$N=10$	$N=3$	$N=4$	$N=5$	$N=7$	$N=10$
1.1	465	469	471	473	474	465	469	471	474	475
1.5	389	406	415	427	436	394	415	428	446	462
2.0	388	413	428	446	459	401	434	454	476	490
2.5	416	442	456	472	482	432	464	480	493	499
3.0	443	465	477	487	493	458	483	493	499	500
5.0	492	497	499	500	500	496	500	500	500	500
10.0	500	500	500	500	500	500	500	500	500	500

Table 2 gives the increase in expected revenue that would follow from a switch from a no-information to a full-information format – that is, the magnitude of the linkage principle effect relative to expected revenue.

Table 2: Increase in expected revenue from observability of bidder dropouts

α	$N=3$	$N=4$	$N=5$	$N=7$	$N=10$
1.1	0.0%	0.1%	0.1%	0.1%	0.2%
1.5	1.3%	2.3%	3.1%	4.5%	5.8%
2.0	3.3%	5.1%	6.1%	6.8%	6.7%
2.5	3.8%	5.1%	5.2%	4.6%	3.4%
3.0	3.4%	3.8%	3.4%	2.3%	1.3%
5.0	0.8%	0.4%	0.2%	0.1%	0.0%
10.0	0.0%	0.0%	0.0%	0.0%	0.0%

I should note that this example has one feature that may exaggerate the magnitude of this effect. Here, θ takes only two values, with signal distributions differing dramatically between the two. If θ varied over more "intermediate" values rather than taking just two extreme ones, correlation would be reduced, and this effect might be significantly smaller. Another example I tried, with $f(s|\theta=1)=\frac{1}{2}+s$ and $f(s|\theta=0)=\frac{3}{2}-s$, generates a revenue difference of only 0.4% when $N=3$.

4 Related Literature

Cho, Paarsch and Rust (2014) test the revenue ranking result empirically, comparing internet auctions with no feedback to auction-house auctions where losing bidders' bids are observed. They find that the latter do indeed give higher revenue – by about 10 to 15% (when considered gross of auction house fees) – although they concede they cannot rule out this being partly due to differences in the (unobserved) number of potential bidders across the two environments.

Aside from the revenue ranking of "closed" versus "open" ascending auctions, Milgrom and

Weber (1982) give two other results which operate through similar mechanisms: that English (or second-price) auctions should give higher expected revenue than first-price auctions, and that expected revenue should increase when sellers commit to revealing any additional information available which is affiliated with bidder valuations. (These three results are often collectively refered to as the "linkage principle"; Krishna (2002) offers a more general formulation which nests all three.) Levin, Kagel and Richard (1996) test the former (the revenue comparison with first-price auctions) experimentally in a common-values setting. They find the opposite result: while bidders in the English auctions do indeed respond to the drop-out prices of losing bidders, overbidding (bidding above equilibrium levels) in first-price auctions overwhelms this effect. The latter result – on the release of seller information – is examined experimentally by Kagel and Levin (1986), and empirically by Tadelis and Zettelmeyer (2015). Kagel and Levin find that the presence of the effect depends on the size of the auction: in large auctions, an increase in information reduces out-of-equilibrium overbidding and therefore reduces revenue. In the setting considered by Tadelis and Zettelmeyer, there are multiple auctions occurring simultaneously, so the disclosed information improves the matching between bidders and auctions, increasing revenue but through a different channel.

Cho, Paarsch and Rust (2014) also offer a more thorough review of related literature.

References

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