Online-Only Appendix for Daniel Quint, “Common Values and Low Reserve Prices”

D1 Proof of Theorem 1

Fix v_0. As in the text, define $\pi_{CV}(r)$ and $\pi_{PV}(r)$ as the seller's expected profit at reserve price r under the two models, $V^{(k)}$ the k^{th}-highest valuation under the private values model, and

$$v(x,y) \quad = \quad \mathbb{E}\left\{u(x,y,X^{(3-N)})|X^{(1)}=x, X^{(2)}=y\right\}$$

Under the private values model, expected profit is $V^{(2)}-v_0$ if $V^{(2)} \geq r$ (so the winner pays the second-highest bid, equal to the second-highest valuation), $r-v_0$ if $V^{(1)} \geq r > V^{(2)}$ (so the winner pays the reserve price), and 0 otherwise; so we can write it as

$$\begin{aligned}
\pi_{PV}(r) &= \mathbb{E}_X\left\{\mathbf{1}_{V^{(1)}\geq r>V^{(2)}}(r-v_0)+\mathbf{1}_{V^{(2)}\geq r}(V^{(2)}-v_0)\right\} \\
&= \mathbb{E}_X\left\{\mathbf{1}_{V^{(1)}\geq r}(r-v_0)-\mathbf{1}_{V^{(2)}\geq r}(r-v_0)+\mathbf{1}_{V^{(2)}\geq r}(V^{(2)}-v_0)\right\} \\
&= \Pr(V^{(1)}\geq r)(r-v_0)+\mathbb{E}_X\left\{\mathbf{1}_{V^{(2)}\geq r}(V^{(2)}-r)\right\} \\
&= \Pr(V^{(1)}\geq r)(r-v_0)+\mathbb{E}_X \max\{0, V^{(2)}-r\}
\end{aligned}$$

(The second line follows from the first because $V^{(2)} \geq r$ implies $V^{(1)} \geq r$, so $\mathbf{1}_{V^{(1)}\geq r>V^{(2)}} = \mathbf{1}_{V^{(1)}\geq r} - \mathbf{1}_{V^{(2)}\geq r}$.)

As for the common values case, the reserve price binds if $X^{(1)} \geq x^* > X^{(2)}$, and does not bind if $X^{(2)} \geq r$ (in which case the winner pays the second-highest bid $v(X^{(2)},X^{(2)}) = V^{(2)}$), so

$$\begin{aligned}
\pi_{CV}(r) &= \mathbb{E}_X\left\{\mathbf{1}_{X^{(1)}\geq x^*>X^{(2)}}(r-v_0)+\mathbf{1}_{X^{(2)}\geq x^*}\left(v(X^{(2)},X^{(2)})-v_0\right)\right\} \\
&= \mathbb{E}_X\left\{\mathbf{1}_{X^{(1)}\geq x^*}(r-v_0)+\mathbf{1}_{X^{(2)}\geq x^*}\left(v(X^{(2)},X^{(2)})-r\right)\right\} \\
&= \Pr(X^{(1)}\geq x^*)(r-v_0)+\mathbb{E}_X\left\{\mathbf{1}_{X^{(2)}\geq x^*}\left(V^{(2)}-r\right)\right\} \\
&\leq \Pr(X^{(1)}\geq x^*)(r-v_0)+\mathbb{E}_X \max\{0, V^{(2)}-r\}
\end{aligned}$$

Thus,

$$\begin{aligned}
\pi_{CV}(r)-\pi_{PV}(r) &\leq \Pr(X^{(1)}\geq x^*)(r-v_0)+\mathbb{E}_X \max\{0, V^{(2)}-r\} \\
&\quad -\Pr(V^{(1)}\geq r)(r-v_0)-\mathbb{E}_X \max\{0, V^{(2)}-r\} \\
&= \left(\Pr(X^{(1)}\geq x^*)-\Pr(V^{(1)}\geq r)\right)(r-v_0)
\end{aligned}$$

Finally, I show that $X^{(1)} \geq x^* \longrightarrow V^{(1)} \geq r$, and therefore $\Pr(X^{(1)} \geq x^*) \leq \Pr(V^{(1)} \geq r)$. Recall from the text that x^* solves

$$r \quad = \quad \mathbb{E}_{X^{(2)}|X^{(1)}=x^*} v(x^*, X^{(2)})$$

Due to affiliation, $\mathbb{E}\{v(x, X^{(2)})|X^{(1)} = x\}$ is increasing in x; so

$$\begin{array}{rcl} X^{(1)} & \geq & x^* \\ & \downarrow & \\ \mathbb{E}_{X^{(2)}|X^{(1)}}\left\{v\left(X^{(1)}, X^{(2)}\right)\right\} & \geq & r \\ & \downarrow & \\ \mathbb{E}_{X^{(2)}|X^{(1)}}\left\{v\left(X^{(1)}, X^{(1)}\right)\right\} & \geq & r \\ & \downarrow & \\ v\left(X^{(1)}, X^{(1)}\right) & \geq & r \\ & \downarrow & \\ V^{(1)} & \geq & r \end{array}$$

Thus, $\Pr(X^{(1)} \geq x^*) \leq \Pr(V^{(1)} \geq r)$. This means that in the common values setting, any reserve price r is more likely to prevent a sale than in the private-values setting (part (i) of the theorem). If $r \geq v_0$, then

$$\pi_{CV}(r) - \pi_{PV}(r) \quad \leq \quad \left(\Pr(X^{(1)} \geq x^*) - \Pr(V^{(1)} \geq r)\right)(r - v_0) \quad \leq \quad 0$$

proving part (iii); repeating the argument with $v_0 = 0$ gives part (ii). □

D2 Proof of Theorem 2

As in the second-price auction, profits in the private values case are

$$\pi_{PV}(r) \quad = \quad \Pr(V^{(1)} \geq r)(r - v_0) + \mathbb{E}_X \max\{0, V^{(2)} - r\}$$

As for the common values case, x^* is the same as in the second-price auction,

$$r \quad = \quad \mathbb{E}\left\{u\left(x^*, \{X_j\}_{j\neq i}\right) \,\middle|\, X_i = x^*, \max_{j\neq i} X_j < x^*\right\}$$

with a sale occurring if and only if $X^{(1)} \geq x^*$. If $X^{(1)} \geq x^* > X^{(2)}$, the sale occurs at price r. If $X^{(2)} \geq x^*$, then the sale occurs at the price where bidder $i(2)$ drops out; but that price depends on the realizations of the lower signals, and also depends on x^*, since bidder 2 will have learned (from equilibrium bidding) the signals of those bidders above x^*, but not those below x^*.

To capture all this in notation, for $k > 2$, let $Z^{(k)}$ denote a garbling of $X^{(k)}$ which is equal to $X^{(k)}$ if $X^{(k)} \geq x^*$, and equal to 0 otherwise. That is, $Z^{(k)}$ tells you exactly $X^{(k)}$ if $X^{(k)} \geq x^*$; but if $X^{(k)} < x^*$, $Z^{(k)}$ only tells you that $X^{(k)} < x^*$, not its exact value. Thus, the realization of $Z^{(k)}$ is exactly the information the other bidders learn about bidder $i(k)$'s signal from equilibrium bidding in the common-values case. Let $X^{(3-N)} = (X^{(3)}, \ldots, X^{(N)})$ and $Z^{(3-N)} = (Z^{(3)}, \ldots, Z^{(N)})$.

Finally, define

$$R(x, z^{(3-N)}) \quad = \quad \mathbb{E}\left\{u(x, x, X^{(3-N)})|X^{(1)} = X^{(2)} = x, Z^{(3-N)} = z^{(3-N)}\right\}$$

For a given realization $x \geq x^*$ of $X^{(2)}$ and $z^{(3-N)}$ of the "garbled" losing bids $Z^{(3-N)}$, this is the price at which the second-highest bidder will drop out, and therefore the price that is paid. Then we can write

$$\begin{aligned}
\pi_{CV}(r) &= \mathbb{E}_X \left\{ \mathbf{1}_{X^{(1)} \geq x^* > X^{(2)}}(r - v_0) + \mathbf{1}_{X^{(2)} \geq x^*} \left(R(X^{(2)}, Z^{(3-N)}) - v_0 \right) \right\} \\
&= \mathbb{E}_X \left\{ \mathbf{1}_{X^{(1)} \geq x^*}(r - v_0) - \mathbf{1}_{X^{(2)} \geq x^*}(r - v_0) + \mathbf{1}_{X^{(2)} \geq x^*} \left(R(X^{(2)}, Z^{(3-N)}) - v_0 \right) \right\} \\
&= \Pr(X^{(1)} \geq x^*)(r - v_0) + \mathbb{E}_X \left\{ \mathbf{1}_{X^{(2)} \geq x^*} \left(R(X^{(2)}, Z^{(3-N)}) - r \right) \right\} \\
&= \Pr(X^{(1)} \geq x^*)(r - v_0) + \int_{x^*}^{\infty} \left(\mathbb{E}_{Z^{(3-N)}|X^{(2)}=x} \left\{ R(x, Z^{(3-N)}) \right\} - r \right) dF^{(2)}(x)
\end{aligned}$$

where $F^{(2)}(\cdot)$ is the distribution of $X^{(2)}$.

Next, I make a standard "linkage principle" argument that for any x, the "garbling" of losing bids due to the reserve price reduces the expected price paid conditional on $X^{(2)}$. To prove this, note that

$$\begin{aligned}
\mathbb{E}_{Z^{(3-N)}|X^{(2)}=x} R(x, Z^{(3-N)}) &= \mathbb{E}_{Z^{(3-N)}|X^{(2)}=x} \left\{ \mathbb{E}_{X^{(3-N)}|X^{(1)}=x, X^{(2)}=x, Z^{(3-N)}} u(x, x, X^{(3-N)}) \right\} \\
&\leq \mathbb{E}_{Z^{(3-N)}|X^{(2)}=x} \left\{ \mathbb{E}_{X^{(3-N)}|X^{(2)}=x, Z^{(3-N)}} u(x, x, X^{(3-N)}) \right\} \\
&= \mathbb{E}_{X^{(3-N)}|X^{(2)}=x} u(x, x, X^{(3-N)})
\end{aligned}$$

The inequality is because in the second line, the expectation over $X^{(3-N)}$ is conditional on all the different values $X^{(1)}$ could take conditional on $X^{(2)} = x$ – all of which are above x – while in the first line, the expectation is taken conditional on $X^{(1)} = x$; the third line is simply iterated expectations. This means that

$$\begin{aligned}
\pi_{CV}(r) &= \Pr(X^{(1)} \geq x^*)(r - v_0) + \int_{x^*}^{\infty} \left(\mathbb{E}_{Z^{(3-N)}|X^{(2)}=x} \left\{ R(x, Z^{(3-N)}) \right\} - r \right) dF^{(2)}(x) \\
&\leq \Pr(X^{(1)} \geq x^*)(r - v_0) + \int_{x^*}^{\infty} \left(\mathbb{E}_{X^{(3-N)}|X^{(2)}=x} \left\{ u(x, x, X^{(3-N)}) \right\} - r \right) dF^{(2)}(x) \\
&= \Pr(X^{(1)} \geq x^*)(r - v_0) + \mathbb{E}_{X^{(2)}} \mathbf{1}_{X^{(2)} \geq x^*} \left(\mathbb{E}_{X^{(3-N)}|X^{(2)}} \left\{ u(X^{(2)}, X^{(2)}, X^{(3-N)}) \right\} - r \right) \\
&= \Pr(X^{(1)} \geq x^*)(r - v_0) + \mathbb{E}_X \mathbf{1}_{X^{(2)} \geq x^*} \left(u(X^{(2)}, X^{(2)}, X^{(3-N)}) - r \right) \\
&= \Pr(X^{(1)} \geq x^*)(r - v_0) + \mathbb{E}_X \mathbf{1}_{X^{(2)} \geq x^*} \left(V^{(2)} - r \right) \\
&\leq \Pr(X^{(1)} \geq x^*)(r - v_0) + \mathbb{E}_X \max \left\{ 0, V^{(2)} - r \right\}
\end{aligned}$$

and so subtracting,

$$\begin{aligned}\pi_{CV}(r) - \pi_{PV}(r) &\leq \Pr(X^{(1)} \geq x^*)(r - v_0) + \mathbb{E}_X \max\{0, V^{(2)} - r\} \\ &\quad - \Pr(V^{(1)} \geq r)(r - v_0) - \mathbb{E}_X \max\{0, V^{(2)} - r\} \\ &= \left(\Pr(X^{(1)} \geq x^*) - \Pr(V^{(1)} \geq r)\right)(r - v_0)\end{aligned}$$

Now, recall that x^* is the value of x solving

$$r = \mathop{\mathbb{E}}_{\{X_j\}_{j\neq i}|X_i=x} \left\{ u\left(x, \{X_j\}_{j\neq i}\right) \middle| \max_{j\neq i} X_j < x \right\}$$

and that due to affiliation, the right-hand side is strictly increasing in x. Thus, $X^{(1)} \geq x^*$ is equivalent to

$$r \leq \mathop{\mathbb{E}}_{\{X_j\}_{j\neq i}|X^{(1)}} \left\{ u\left(X^{(1)}, \{X_j\}_{j\neq i}\right) \middle| \max_{j\neq i} X_j < X^{(1)} \right\} = R^{(1)}$$

as defined in the text; so

$$\begin{aligned}\pi_{CV}(r) - \pi_{PV}(r) &\leq \left(\Pr(R^{(1)} \geq r) - \Pr(V^{(1)} \geq r)\right)(r - v_0) \\ &= \left((1 - F_{R^{(1)}}(r)) - (1 - F_{V^{(1)}}(r))\right)(r - v_0) \\ &= \left(F_{V^{(1)}}(r) - F_{R^{(1)}}(r)\right)(r - v_0)\end{aligned}$$

Recall that $\tilde{r}$ was defined in the text such that $r \geq \tilde{r}$ implies $F_{V^{(1)}}(r) \leq F_{R^{(1)}}(r)$, proving part (i). If $r \geq \tilde{r}$ and $r \geq v_0$, then $\pi_{CV}(r) - \pi_{PV}(r) \leq \left(F_{V^{(1)}}(r) - F_{R^{(1)}}(r)\right)(r - v_0) \leq 0$ (part (iii)). Repeating the argument with $v_0 = 0$ establishes $Rev_{CV}(r) \leq Rev_{PV}(r)$ (part (ii)). □

D3 Proof of Result 1

Let $Rev_\lambda(r)$ and $\pi_\lambda(r)$ denote expected revenue and profit, respectively, at reserve price r given a value of λ. First, note that $Rev_\lambda(0)$ does not depend on λ. This is because given realizations $(x^{(1)}, \ldots, x^{(N)})$ of the signals, the second-highest bidder drops out at the price

$$\begin{aligned}&E\left(V_i | X^{(1)} = X^{(2)} = X_i = x^{(2)}, X^{(3-N)} = x^{(3-N)}\right) \\ &= (1 - \lambda)\frac{N+2}{2N}x^{(2)} + \lambda\frac{1}{N}\left(x^{(2)} + x^{(2)} + x^{(3)} + \ldots + x^{(N)}\right)\end{aligned}$$

Since, conditional on the realization $x^{(2)}$ of $X^{(2)}$, the types of the bidders who dropped out before him are independently uniform on $[0, x^{(2)}]$, this has expected value, given $x^{(2)}$, of

$$(1-\lambda)\frac{N+2}{2N}x^{(2)} + \lambda\frac{1}{N}\left(2x^{(2)} + E\left\{X^{(3)} + \ldots + X^{(N)}|X^{(2)} = x^{(2)}\right\}\right)$$

$$= \quad (1-\lambda)\frac{N+2}{2N}x^{(2)} + \lambda\frac{1}{N}\left(2x^{(2)} + (N-2)\frac{1}{2}x^{(2)}\right)$$

$$= \quad (1-\lambda)\frac{N+2}{2N}x^{(2)} + \lambda\frac{N+2}{2N}x^{(2)} \quad = \quad \frac{N+2}{2N}x^{(2)}$$

which does not depend on λ.

Second, as noted in the text, $x^* = \frac{2N}{N+2-\lambda}r$ is increasing in λ, so the probability of a sale, which is $\Pr(X^{(1)} > x^*)$, is decreasing in λ, giving part (i).

Third, note that conditional on a realization $x^{(2)} > x^*$ of $X^{(2)}$, the expected price at which the second-highest bidder drops out is independent of x^*. To see this, consider the case of $N = 3$. Given a realization of X, renumber the bidders such that $X_1 > X_2 > X_3$. When bidder 2 drops out, bidder 3 is already out. If he bid and then dropped out, the realization $x^{(3)}$ of $X^{(3)}$ would be inferred, and bidder 2 would drop out at price

$$B(x^{(2)}, x^{(3)}) \quad = \quad (1-\lambda)\frac{5}{6}x^{(2)} + \lambda\frac{1}{3}(x^{(2)} + x^{(2)} + x^{(3)}) \quad = \quad \left(\frac{5}{6} - \frac{1}{6}\lambda\right)x^{(2)} + \frac{\lambda}{3}x^{(3)}$$

If bidder 3 did not bid because of a reserve price r, then bidder 2 takes the expectation of $B(x^{(2)}, x^{(3)})$ over the values of $X^{(3)}$ at which bidder 3 would not have bid. Importantly, conditional on $X^{(2)} = x^{(2)}$, the distribution of $X^{(3)}$ is uniform over $[0, x^{(2)}]$; so overall, the expected price at which bidder 2 drops out, given $X^{(2)} = x^{(2)} > x^*$, is

$$\begin{aligned}
\overline{B}(x^{(2)}) \quad &\equiv \quad \frac{x^*}{x^{(2)}}\mathbb{E}_{X^{(3)}<x^*}B(x^{(2)}, X^{(3)}) + \frac{x^{(2)} - x^*}{x^{(2)}}\int_{x^*}^{x^{(2)}} B(x^{(2)}, x)\frac{dx}{x^{(2)} - x^*} \\
&= \quad \frac{x^*}{x^{(2)}}\left(\left(\frac{5}{6} - \frac{1}{6}\lambda\right)x^{(2)} + \frac{\lambda}{3}\frac{x^*}{2}\right) + \frac{1}{x^{(2)}}\int_{x^*}^{x^{(2)}}\left(\left(\frac{5}{6} - \frac{1}{6}\lambda\right)x^{(2)} + \frac{\lambda}{3}x\right)dx \\
&= \quad \left(\frac{5}{6} - \frac{1}{6}\lambda\right)x^* + \frac{\lambda}{6}\frac{(x^*)^2}{x^{(2)}} + \frac{x^{(2)} - x^*}{x^{(2)}}\left(\frac{5}{6} - \frac{1}{6}\lambda\right)x^{(2)} + \frac{1}{x^{(2)}}\frac{\lambda}{6}\left((x^{(2)})^2 - (x^*)^2\right) \\
&= \quad \left(\frac{5}{6} - \frac{1}{6}\lambda\right)x^{(2)} + \frac{1}{x^{(2)}}\frac{\lambda}{6}\left((x^{(2)})^2\right) \\
&= \quad \frac{5}{6}x^{(2)}
\end{aligned}$$

which does not depend on x^* (or λ) at all.[22] The same holds for general N: conditional on

[22]This is in contrast to the affiliated case, such as in the proof of Theorem 2, where truncation of losing

$X^{(2)} = x^{(2)} > x^*$, the expected price at which the second-highest bidder drops out is $\frac{N+2}{2N} x^{(2)}$, and does not depend on x^*. Thus, if we let F_2 denote the CDF of $X^{(2)}$ and $B(X|x^*)$ denote the price at which the second-highest bidder will drops out on the equilibrium path, we can write expected profit as

$$\begin{aligned}
\pi_\lambda(r) &= \mathbb{E}_X \left\{ \mathbf{1}_{X^{(1)} \geq x^* > X^{(2)}} (r - v_0) + \mathbf{1}_{X^{(2)} \geq x^*} (B(X|x^*) - v_0) \right\} \\
&= \mathbb{E}_X \left\{ \mathbf{1}_{X^{(1)} \geq x^*} (r - v_0) + \mathbf{1}_{X^{(2)} \geq x^*} (B(X|x^*) - r) \right\} \\
&= \Pr(X^{(1)} \geq x^*)(r - v_0) + \int_{x^*}^1 \left(\mathbb{E}_{X|X^{(2)}=x} B(X|x^*) - r \right) dF^{(2)}(x) \\
&= \Pr(X^{(1)} \geq x^*)(r - v_0) + \int_{x^*}^1 \left(\overline{B}(x) - r \right) dF^{(2)}(x)
\end{aligned}$$

where $\overline{B}(x) = \frac{N+2}{2N} x$.

Now, if $r \geq v_0$, the first term is decreasing in x^*. $\overline{B}(x^*) = \frac{N+2}{2N} \frac{2N}{N+2-\lambda} r = \frac{N+2}{N+2-\lambda} r > r$, so the second term is strictly decreasing in x^* (since the integrand is strictly positive at $x = x^*$). Since $\overline{B}$ is independent of λ, λ effects π_λ only through x^*, which is strictly increasing in λ; so $\pi_\lambda(r)$ is decreasing in λ for $r \geq v_0$. That gives part (iii) of the result; repeating with $v_0 = 0$ gives part (ii).

To prove the last part of the result, let $F^{(1)}$ denote the CDF of $X^{(1)}$, and differentiate π_λ to get

$$\begin{aligned}
\pi_\lambda(r) &= (1 - F^{(1)}(x^*))(r - v_0) + \int_{x^*}^1 (\overline{B}(x) - r) dF^{(2)}(x) \\
&\downarrow \\
\pi'_\lambda(r) &= 1 - F^{(1)}(x^*) - (x^*)' f^{(1)}(x^*)(r - v_0) - (1 - F^{(2)}(x^*)) - (x^*)' \left(\overline{B}(x^*) - r \right) f^{(2)}(x^*) \\
&= F^{(2)}(x^*) - F^{(1)}(x^*) - (x^*)' f^{(1)}(x^*)(r - v_0) - (x^*)' \left(\overline{B}(x^*) - r \right) f^{(2)}(x^*)
\end{aligned}$$

where $(x^*)'$ is the derivative of x^* with respect to r. If we are in the range of r where $x^* < 1$, then $x^* = \gamma r$, where $\gamma = \frac{2N}{N+2-\lambda}$, and $(x^*)'$ is therefore equal to γ. $\overline{B}(x^*) - r = \frac{N+2}{2N} x^* - r = \frac{N+2}{N+2-\lambda} r - r = \frac{\lambda}{N+2-\lambda} r$; so we can write this as

$$\pi'_\lambda(r) = F^{(2)}(\gamma r) - F^{(1)}(\gamma r) - \gamma f^{(1)}(\gamma r)(r - v_0) - \gamma \left(\tfrac{\lambda}{N+2-\lambda} r \right) f^{(2)}(\gamma r)$$

Since $\{X_i\}$ are independently uniform, $F^{(2)}(x) = N x^{N-1} - (N-1) x^N$ and $F^{(1)}(x) = x^N$

bids reduces expected revenue. When a losing bid is not observed, its expectation is taken by the second-highest bidder, conditional on $X^{(1)} = X^{(2)} = x^{(2)}$. With affiliated signals, this is pessimistic relative to the truth that $X^{(1)} > X^{(2)} = x^{(2)}$; but with independent signals, once we're already conditioning on the value of $X^{(2)}$, the distribution of the lower signals does not depend on the value of $X^{(1)}$, and therefore this effect vanishes. Similarly, with independent signals, second-price sealed-bid auctions are revenue-equivalent to English auctions, while with affiliated signals, the latter are strictly revenue-superior.

(properties of order statistics), so

$$\begin{aligned}\pi'_\lambda(r) &= N(\gamma r)^{N-1}(1-\gamma r) - \gamma N(\gamma r)^{N-1}(r-v_0) - \gamma r\left(\tfrac{\lambda}{N+2-\lambda}\right) N(N-1)(\gamma r)^{N-2}(1-\gamma r) \\ &= N(\gamma r)^{N-1}(1-\gamma r)\left[1 - \tfrac{\gamma(r-v_0)}{1-\gamma r} - \tfrac{\lambda}{N+2-\lambda}(N-1)\right]\end{aligned}$$

Dividing by $N(\gamma r)^{N-1}(1-\gamma r)$ preserves sign, so

$$\pi'_\lambda(r) \overset{\text{sign}}{=} 1 - \frac{\gamma(r-v_0)}{1-\gamma r} - \frac{\lambda(N-1)}{N+2-\lambda}$$

This is strictly decreasing in r, so π is strictly quasiconcave. As long as $\gamma v_0 < 1$ (or $x^* < 1$ at $r = v_0$), this is strictly positive at $r = v_0$ (since the second term vanishes and the third term is strictly smaller than 1), and strictly negative as $r \to \frac{1}{\gamma}$ (since the denominator of the second term goes to 0 while the numerator remains positive), so $\pi_\lambda(r)$ has a unique maximizer characterized by $\pi'_\lambda(r) = 0$. And finally, recalling that $\gamma = \frac{2N}{N+2-\lambda}$ is increasing in λ, this last expression is strictly decreasing in λ; so where $\pi'_\lambda(r) = 0$, $\pi'_{\lambda'}(r) < 0$ for $\lambda' > \lambda$, meaning that $\arg\max \pi_{\lambda'}(r) < \arg\max \pi_\lambda(r)$, proving the final claim.

Also note that evaluating the integral in the expression for π_λ, plugging in the expressions for $\overline{B}$, F_2, and F_1, and simplifying gives $\pi_\lambda(r) =$

$$N(x^*)^{N-1}(1-x^*)r + \frac{(N+2)(N-1)}{2N(N+1)}\left(1-(N+1)(x^*)^N + N(x^*)^{N+1}\right) - (1-(x^*)^N)v_0$$

which was used for the charts in Figure 11. □

D4 Unverifiable Seller Information

Recall that we're in the discrete θ example, with $N = 3$, $\alpha = 2$, and $v_0 = 0.7\theta$; we're supposing that the seller gets a signal S with $\Pr(S = 0|\theta = 0) = \Pr(S = 1|\theta = 1) = 0.75$ and $\Pr(S = 1|\theta = 0) = \Pr(S = 0|\theta = 1) = 0.25$, and can choose to condition the reserve price on the value of S, using reserve as a costly signal. I consider equilibria where the seller plays a pure strategy; there are two types of equilibria.

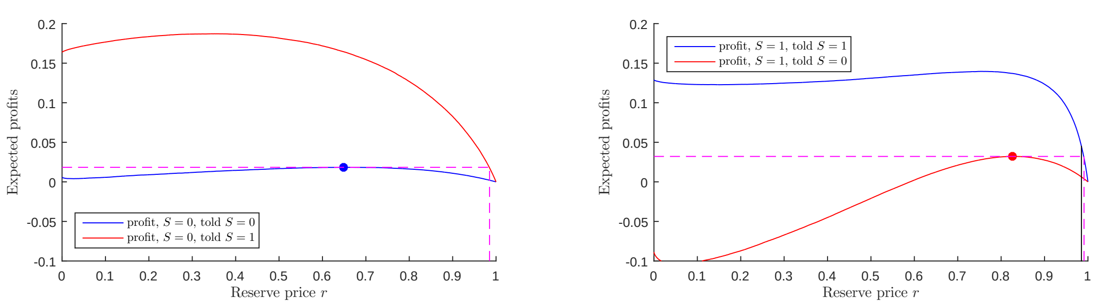
Separating Equilibria

Here, a seller reveals his signal through his choice of reserve price, so bidders correctly infer the value of S on the equilibrium path. Let $\widehat{r}_0$ denote the equilibrium reserve price set when $S = 0$, and $\widehat{r}_1$ the reserve set when $S = 1$. To deter other reserve prices, I set off-equilibrium-path beliefs to $\Pr(S = 0|r \notin \{\widehat{r}_0, \widehat{r}_1\}) = 1$. For $(\widehat{r}_0, \widehat{r}_1)$ to be an equilibrium, a seller with $S = 1$ must prefer setting $r = \widehat{r}_1$ (and the subsequent belief that $S = 1$) to setting any other reserve price and being met with the belief that $S = 0$. A seller with $S = 0$ must prefer setting $r = \widehat{r}_0$ and revealing that $S = 0$ to any other reserve price with the belief $S = 0$, and also to the reserve price $\widehat{r}_1$ with the belief that $S = 1$. If we let $\pi(r, p, q)$ denote the seller's expected profit at reserve price r when $S = p$ and the buyers believe that $S = q$, separating equilibrium requires

$$\begin{aligned} \pi(\widehat{r}_0, 0, 0) &= \max_r \pi(r, 0, 0) \\ \pi(\widehat{r}_0, 0, 0) &\geq \pi(\widehat{r}_1, 0, 1) \\ \pi(\widehat{r}_1, 1, 1) &\geq \max_r \pi(r, 1, 0) \end{aligned}$$

Figure 15 illustrates these constraints for our numerical example. On the left pane, the blue curve is $\pi(r, 0, 0)$, which is maximized at $r = 0.648$, which the low-type seller must set in equilibrium. This gives expected profit of 0.0184. The red curve is $\pi(r, 0, 1)$ – the low-type seller's expected profit if he could convince the buyers that $S = 1$. The magenta (dashed) line shows that if $\widehat{r}_1 < 0.985$, the low-type seller would choose to imitate the high type; so equilibrium requires $\widehat{r}_1 \geq 0.985$.

Figure 15: IC constraints for separating equilibria



On the right pane, the blue curve is $\pi(r, 1, 1)$ – the high-type seller's payoff on the equilibrium path. The red curve is $\pi(r,1,0)$ – the payoff a high-type seller could get by deviating to a reserve $r \neq \widehat{r}_1$. If he chose to deviate, the red dot shows his optimal deviation would be to $r = 0.826$, giving expected profit of 0.0321. So $\widehat{r}_1$ (combined with beliefs that $S = 1$) must give him a payoff higher than that. The magenta (dashed) line shows that this requires $\widehat{r}_1 \leq 0.991$, and the solid black line shows $r = 0.985$.

So there is a continuum of separating equilibria, all with $\widehat{r}_0 = 0.648$, and with $\widehat{r}_1 \in [0.985, 0.991]$. The best of these ($\widehat{r}_1 = 0.985$) gives the high type an expected payoff of 0.045. Thus, the best separating equilibrium gives expected profit of $\frac{1}{2}0.0184 + \frac{1}{2}0.045 = 0.0317$, about *half* the expected payoff the seller would get if he had not learned S and set r optimally. What's happending here is that buyer beliefs about S have a strong effect on seller profit regardless of the true realization of S, and as a result, the high type seller must dissipate a large part of all his rents by setting a reserve high enough that the low type won't imitate it.

Pooling Equilibria

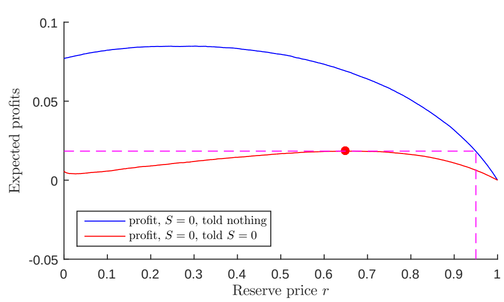
Second, I consider pooling equilibria. To support a pooling equilibrium with reserve price $\widehat{r}$, I again assign beliefs $\Pr(S = 0) = 1$ if the seller sets any other (off-equilibrium-path) reserve price. For $\widehat{r}$ to be a pooling equilibrium, two conditions must hold:

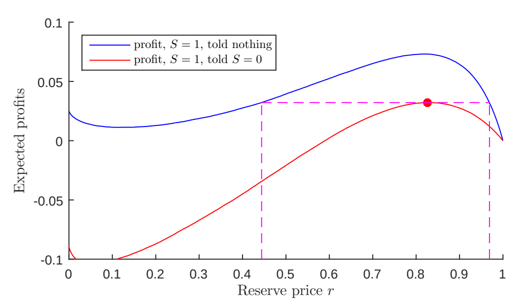
$$\begin{aligned} \pi(\widehat{r}, 1, -) &\geq \max_r \pi(r, 1, 0) \\ \pi(\widehat{r}, 0, -) &\geq \max_r \pi(r, 0, 0) \end{aligned}$$

where $\pi(r, p, -)$ denotes expected profit to a seller with signal $S = p$ when buyers do not infer anything about S. Thus, the two conditions say that both buyer types prefer to pool at $r = \widehat{r}$ (with buyers inferring nothing) to deviating to any other reserve price and having buyers infer that $S = 0$.

Figure 16 illustrates the two types' incentive constraints for a pooling equilibrium. In the

Figure 16: IC constraints for pooling equilibria



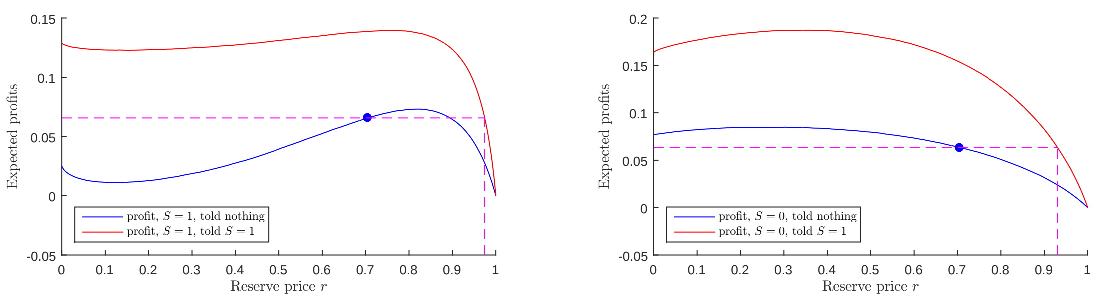


left pane, the blue (top) curve is $\pi(r, 0, -)$, and indicates the low type of seller's equilibrium-path payoff at each potential value of $\widehat{r}$; the red (bottom) curve is $\pi(r, 0, 0)$, and shows the profit he could get from deviating from $r = \widehat{r}$. If he were to deviate, his optimal deviation would be to $r = 0.648$, giving expected profit 0.0184 when buyers believe $S = 0$; thus, equilibrium requires $\pi(\widehat{r}, 0, -) \geq 0.0184$. The magenta (dahsed) line shows that this holds for any $\widehat{r} \leq 0.949$.

The right pane shows the same exercise for the high type. Again, the blue (top) curve is the equilibrium payoff at various potential $\widehat{r}$; the red is the payoff from deviating. In this case, the incentive constraint is satisfied for any $\widehat{r} \in [0.445, 0.968]$. Intersecting the two, any reserve $\widehat{r} \in [0.445, 0.949]$ is a pooling equilibrium. This includes the "optimal" pooling equilibrium, $\widehat{r} = 0.705$, which gives expected profit of 0.0646.

However, Figure 17 shows that this optimal pooling equilibrium fails the "intuitive criterion" of Cho and Kreps (1987).[23] In the left pane, the blue dot shows the high type's

Figure 17: Failure of the Cho Kreps condition by the optimal pooling equilibrium



equilibrium payoff in the optimal pooling equilibrium, which is 0.0657. The red curve is his expected payoff if buyers believed $S = 1$. The magenta dashed line shows that he would happily set any reserve price up to 0.974 if it would convince buyers that $S = 1$.

The right pane shows the same exercise for the low type. The low type's equilibrium payoff at $\widehat{r} = 0.705$ is 0.0635. The red curve, and magenta line, show that even with buyer beliefs $S = 1$, the low type's expected payoff for $r > 0.93$ would be below 0.0635.

Thus, in a pooling equilibrium with $\widehat{r} = 0.705$, a high type could set a reserve of, say, 0.95, and try to convince bidders that S must be 1, because if $S = 0$, he wouldn't have been willing to set such a high reserve *even if he thought it would convince bidders that* $S = 1$; hence, the pooling equilibrium fails the Cho-Kreps criterion.

While I haven't done the same exercise for every possible pooling equilibrium, it appears they will all fail Cho-Kreps, because the right side of the red curve is so much steeper for the high type than for the low, making him "more willing" to set an extremely high reserve, thus making it likely that some such deviation would exist for every pooling equilibrium. (It is clear that any pooling equilibrium with $\widehat{r} < 0.705$ would fail the condition, since the high type does worse and is therefore willing to go even higher with his deviation, while the low type does better on the equilibrium path.)

[23] Cho, In-Koo, and David Kreps (1987), "Signaling Games and Stable Equilibria," *Quarterly Journal of Economics* 52(2).