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ScienceDirect
Journal of Economic Theory 132 (2007) 147–166
JOURNAL OF Economic Theory
www.elsevier.com/locate/jet

Multilateral bargaining with concession costs☆

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Received 23 June 2005; final version received 14 September 2005
Available online 18 November 2005

Abstract

This paper presents a new non-cooperative approach to multilateral bargaining. We consider a demand game with the following additional ingredients: (i) there is an exogenous deadline, by which bargaining has to end; (ii) prior to the deadline, players may sequentially change their demands as often as they like; (iii) changing one's demand is costly, and this cost increases as the deadline gets closer. The game has a unique subgame perfect equilibrium prediction in which agreement is reached immediately and switching costs are avoided. Moreover, this equilibrium is invariant to the particular order and timing in which players make demands. This is important, as multilateral bargaining models are sometimes too sensitive to these particular details. In our context, players with higher concession costs obtain higher shares of the pie; their increased bargaining power stems from their ability to credibly commit to a demand earlier. We discuss how the setup and assumptions are a reasonable description for certain real bargaining situations.
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JEL classification: C72; C73; C78

Keywords: Bargaining; Commitment; Switching costs

1. Introduction

Non-cooperative analyses of bargaining are sometimes criticized for being sensitive to the exact description of the extensive form game. This concern is especially acute in the case of multilateral bargaining. In general, one needs to precisely specify the rules and timing for offers, counter-offers,

☆ We thank an anonymous referee, the associate editor, and seminar participants at Universidad Complutense and at the Second World Congress of the Game Theory Society in Marseille for helpful comments.

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doi:10.1016/j.jet.2005.09.001

vetoes, exit options, and so on. All these details may have an impact on the outcome of the bargaining process. But this is at odds with the perception that negotiations are by nature very amorphous processes in which these details seem not to play such an important role. [1] The relevant aspects have more to do with the ability to credibly commit to certain threats or promises. This paper tries to construct a model that abstracts from the former technical aspects and focuses more on the latter commitment opportunities. [2]

Bargaining is normally studied using either the axiomatic approach introduced by Nash [21], [3] or the strategic approach, for which Rubinstein's [23] alternating offer model is probably the most influential. Rubinstein's model is constructed for two players and has a unique equilibrium which implements Nash's axiomatic outcome in a non-cooperative way. The uniqueness of the equilibrium prediction, however, is lost when more players are introduced. Herrero [10] and Haller [9] show that any feasible agreement can be obtained in a subgame perfect equilibrium of the most natural extension of Rubinstein's bargaining protocol to three players. [4] This is due to the veto power that each player possesses: he can void any agreement made by the other players.

Krishna and Serrano [11] illustrate that multilateral versions of Rubinstein's model can get around this problem and restore uniqueness by introducing an "exit option." After a proposal has been made, the exit option allows any player to accept his offered share, leave the bargaining table, and let the remaining $N-1$ players bargain over the rest of the pie. [5] Such an exit option is a realistic description of some, but not all, bargaining situations. In legislative budget negotiations, for example, no party can secure funding until the entire budget is approved. Another feature of Krishna and Serrano's model (and of many other models of multilateral bargaining) is that when a player makes a proposal, he specifies the exact division of the pie among all parties. We will refer to this as an offer. An alternative approach, and the one used in this paper, is that parties can only express how much they demand for themselves, without stipulating the division among the others. In many situations this seems more realistic, as for example in the financing of public goods. [6] Selten [26] and Winter [28,29] use the demand approach for the study of multilateral bargaining, but their outcomes depend on the exogenously pre-specified order of play. [7]

The model we present builds on the framework proposed in Caruana and Einav [3]. Bargaining must end before a fixed deadline, and players make demands sequentially in nearly continuous time. They can revise their demands as often as they like, but this is costly, and this cost increases as the deadline gets closer. The result of the bargaining process is successful only if the final demands are compatible with each other. Earlier demands, however, serve as a commitment mechanism, as reducing one's demand becomes increasingly expensive. This assumption imposes some aspect of irreversibility to past actions. This is similar to Admati and Perry's [2] contribution game and to

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[1] See also the introduction of Perry and Reny [22] for more on this issue.

[2] The idea that commitment plays an important role in bargaining goes back to Schelling [25]. More recently, Myerson [20] and Abreu and Gul [1] provide a formal treatment using asymmetric information and reputation. As discussed further in Section 4, the approach and results in this paper are very different.

[3] See Lensberg [13] for a modern treatment of the multilateral case.

[4] See Sutton [27] for a more general review.

[5] A similar result is obtained by Chae and Yang [4], who achieve uniqueness by modeling multilateral bargaining as a sequence of bilateral agreements.

[6] This distinction has no bite in bilateral situations, where a demand uniquely determines the offer to the other party.

[7] Morelli [18] extends their framework to allow for an endogenously determined order, but still within a particularly defined protocol.

other papers that consider irreversible actions. [8] All these papers, however, impose a very drastic switching cost structure: from the very beginning players are fully committed not to increase their demands from previously demanded levels (they face infinite costs of doing so), but are always free to decrease their demands. In contrast, our paper deals with finite concession costs that increase in a smooth way. In this manner we can focus on the process by which players achieve commitment. Moreover, as we show later on, all that is needed in the model is that the costs of conceding (reducing one's demand) increase over time; no structure is imposed on the costs of increasing one's demand. Thus, one can use our setting to analyze both the case in which players cannot back out of previous promises, as in the case of irreversible actions; and the case in which one is always free to withdraw from earlier promises. At first this latter case may seem striking: it is not obvious how commitment is achieved when players are not bound by previous offers. Note, however, that while demanding more appears to be costless, dynamically it is not. If one party fancifully makes an extremely high demand, he will later have to concede, incurring a cost.

The game has a unique equilibrium in which agreement is reached immediately and switching costs are avoided. Theorem 2 presents our main result: if players can revise their demands as often as they want, or more precisely, as the time between consecutive moves goes to zero, this equilibrium converges to a split of the pie which is invariant to the specific order and timing of moves. It is in this sense that we claim that the model abstracts from the details of the protocol. In the case of symmetric players the model predicts equal shares for all players. If players are asymmetric, those with higher concession costs obtain higher shares of the pie, as they are able to commit faster to particular demands. [9] Thus, in our setting higher concession costs imply higher bargaining power. While the capability of an organization to be flexible is generally considered a positive feature, in this setting it results in a loss of bargaining power. This suggests a rationale for rigid structures as bargaining devices. The difficulty of organizing a board of directors meeting, complex bureaucratic structures, posted prices, or having a clerk with no discretion at the shop counter are only some examples.

To gain intuition for the equilibrium outcome, consider two identical players bargaining over a dollar. We argue that it is perfectly credible for a player to hold firm to a demand of 50 cents. Just after the point in time at which concession costs increase above 50 cents, this player is committed to never reduce his demand below 50 cents. Thereby, just before this critical point in time, if the other player had started by asking "too much," he is better off conceding and scaling down his demand to 50 cents as well. In this manner, he will obtain positive payoffs, compared to payoffs of zero if no adjustment is made (resulting in no agreement). This argument can be made for both players, and thus, in equilibrium, each starts by demanding 50 cents and never changes thereafter, avoiding any switching costs. The same logic extends to situations with more than two players, and when players are not identical.

We are aware, of course, that our dynamic structure does not adequately describe all bargaining situations. If the reader imagines a series of rounds at a bargaining table, it is difficult to justify the presence of increasing switching costs. Nevertheless, in our view, there are many relevant

[8] See, for example, Saloner [24], Gale [8], Lockwood and Thomas [15], and Compte and Jehiel [5]. In the latter two, as well as in the *contribution* game of Admati and Perry [2], gradualism is an important feature of equilibrium. In our setup agreement is achieved immediately because actions are reversible, and do not directly affect commitment opportunities of opponents. This is somewhat similar to Admati and Perry's [2] *subscription* game.

[9] Muthoo [19] presents a different two-player two-period bargaining model with commitment in which he obtains a similar qualitative result.

situations which are characterized by both an exogenous deadline and an increasing switching cost structure. Consider, for example, New York's (failed) bid to host the 2012 Olympic Games. The deadline was provided by the International Olympic Committee, which selected the host in July 2005. The bid involved negotiations among multiple interested parties—New York City, New York State, the U.S. Federal Government, and various representatives of the private sector—over their relative contributions to improve New York's chances. Such contributions required specific investments which could not be fully recouped should the bid fail. To the extent that adding or subtracting from such investments would become more costly as the July 2005 deadline approached, our framework may fit this situation. More generally, switching costs could represent actual costs of revising contracts, financial or legal costs, inconvenience, or reputational concerns. These are discussed in more detail in Section 5.

In Section 4 we argue that the scope of the main result of the paper applies more broadly. First, we extend our basic bargaining model to cover a public good game. Second, we show (for the case of two players) that the main equilibrium prediction does not rely on the knowledge of the particular order and timing of moves. Even if players do not know exactly when they or their opponents will play, common knowledge that players can play very often suffices. Third, we present a similar bargaining model, but with flow payoffs and constant (but small) switching costs. [10] Such a situation better describes an ongoing bargaining relationship. For example, one can think of the daily division of labor among members of a household, or of the decision on how to share the flow of profits among members of a patent pool. While this model is of a distinct nature from the main model of the paper, it shares an analogous equilibrium structure and comparative statics.

2. The model

The model is an application of the framework proposed by Caruana and Einav [3]. Consider N players who bargain over a pie of size 1. Time is discrete. The game starts at $t = 0$ and ends at a predetermined deadline $t = T$. Each player i acts at a large but finite time grid $g_i = \{t_1^i, t_2^i, \ldots, t_{L_i}^i\}$ where $t_k^i \in [0, T]$ for all k and $t_l^i < t_m^i$ if $l < m$. Players play sequentially, so $g_i \cap g_j = \emptyset$ for any $i \neq j$. When player i acts at $t \in g_i$, he states some demand $a_i(t) \in A_i = [0, 1]$. At every point in time all previous actions are common knowledge. For any point in time $t \in [0, T]$, denote the time of player i's next move by $next_i(t) = \min\{t' \in g_i \mid t' \geqslant t\}$, and the time of player i's last move by $prev_i(t) = \max\{t' \in g_i \mid t' < t\}$. Let also $next(t) = \min\{t' \in \bigcup_i g_i \mid t' > t\}$ be the time of the next move after t.

The first move by player i, taken at $t_1^i = next_i(0)$, is costless. However, if he later (at $t > t_1^i$) changes his action, he has to pay a switching cost. If he concedes by changing his demand downwards, he pays a concession cost $c_i(t)$. If he demands more by changing his demand upwards, he pays demand costs $d_i(t)$. We place no restriction on demand costs, except that $d_i(t) > 0$ for any t. [11] We impose the following assumptions on the concession cost function: $c_i(t)$ is strictly increasing in t with $c_i(0) = 0$ and $c_i(T) > 1$. These assumptions capture the idea that conceding is very cheap early in the process, but prohibitively expensive just before the deadline.

[10] This is based on the game structure proposed by Lipman and Wang [14]. See also Marx and Matthews [17], who analyze a finite-horizon public good game situation with flow payoffs.

[11] The assumption that demand costs, $d_i(t)$, are strictly positive is only made for convenience. Assuming weak inequality, i.e. $d_i(t) \geqslant 0$, does not change the equilibrium outcome and payoffs, but slightly complicates the analysis.

Finally, we specify payoffs. Denote player i's actions by $\overline{a}_i = (a_i(t))_{t\in g_i}$, all actions of all players by $\overline{a} = (\overline{a}_i)_{i\in N}$, and the final actions by all players by $a^* = (a_i(t^i_{L_i}))_{i\in N}$. Player i's payoffs are

$$u_i(\overline{a}) = \pi_i(a^*) - \sum_{\{t\in g_i-\{t^i_1\}:a_i(t)<a_i(prev_i(t))\}} c_i(t) - \sum_{\{t\in g_i-\{t^i_1\}:a_i(t)>a_i(prev_i(t))\}} d_i(t), \qquad (1)$$

where $\pi_i(a^*)$ is the usual demand game payoff

$$\pi_i(a^*) = \begin{cases} a^*_i & \text{if } \sum a^*_j \leqslant 1, \\ 0 & \text{if } \sum a^*_j > 1 \end{cases} \qquad (2)$$

evaluated at the players' final demands.

The solution concept that we use is subgame perfect equilibrium (spe). While much of the analysis is carried out for arbitrary grids, our main interest lies in fine, nearly continuous grids. Thus, we define the *fineness* of a player's grid as $\varphi(g_i) = \max\{t^i_1, t^i_2 - t^i_1, t^i_3 - t^i_2, \ldots, T - t^i_{L_i}\}$, and denote the *game grid* by $g = \{g_i\}_{i=1}^N$ and its fineness by $\varphi(g) = \max_i \{\varphi(g_i)\}$. Our main result (Theorem 2) is a limiting result, when $\varphi(g)$ goes to zero.

3. Results and discussion

3.1. Subgame perfect equilibrium

In this section we first solve for the equilibrium path of the game given a specific grid g.[12] We show that on the equilibrium path an agreement is reached immediately and therefore switching never occurs. Each player's share of the surplus is uniquely determined by the game grid and the switching cost structure. Later we will focus on the limit of the equilibrium outcomes as the fineness of the grid tends to zero. In this manner we will be able to abstract from the grid and show that the equilibrium does not depend on the particular order in which players get to play.

Given a game with cost structure $\{c_i(\cdot), d_i(\cdot)\}_{i=1}^N$ and grid g, define

$$t^* \equiv \max\left\{ t \in \bigcup_i g_i \,\middle|\, \sum_j c_j(next_j(t)) \leqslant 1 \right\} \qquad (3)$$

and

$$\theta_i \equiv \begin{cases} c_i(next_i(t^*)) & \text{if } t^* \notin g_i, \\ 1 - \sum_{j\neq i} c_j(next_j(t^*)) & \text{if } t^* \in g_i. \end{cases} \qquad (4)$$

Note that by construction $\sum_j \theta_j = 1$ and that $c_i(t^*) \leqslant \theta_i < c_i(next_i(next(t^*)))$ for player i who moves at $t^* \in g_i$.

Our main result is that the equilibrium path of the bargaining game involves each player i demanding θ_i the first time he plays and never switching thereafter. Since the path of play does not depend on the costs of increasing one's demand, $d_i(t)$, we largely ignore these cost functions.

[12] Strictly speaking, the equilibrium need not be unique, due to the fact that players are sometimes indifferent between two actions. Still, as we show below, the important elements of the equilibrium, namely, actions on the equilibrium path and payoffs, are indeed unique.

Much of our result is proved in the appendix. However, a few definitions and results which aid in the proof are instructive. Throughout, we abuse notation by describing each subgame by (a, t), where $t \in \bigcup_i g_i$ is the point in time and $a \in [0, 1]^N$ are the most recent demands made by each player. Strictly speaking, when a player is indifferent, his strategy may also depend on the history of play. We ignore this as all our statements about a subgame (a, t) will hold for any history.

Definition 1. Player i is flexible at (a, t) if $c_i(next_i(t)) < 1 - \sum_{j \neq i} \min(a_j, c_j(next_j(t)))$.

In essence, a player is flexible if he could potentially earn a positive continuation payoff by revising his demand downwards.[13] Note that whether player i is flexible depends on t and a_{-i}, but not on a_i. At $t < t^*$ all players are flexible (this follows from the definition of t^*). Further, late in the game (after time $\bar{t} \equiv \max_j(c_j^{-1}(1))$), no player is flexible.

Definition 2. A demand profile a is compatible if and only if $\sum_j a_j \leqslant 1$.

Clearly, an agreement is reached when final demands are compatible. We now give the following result, which will lead us to the equilibrium play.

Proposition 1. *Consider a subgame (a, t), in which there exists a player i with $a_i \leqslant c_i(next_i(t))$. If a is compatible or some player is flexible at (a, t), then if player i never switches and the other players play their equilibrium strategies, an agreement will be reached.*

The proof is in the appendix. Proposition 1 implies that under the conditions stated above player i is guaranteed a continuation value of at least a_i. This leads to the following result, and its subsequent implication.

Proposition 2. *In any spe, every player i must get a payoff of at least θ_i.*

Proof. For any given player i, let $\tilde{t} = next(t^*)$ if $t^* \in g_i$, and $\tilde{t} = t^*$ otherwise. Note that in either case, $\theta_i + \sum_{j \neq i} c_j(next_j(\tilde{t})) \leqslant 1$. Thus, if player i demands any $a_i < \theta_i$ at time $\tilde{t}$ then all players $j \neq i$ will be flexible. Since $a_i < \theta_i \leqslant c_i(next_i(\tilde{t}))$, Proposition 1 holds, so player i is guaranteed agreement without switching. Thus, for any $\varepsilon > 0$, the strategy "Demand $\theta_i - \varepsilon$ at the beginning and never switch" earns a payoff of $\theta_i - \varepsilon$. Now, if in equilibrium player i earned less than θ_i, then for ε sufficiently small, this strategy would represent a profitable deviation. □

Theorem 1. *In any spe of this game, every player i demands θ_i in the first round and never switches on the equilibrium path.*

Proof. Since $\sum_j \theta_j = 1$, any other equilibrium play would result in a payoff of less than θ_i for some player i, violating Proposition 2. □

The main qualitative features of the equilibrium are the following. First, equilibrium is efficient; in order to avoid switching costs, an agreement is achieved immediately. Second, the higher the

[13] Once a switch is done, its costs are sunk; throughout the paper, we frequently consider players' payoffs net of previous switching costs, since at any point in the game, each player in equilibrium acts to maximize his continuation payoffs.

concession costs of one player, the higher the share of the pie he obtains. Having higher costs allows a player to commit not to lower his demand any longer while other players are still flexible to do so. Thus, the source of bargaining power in this model stems from the ability to commit to certain demands. Third, note from Eq. (4) that around t^* there is a local first mover advantage: each player is guaranteed at least $c_j(next_j(t^*))$, but if these sum to less than 1, the player who moves at t^* captures the remainder. This advantage is more likely to benefit players who move less frequently; it is an important advantage when the grid is coarse, [14] but it vanishes as the grid gets finer. Finally, given t^* (which is determined endogenously), the equilibrium is invariant to changes to the concession cost functions $c_i(\cdot)$ at any $t \neq next_i(t^*)$, as well as to changes in the players' costs of increasing their demands. Thus, the relative flexibility of different players only matters in the neighborhood of t^*.

3.2. The main result

We can now abstract from the specific grid chosen and think generally on situations in which players can revise their demands as often as they want. We study this case by taking limits on the fineness of the grid. In other words, we consider the distance between any two consecutive decisions by the same player going to zero. As one can notice, we only need to add the requirement that the concession cost functions are continuous for the result to hold.

Theorem 2. *If $c_i(t)$ is continuous for all i, then taking $\varphi(g) \to 0$, the limit of the equilibrium path exists and converges to each player i demanding $\theta_i = c_i(t^*)$ throughout the game, where t^* solves $\sum_j c_j(t^*) = 1$.*

Note that this outcome is independent of the order in which the players get to play. It is in this sense that we argue that this multilateral bargaining model is robust to changes in the protocol. For the rest of this section, we ignore the grid and focus on the limit case, when $\varphi(g) \to 0$.

Next we consider a few special cases which lead to simple comparative statics and provide intuition for the forces at play. First, we consider the family of cost functions $c_i(t) = \lambda_i c(t)$. That is, all players share the same concession cost technology, up to a multiplicative constant. In this case $\theta_i = \frac{\lambda_i}{\sum_j \lambda_j}$. Thus, the vector of λ_i's is a sufficient statistic for the equilibrium allocation, and the players receive shares of the pie proportional to their λ_i's. That is, the higher the (relative) concession costs, the bigger the share of the pie obtained. Moreover, in this case the allocation is independent of the choice of the cost function $c(t)$ and of the actual size of the pie. Interestingly, this invariance with respect to the size of the pie does not hold if one considers more general cost functions. Recall that the solution depends only on the relative value of the cost functions at a particular point, t^*. If the size of the pie is k, then t^* would be defined by $\sum_j c_j(t^*) = k$. Thus changes in the size of the pie result in a different t^*, which in principle could result in different relative costs.

Consider next the case in which all players share the same concession cost technology but differ in their marginal valuation for the pie. That is, if there is an agreement and the shares are a, player i values it $\pi_i(a) = \gamma_i a_i$. In other words, what we are considering now are (linear) changes in the utility bargaining sets. It is easy to see that this is equivalent to the case in which all players value the pie equally but have cost functions $c_i(t) = c(t)/\gamma_i$. Thus, the outcome of the bargaining

[14] For example, if one of the players only moves at $t = 0$, then $t^* = 0$ and Theorem 1 implies that in equilibrium he would capture the entire pie. This game essentially collapses to a standard ultimatum game.

process would deliver a higher share of the pie to those who value the object less, as these are the ones who will get committed more quickly. Graphically, the solution of the game would be the point on the Pareto frontier that intersects with the ray with direction θ. If players have the same cost technology, the solution would be the egalitarian one,[15] independent of the values of the γ_i's.

Finally, we consider the impact of introducing a discount factor into the model. Suppose that each player discounts the future at a discount rate of δ_i; that is, costs incurred at time t are discounted by δ_i^t, and the agreement by δ_i^T. By dividing each player's utility function u_i by δ_i^T, it is clear that this model is equivalent to one without discounting where each player's concession costs are $\tilde{c}_i(t) = \delta_i^{t-T} c_i(t)$. Thus, introducing discounting is equivalent to a change in the cost function. Since δ_i^{t-T} is decreasing in δ_i, more patient players have less bargaining power. Since players at any point in time compare their switching costs (incurred now) to their eventual payoff gains (received later), a lower valuation of the latter makes switching costs effectively higher, giving less patient players more commitment power. Since bargaining shares depend on the relative sizes of $\delta_i^{t^*-T} c_i(t^*)$, this effect is more pronounced for earlier t^*, corresponding to a smaller surplus being divided. Of course, since less-patient players discount their consumption more steeply, their bargaining advantage does not lead them to a higher utility level.

4. Extensions

4.1. The public good game

In the introduction we used the leading example of different parties trying to jointly fund New York's bid to host the 2012 Olympic Games. Below we show how our previous result is useful in analyzing this sort of public good problem as well.

Public good games are strategically very similar to bargaining games. Consider the following payoff structure. N players have to simultaneously decide how much to contribute towards a public good. The public good is provided only if a minimal amount, which we normalize to 1, is collected. If each player contributes $b_i \in [0, 1]$, payoffs are

$$\pi_i(b) = \begin{cases} v_i - b_i & \text{if} \quad \sum b_j \geqslant 1, \\ 0 & \text{if} \quad \sum b_j < 1, \end{cases} \tag{5}$$

where v_i is player i's valuation of the public good.

We can reinterpret this model as a bargaining one in which demands are $a_i = v_i - b_i$ and the size of the pie is $\sum v_j - 1$. The only difference is that now demands are constrained to lie in the interval $a_i \in [v_i - 1, v_i]$ because contributions cannot be negative. As before, higher concession costs result in a higher share of the pie, which corresponds to lower contributions. Since equilibrium demands $a_i^* = v_i - b_i^*$ are derived directly from the cost structure, a player's contribution is increasing in his own valuation of the public good.

One new feature is that when asymmetries among players become sufficiently acute the equilibrium results in a corner solution. This happens when the solution to the analogous bargaining game involves a demand a_i^* which is greater than v_i. Since we restrict contributions to be non-negative, player i would demand in equilibrium v_i, which corresponds to a contribution of $b_i = 0$. If a player's relative interest in the public good is sufficiently low, or his concession costs sufficiently high, he can commit not to contribute at all, forcing others to do all the funding. The

[15] That is, all the players receiving the same utility, not the same share of the pie.

actual division among the active contributors could be computed by analyzing the reduced game in which the free riders are ignored.

4.2. Uncertainty about the protocol

One of the key results in this paper is that the equilibrium is robust to changes in the protocol as long as players play often enough. We derived this result assuming players had perfect information about the protocol. Here we argue that this is not necessary; we show that models with symmetric or asymmetric uncertainty about the exact timing of players' moves still exhibit the same feature. In particular, we only require that there is common knowledge that players play sufficiently often. While it is well known that information imperfections in this context may lead to inefficiencies through delays (and the occurrence of switching costs in our context) and possibly disagreements,[16] such inefficiencies can be made arbitrarily small by having players play sufficiently often. The theorem below illustrates this point for the case of two players.[17]

For $i = 1, 2$, let concession costs $c_i(t)$ be continuous and strictly increasing in t, let t^* (uniquely) solve $c_1(t^*) + c_2(t^*) = 1$, and let $\theta_i = c_i(t^*)$. Assume also for simplicity that $d_i(t) = c_i(t)$ (switching costs are equal in either direction), and that each player moves at least once before t^* and initial demands are costless. For this section, the equilibrium concept is a Perfect Bayesian Equilibrium. Under these conditions, the following theorem provides a lower bound for expected equilibrium payoffs.

Theorem 3. *Let $\varepsilon_1, \varepsilon_2 > 0$ and $p_1, p_2 \leqslant 1$. If it is common knowledge*[18] *that the probability that player j moves at least once while $c_j(t) \in \left(\theta_j, \theta_j + \varepsilon_j\right)$ is at least p_j, then in any equilibrium player i's expected payoff is at least $p_j\left(\theta_i - \varepsilon_j\right)$.*

The proof is in the appendix. Its intuition is similar to the one provided for the main model. By demanding $\theta_i - \varepsilon_j$ initially and never switching thereafter, player i can guarantee an agreement with probability p_j, since his demand will be made compatible if player j has an opportunity to move in the interval $\left(t^*, c_j^{-1}(\theta_j + \varepsilon_j)\right)$. Thus, this strategy guarantees player i a payoff of $p_j\left(\theta_i - \varepsilon_j\right)$ and therefore any equilibrium must give player i at least this much.

It is easy to see that as the grid gets finer, namely as $\varepsilon_1, \varepsilon_2$ tend to zero and p_1, p_2 tend to one, the lower bound for player i's payoff approaches θ_i, which is his equilibrium share in the perfect information game with fine grids. Given that the total surplus approaches one, this implies that inefficiencies vanish. Therefore, in the limit there are no disagreements or delays in reaching an agreement, which exhibits the same division of the pie as with perfect information.

Let us emphasize the scope of the previous theorem: *any* protocol satisfying the conditions above (simply that players play very often) would result in the same split of the pie, namely the one provided in Theorem 2. Here we mention several interesting cases. First, consider the case in which the grid g is known to both players, they both play at every point on the grid, but do

[16] Crawford [6] models bargaining with stochastic switching costs, and shows that this uncertainty sometimes leads to disagreements. Ma and Manove [16] obtain a similar result when the uncertainty is about the arrival of offers. Myerson [20] and Abreu and Gul [1] obtain inefficient delays in the presence of obstinate types due to reputation effects.

[17] We believe that a similar result holds for more players. However, when $N > 2$, it is difficult to prove Lemma 10 (the analog to Proposition 1) without common knowledge about the "final" round of moves before agreement becomes impossible.

[18] In fact, common knowledge is not important. All one needs is that this information is known by player i.

so sequentially and the order at each point is randomly determined. This fits the conditions of Theorem 3, with $\varepsilon_1 = \varepsilon_2 = \varphi(g)$ and $p_1 = p_2 = 1$. Similarly, if at each point of the grid only one player gets to move, but the identity of that player is determined randomly, Theorem 3 applies with $p_1, p_2 < 1$. This is also similar to a model where the timing of players' moves is randomly determined by a Poisson process (as in Lagunoff and Matsui [12]). The previous examples were cases of symmetric uncertainty. The result extends also to protocols with asymmetric information. Consider for instance the case where the timing of each player's moves is private information, learned at the start of the game. Theorem 3 still applies. In all these cases, as the arrival rate of opportunities to move increases, the equilibria of these stochastic games approach that of the complete information case analyzed in the previous section. This should not come as a surprise: after all, if the details of the fully known protocol do not matter (as established by Theorem 2), then uncertainty or private information about these "irrelevant" details should not matter either.

As discussed earlier, even with perfect information, if the grid is not fine, the particular details of the grid effect the commitment power and equilibrium outcomes in important ways. In the case of symmetric or asymmetric uncertainty about a (coarse) grid these would matter even more. For instance, protocols with private information about the grid will likely give rise to reputation issues similar to those in Myerson [20] and Abreu and Gul [1]. As an example, a player with private information that he will only move once would be similar to an obstinate type who always makes the same demand, which is how a "tough" reputation is often modeled in bargaining situations. The analogy is not perfect, however, since in our model this player would still have a choice over his initial demand. [19]

4.3. *Flow payoffs with constant switching costs*

In this section, we extend our analysis to a different bargaining model. While its underlying economic structure is very different from the one studied before, its analysis is quite similar. The model builds on the framework studied by Lipman and Wang [14]. There are two key differences from the previous model. First, rather than a one-shot payoff in the end of the game, players collect a flow of payoffs from their bargaining interaction. Second, switching costs are small and constant over time.

Patent pools may provide a good application for this setup. Suppose that two firms held complementary patents which could only be used together. Should they agree on a way to share revenues, they can collect a flow of revenues as the technology is used; as long as they disagree, nobody can use the patents, so potential revenues are lost. In this context, switching costs can be arbitrarily small, so the additional lawyers' fees of re-drafting an agreement would suffice.

As before, players make demands on a finite grid g in the interval $[0, T]$, and our focus lies on fine grids. When player i plays he decides on his demand $a_i \in [0, 1]$. If he changes his demand from his previous level, he pays a small switching cost $\varepsilon_i > 0$. [20] In between two decision periods demands are fixed. Thus, flow payoffs are the standard payoffs of the demand game with respect to the most recent announcements made. With some abuse of notation, we denote $a_i(t) = a(prev_i(t))$

[19] Since the grid only matters through the level of concession costs at each point on the grid, private information about grids that are not fine is analogous to private information about the cost structure.

[20] To simplify notation, we assume throughout this section that switching demand upwards and downwards has the same cost. This is not important. As before, only concession costs will matter, so we could assume any arbitrary non-negative structure on the cost of switching the demand upwards.

for $t \notin g_i$ and obtain the final payoffs as

$$u_i(\overline{a}_i, \overline{a}_j) = \int_0^T \pi_i(a_i(\tau), a_j(\tau))\, d\tau - \sum_{\{t \in g_i - \{t_1^i\} : a_i(t) \neq a_i(prev_i(t))\}} \varepsilon_i, \tag{6}$$

where $\pi_i(a)$ is given by Eq. (2).[21]

This setup is closely related to our main model. Loosely speaking, here one compares the constant switching costs, ε_i, to the remaining future payoffs, $(T-t)\pi_i$. Meanwhile, in the main model the relevant comparison is made between the final payoffs of π_i and the increasing switching costs of $c_i(t) = \frac{\varepsilon_i}{T-t}$. This explains why the analysis is similar. There is an important difference, however, between the two models. In the main model each player only cares about his own actions and his opponents' *final* actions. Here, each player's payoffs depend on the whole sequence of his opponents' *interim* actions, as these determine the flow-payoffs. This makes the analysis of the flow-payoff case more complicated. For this reason, we restrict our attention to the case of two players.

As the proof is quite similar to the one of Theorem 2, we state below only the main result and relegate all the intermediate results to the appendix.

Theorem 4. *Given switching costs $\varepsilon_i, \varepsilon_j$, the limit of the equilibrium path of the flow-payoff game, taking $\varphi(g) \to 0$, exists and converges to players constantly demanding $a_i = \frac{\varepsilon_i}{\varepsilon_i+\varepsilon_j}$ throughout the game.*

The first step of the proof is similar to the two-player version of our main model. The critical point in time, t^*, is now equal (in the limit) to $T - \varepsilon_i - \varepsilon_j$. After this point, the remaining continuation payoffs of the game are less than $\varepsilon_i + \varepsilon_j$, so in equilibrium we cannot expect both players to switch after t^*. This allows each player to obtain his share of the pie, $a_i = \frac{\varepsilon_i}{\varepsilon_i+\varepsilon_j}$, from then on. Unlike our original model, however, this is not enough to finish the proof. Because of the flow payoffs, we need to argue that these shares are obtained throughout the game, and not only in the end. To do so, we use induction on the game tree, and show that by demanding $\frac{\varepsilon_i}{\varepsilon_i+\varepsilon_j}$ at any point, player i can guarantee an agreement almost immediately.

As before, higher switching costs imply higher commitment and higher bargaining power. Therefore, it allows a player to obtain a higher share of the pie. One should note that these results hold even when switching costs are arbitrarily small. As long as the players can change their demands sufficiently often, the absolute level of switching costs does not matter; only the relative costs do.

Finally, let us point out why the result of Theorem 4 does not extend to $N > 2$ players. The key point at which the proof fails is when one argues that a flexible player finds it profitable to switch down and lock himself into a compatible demand profile.[22] With flow payoffs this may not be profitable anymore. In principle, such a switch can induce other opponents to change their own demands as well. This may temporarily result in a period of disagreement. This amount of time, even though short, may be enough to make the original switch not profitable.

[21] For completeness, one may assume that $\pi_i = \pi_j = 0$ before both players make their very first announcements (i.e. for all $t < \max(t_1^i, t_2^i)$).

[22] This happens, for example, in the last paragraph of the proof of Proposition 1.

5. Concluding remarks

We have presented a new bargaining model in which an arbitrary number of players negotiate in nearly continuous time, subject to an exogenous deadline and increasing concession costs. We find that as the time between "rounds" decreases toward zero, our model gives a unique equilibrium prediction which is invariant to the order and exact timing of the players' moves. Delay or disagreement never occur on the equilibrium path, and higher shares go to players with higher concession costs, as they are able to credibly commit to higher demands. As already emphasized, only concession costs matter. Whether concessions are fully reversible or completely irreversible has no impact on the results.

As we highlighted before, there are three key features that describe our model: (i) players must reach an agreement by some external deadline or forfeit the entire prize; (ii) players, in a sequential manner, have many opportunities to change their minds; (iii) the costs of conceding and switching to a lower demand, even a nearby one, are significant, and increase over time.

The second feature is of a more technical nature. We are simply imposing a particular set of rules on how players are allowed to express their demands. As was our goal, we show that these assumptions on the timing and order of moves have no real implications. The important aspect is to ensure the ability of players to react quickly to other players' moves. The other two assumptions are more economic in nature, and deserve a more careful discussion of their applicability.

Examples of bargaining with a fixed deadline are common. [23] During bankruptcy proceedings, management may face a court-assigned deadline by which they must reach new wage agreements with multiple unions or face liquidation. In the Olympic bid example discussed above, multiple parties must agree to provide costly services in order to submit a potentially winning bid. This process is clearly subject to an external deadline. Another example in which these deadlines are ubiquitous is in major sports. Multi-team trades are frequent and there are rules imposing specific deadlines to player trades.

Real-world examples of bargaining with literal switching costs may not seem as natural. But, without any commitment, demands or offers can be seen as simple cheap talk, or be subject to future renegotiation. Thus, an offer only becomes credible once it becomes costly to change it. In the Olympic bid example, parties may begin spending money (hiring architects or planners, scheduling contractors, even beginning construction) to show the seriousness of their offers. Once these steps are taken, changing plans would likely incur additional costs. In the presence of a deadline, such costs are likely to be higher as the deadline approaches. In addition, in many complex situations, there are various frictional costs (lawyers' fees, court fees, costs of preparing a new proposal) associated with each new offer submitted.

One can also interpret switching costs in a more metaphoric way, or in the context of an (unmodeled) larger game. In a wage negotiation setting, each side may be aware that by retreating from their demands, they sacrifice their reputation for being a tough negotiator, hurting them in future negotiations and thus imposing a cost to conceding. This interpretation is especially appealing in negotiations in which parties choose to make public statements in the media as a way

[23] See also Fershtman and Seidmann [7] and Ma and Manove [16] for analyses of bargaining situations with fixed deadlines. In these papers much of the equilibrium play is directly driven by what happens just before the deadline. In this sense, the deadline is less important in our analysis, as equilibrium is primarily dictated by what happens around t^*, which in principle may occur much before the deadline. The existence of a deadline is still meaningful in our setting, since in our view it is the deadline that helps us motivate our assumption of increasing (and eventually high) concession costs.

to tie their hands to future concessions. Union leaders, for instance, may advertise their demands publicly, essentially staking their credibility on achieving the outcome they have promised. In a situation where sports teams try to arrange a multi-team trade, noisy communication could lead to a situation where changing one's demand leads to a positive probability that the deal falls through. Many of these features are likely to become more salient when changes-of-mind happen late in the game, justifying our assumption that switching costs increase.

Appendix

Proof of Proposition 1. We now build up a proof of Proposition 1 through a series of lemmas.

First, similar to the definition of flexible in the text (Definition 1), we define the following:

Definition 3. Player i is locked at (a, t) if $c_i(next_i(t)) > 1 - \sum_{j \neq i} \min(a_j, c_j(next_j(t)))$.

As we will establish in Lemma 3, a player is locked when, subject to other players playing equilibrium strategies, he prefers disagreement to further concessions. At $t \leqslant t^*$, no player is locked, and late in the game, after time $\bar{t} \equiv \max_j(c_j^{-1}(1))$, all players are locked.

Lemma 1. *Given any (a, t), the final demands arising in any equilibrium of the continuation game starting at (a, t), a^*, satisfy $a_i^* \geqslant \min(a_i, c_i(next_i(t)))$ for all i. Moreover, if a is compatible or at least one player switches, then a^* is compatible.*

Proof. If a player switches downward then he must at least recoup his switching costs ($a_i \geqslant c_i\ (next_i(t))$), otherwise he would have been better off not switching and receiving at least a continuation value of zero (net of switching costs incurred prior to t). If a player switches only upward then $a_i^* \geqslant a_i$. Since switching costs are positive in either direction, a switch is rationalized only if a^* is compatible. If a is compatible and nobody switches, then $a^* = a$ is compatible. □

Lemma 2. *Let a^* be the spe outcome of the subgame beginning at (a, t). If a^* is compatible then $a_i^* \leqslant 1 - \sum_{j \neq i} \min(a_j, c_j(next_j(t)))$ for all i.*

Proof. This follows directly from the fact that if a^* is compatible then $a_i^* \leqslant 1 - \sum_{j \neq i} a_j^*$, and Lemma 1. □

Lemma 3. *If player i is locked at (a, t), in equilibrium he will not switch downwards thereafter.*

Proof. If i switched on equilibrium, the final demand profile a^* would be compatible (Lemma 1), implying that $a_i^* \leqslant 1 - \sum_{j \neq i} \min(a_j, c_j(next_j(t))) < c_i(next_i(t))$ (Lemma 2). But this would imply that player i's continuation payoff is negative. □

Lemma 4. *If there exists a player who is flexible at (a, t) then any player k who is not flexible must have $a_k < c_k(next_k(t))$.*

Proof. Let i be a player who is flexible and k the player who is not flexible. Then

$$\min\{a_i, c_i(next_i(t))\} \leqslant c_i(next_i(t)) < 1 - \sum_{j \neq i} \min\{a_j, c_j(next_j(t))\} \tag{7}$$

and rearranging,

$$\min\{a_k, c_k(next_k(t))\} < 1 - \sum_{j \neq k} \min\{a_j, c_j(next_j(t))\}. \tag{8}$$

If $a_k \geqslant c_k(next_k(t))$ then the left-hand side above is $c_k(next_k(t))$ and the assumption that k was not flexible is violated. □

Lemma 5. *If none of the players is flexible at (a, t) then any upward switch leaves demands compatible.*

Proof. Suppose player j moves at t. Then $c_j(next_j(next(t))) > c_j(next_j(t))$. If a_j also increases at time t (to a'_j) then $\min(a'_j, c_j(next_j(next(t)))) > \min(a_j, c_j(next_j(t)))$. In that case, all players $i \neq j$ who were not flexible become locked. Since $c_j(next_j(next(t))) > c_j(next_j(t))$, player j also becomes locked. Thus, if player j increases his demand when no players are flexible, all players are locked after his move, so by Lemma 3, nobody switches down in the future; if he leaves a incompatible then no agreement is reached, giving j negative continuation value. □

Lemma 6. *Consider the continuation game (a, t) where player i moves at t. If $a_i \leqslant c_i(t)$, a is not compatible, and no player is flexible then in equilibrium, player i does not switch at t.*

Proof. By Lemma 5, player i does not switch upwards. If i is not flexible,

$$c_i(t) \geqslant 1 - \sum\nolimits_{j \neq i} \min(a_j, c_j(next_j(t))),$$

so by Lemma 2, $a_i^* \leqslant c_i(t)$. Since $a_i \leqslant c_i(t)$, if i switches downwards at t, he incurs a cost $c_i(t)$, and either receives $a_i^* < c_i(t)$, or receives $a_i^* = c_i(t)$ but incurs a cost $d_i(t')$ of switching upwards later. In either case, the continuation payoff is negative. □

We are now ready to prove Proposition 1. The proof is by induction on t. Late in the game, no players are flexible. At the latest point in the game grid, $t = \max(\tau \mid \tau \in \cup g_i)$, if a is incompatible, the proposition is vacuously true. Suppose a is compatible. If i moves at t and does not switch, the game ends with compatible demands. If $j \neq i$ moves at t, j will not switch to incompatible demands (doing so would give him negative continuation value), so the game ends in agreement with $a_i^* = a_i$.

Now suppose the proposition is proven for all $t' \geqslant next(t)$; we prove it for t. There are two cases:

1. First, suppose player i moves at t. If a is compatible then by not switching, i leaves a compatible and the induction assumption proves the proposition. Suppose a is not compatible but some player other than i is flexible. Since $a_i \leqslant c_i(next_i(t))$, if i does not switch, $\min(a_i, c_i(next_i(\cdot)))$ does not change, so that player remains flexible so the induction assumption proves the proposition. Finally, if only player i is flexible then by Lemma 4 and the definition of flexible

$$c_i(t) \leqslant 1 - \sum_{j \neq i} \min(a_j, c_j(next_j(t))) = 1 - \sum_{j \neq i} a_j. \tag{9}$$

Then since $a_i \leqslant c_i(t) \leqslant 1 - \sum_{j \neq i} a_j$, a is compatible, in which case we have already proven the proposition.

2. Now suppose player $j \neq i$ moves at t. If after j's move, the new demands are compatible or some player is flexible, the induction assumption proves the proposition. Thus, we only need to consider the case where after j's move, no player is flexible and demands are incompatible. First, suppose j switches at time t. Assume that thereafter player i does not switch and the other players play equilibrium strategies, and consider all the times that player i moves after t. If a is compatible or some player is flexible at any of these times, then the induction assumption proves the proposition. If a is incompatible and no player is flexible at all of these times, then by Lemma 6, player i is playing his equilibrium strategy by not switching. Then starting at (a, t), player j switches and every player plays his equilibrium strategy after t; by Lemma 1, an agreement is reached.

   We are left with the case where j does not switch at t but after j's move, all players are not flexible and a is not compatible. As we argued above, if these conditions do not remain at each of player i's subsequent moves, then the induction assumption proves the proposition; if they do, then all players are playing equilibrium strategies, so player j's inaction led to a subgame where, in equilibrium, agreement is not reached, giving a continuation payoff of 0 to player j. If j was the only player flexible at t, then switching down to $a_j^* = 1 - \sum_{k \neq j} a_k > c_j(t)$ would have locked all players and led to positive continuation payoff. If j was not flexible at t, then some other player was; since by Lemma 4, $a_j < c_j(t)$, not switching could not have changed another player's flexibility, so by the induction assumption, an agreement would have been reached. Finally, if j and another player were flexible at t, then j switching down to $c_j(t) + \varepsilon$ would have left the other player flexible for ε small enough; since $c_j(t) + \varepsilon < c_j(next_j(next(t)))$, player j would have been ensured agreement without switching again (by the induction assumption) for a continuation payoff of $\varepsilon > 0$.

Thus, regardless of who moves at t, the proposition is proved. □

Proof of Theorem 3. We prove Theorem 3 using the following lemmas. Throughout we assume that $\theta_1 = c_1(t^*) \geqslant \frac{1}{2} \geqslant c_2(t^*) = \theta_2$. This is without loss of generality since $c_1(t^*) + c_2(t^*) = 1$. Recall also that $d_i(t) = c_i(t)$.

Lemma 7. *For any $t > t^*$, (i) if, in equilibrium, player 1 switches at t then he switches to $a_1' \geqslant c_1(t)$; (ii) if $a_1 \geqslant \theta_1 - \theta_2$ and, in equilibrium, player 2 switches at t then he switches to $a_2' \geqslant c_2(t)$.*

Proof. To prove part (i) note that since $c_1(t) \geqslant \frac{1}{2}$ player 1 will obtain negative continuation payoffs by switching twice, since switching costs would exceed the size of the entire pie. Thus, if player 1 switches at t, his new demand a_1' must compensate him for his switching cost, so $a_1' \geqslant c_1(t)$. To prove part (ii), note that because of (i) any future switch by player 1 will be to $a_1' \geqslant c_1(t) \geqslant \theta_1$, so player 2 knows player 1's final demand will be at least $\theta_1 - \theta_2$. Thus, if agreement is reached, player 2's final payoff will be at most $1 - (\theta_1 - \theta_2) = 2\theta_2$. Since switching at $t > t^*$ costs more than θ_2, this cannot be enough to compensate player 2 for two switches. Thus, if player 2 switches to a_2' at t, it must be with the intention of capturing a_2' of the pie, and so $a_2' \geqslant c_2(t)$. □

Lemma 8. *For any $t > t^*$, if $a_j \geqslant \theta_j$ then in equilibrium player i does not switch.*

Proof. Since $a_j \geqslant \theta_j \geqslant \theta_j - \theta_i$, Lemma 7 holds: if i switches at t, it is to $a'_i \geqslant c_i(t) > \theta_i$. But then Lemma 7 will hold for both players, so all future switches will leave $a''_i > \theta_i$ and $a''_j \geqslant \theta_j$. Since $\theta_1 + \theta_2 = 1$ demands will never become compatible; player i therefore cannot recover his switching costs, so he does not switch. □

Lemma 9. *In equilibrium, any switch at $t > t^*$ leaves demands compatible.*

Proof. First, suppose it is player 1 who moves at t, so $c_1(t) > \frac{1}{2}$. Part (i) of Lemma 7 holds, so player 1 switches to at least $c_1(t) > \theta_1$. By Lemma 8, he does not expect player 2 to switch in response and he cannot plan to switch again, since switching twice exhausts the entire available surplus. So player 1 only gets a non-negative continuation payoff by switching to a compatible demand.

Next, suppose player 2 moves at t and $c_2(t) > 1 - a_1$. We argue that player 2 will never switch. From part (i) of Lemma 7, player 1's final demand will be at least $c_1(t)$ if he switches and exactly a_1 if he does not. Thus, if agreement is reached, player 2's final payoff will be either $a'_2 \leqslant 1 - c_1(t)$ or $a'_2 \leqslant 1 - a_1$. In either case, $a'_2 < c_2(t)$, so player 2 cannot recover his switching costs if he switches at t.

Finally, suppose player 2 moves at t and $c_2(t) \leqslant 1 - a_1$. If player 2 switches to $a'_2 = 1 - a_1 \geqslant c_2(t) > \theta_2$ and never switches again, player 1 will never switch (by Lemma 8), so player 1 is guaranteed a continuation payoff of $1 - a_1 - c_2(t) \geqslant 0$. We show that this is strictly more than player 2 can get if he switches to an incompatible demand. Since $t > t^*$, $c_1(t) + c_2(t) > 1$, and so $a_1 \leqslant 1 - c_2(t) < c_1(t)$. From part (i) of Lemma 7, any switch by player 1 will be to $a'_1 \geqslant c_1(t) > a_1$. If player 2 switches to an incompatible demand at time t, then he will have to switch again later to get agreement, and will still get a final demand of at most $1 - a_1$. Therefore, his continuation payoff would be less than $1 - a_1 - 2c_2(t)$, which is strictly less than he could get by switching to the compatible demand $a'_2 = 1 - a_1$ at t. □

Lemma 10. *Suppose that player j moves at time $t > t^*$, and that $a_i < \theta_i$ and $c_j(t) < 1 - a_i$. Then after j's move, in equilibrium demands will be compatible.*

Proof. Suppose first that demands are already compatible before player j's move. Since $t > t^*$, by Lemma 9 any switch leaves demands compatible. Therefore, suppose that $a_i + a_j > 1$ before j moves at t. If player j switches, then by Lemma 9 he switches to a compatible demand. Thus, we need only show that in equilibrium player j must switch at t. We treat two cases separately.

First, suppose player j is player 1, so $c_j(t) \geqslant \frac{1}{2}$. Since $a_i < \theta_i$ and demands are incompatible, we must have $a_j > \theta_j$. By Lemma 7, any switch by player j (either at t or later) will leave $a'_j > \theta_j$. By Lemma 8, player i will never switch again in equilibrium. So if agreement is to be reached, player j must switch at some point to a final demand less than or equal to $1 - a_i$; player j gets strictly higher payoff by switching to $1 - a_i$ at time t than he could get by switching later. Since $1 - a_i > c_j(t)$, this is also strictly better than he gets by never switching. Thus, in equilibrium player j must switch at t.

Second, suppose player j is player 2 and does not switch at t. From part (i) of Lemma 7, player j must expect i's final demand to be greater than $c_i(t) > \theta_i > a_i$ if he switches, and equal to a_i if he does not. Thus, if agreement is to be reached, player j's share will be at most $1 - a_i$. Further, since demands were assumed to be incompatible at t and player i's demand will not fall, j will have to switch at $t' > t$ to reach agreement. Thus, if agreement is to be reached, player j's continuation payoff will be at most $1 - a_i - c_j(t')$, and if agreement is not reached, it is at most 0; both of these are strictly less than what j would get by switching to $a'_j = 1 - a_i$ at t. So not switching at t cannot occur in equilibrium. □

We can now prove Theorem 3. Consider the following strategy for player i: demand $\theta_i - \varepsilon_j$ at every point, ignoring any new information which may be received over the course of the game. With probability at least p_j, player j would move at some $t > t^*$ with $a_i = \theta_i - \varepsilon_j < \theta_i$ and $c_j(t) < \theta_j + \varepsilon_j = 1 - a_i$. By Lemma 10, this would lead to compatible demands after j's move. By Lemma 9, any subsequent switch by j would still leave demands compatible. Thus, with probability at least p_j, this strategy would lead to agreement, so player i's (ex ante) expected payoff would be at least $p_j\left(\theta_i - \varepsilon_j\right)$. This would therefore be a profitable deviation from any supposed equilibrium giving player i a lower expected payoff. □

Proof of Theorem 4. We use a similar approach to the one we use to prove Theorem 2. Given a grid g, we define

$$t^* = \max\left\{t \left| \sum_j \frac{\varepsilon_j}{T - next_j(t)} \leqslant 1 \right.\right\} \tag{10}$$

and

$$\theta_i = \begin{cases} \dfrac{\varepsilon_i}{T - next_i(t^*)} & \text{if } t^* \notin g_i, \\ 1 - \theta_j & \text{if } t^* \in g_i. \end{cases} \tag{11}$$

As before, we will abstract from the grid and show through a series of lemmas and propositions that in equilibrium players immediately demand θ_i and do not change these demands thereafter. We need to introduce some more notation. We will refer to

$$a_i^* = \frac{1}{T - t} \int_t^T a_i(\tau)\, d\tau \tag{12}$$

as the average continuation *demand*. This does not need to coincide with average payoffs, as it may be the case that the demand profiles are not compatible, so at least temporarily no agreement is reached.

Lemma 11. *Given any (a, t), the average demands arising in any equilibrium of the continuation game starting at (a, t), a^*, satisfy $a_i^* \geqslant \min(a_i, \frac{\varepsilon_i}{T-t})\ \forall i$.*

Proof. Consider the equilibrium path of (a, t). If player i never switches on equilibrium then $a_i^* = a_i$. If player i switches in equilibrium (at time t or after), he must obtain non-negative payoffs. This can only be possible if $\varepsilon_i \leqslant (T - t)a_i^*$, otherwise he cannot hope to cover his switching costs. Therefore, we obtain that $a_i^* \geqslant \min(a_i, \frac{\varepsilon_i}{T-t})$. □

Proposition 3. *Given (a, t) such that $a_i \geqslant \theta_i$ and $t > next_i(t^*)$, player j never switches in any equilibrium of the continuation game (a, t).*

Proof. By Lemma 11 we know that player j's continuation payoffs are at most $(T - t)(1 - \min(a_i, \frac{\varepsilon_i}{T-t}))$. By switching he pays ε_j. Therefore, player j never switches if $\frac{\varepsilon_j}{T-t} > 1 -$

$\min(a_i, \frac{\varepsilon_i}{T-t})$. But it is easy to check that this inequality is now satisfied. Note that for $t > next_i(t^*)$ we have that $\frac{\varepsilon_i}{T-t} > \theta_i$ and $\frac{\varepsilon_j}{T-t} > \theta_j$. With $a_i \geqslant \theta_i$ we have that $\frac{\varepsilon_j}{T-t} > \theta_j = 1 - \theta_i \geqslant 1 - \min(a_i, \frac{\varepsilon_i}{T-t})$. □

Lemma 12. *Without loss of generality, let $t^* \in g_i$. In equilibrium, player j plays $a'_j \leqslant \theta_j$ at $((\theta_i, a_j), next_j(t^*))$.*

Proof. If player j plays $a'_j > \theta_j$ then by Proposition 3 we know that none of the players will switch thereafter, the demands are not compatible, so continuation values are at most zero. By playing $a'_j = \theta_j$, we know by Proposition 3 that none of the players will switch thereafter, the demands are now compatible, so continuation values are $(T - next_j(t^*))\theta_j$, which are equal to ε_j. Thus, playing θ_j weakly dominates playing $a'_j > \theta_j$.[24] □

Corollary 1. *Player i, by playing $a_i = \theta_i$ at t^* and never switching thereafter, can guarantee himself an agreement from $next_j(t^*)$ on.*

Proposition 4. *Let the fineness of the grid satisfy $\varphi(g) < \min(\varepsilon_i, \varepsilon_j)$. Consider a subgame $((\theta_i, a_j), t)$ for $t \leqslant t^*$ and $t \in g_i$. Player i can guarantee himself continuation payoffs of $\theta_i(T - next_j(t))$ by never switching.*

Proof. We prove it by induction on the game tree. The base is proved above for t^* (Corollary 1). Suppose now that the proposition is true for $next(t)$ and we need to show it for t. By applying the induction assumption for player i's subsequent move (at $next_i(next(t))$), we already know that i can guarantee himself continuation value of $\theta_i(T - next_i(next(t)))$. Therefore, all we need to show is that player j will accommodate immediately, namely that at $t' = next_j(t)$ player j will play $a'_j \leqslant \theta_j$. Note that if player j plays θ_j at t', because of the induction assumption, his continuation value would be at least $\theta_j(T - t') - \varepsilon_j$: player i will immediately accommodate, and agreement will be achieved already at t' because the demands are compatible.

We will now show that for player j playing θ_j dominates not accommodating, namely setting $a'_j > \theta_j$. Let $t'' = next_i(t')$ and $t''' = next_j(t'')$. Let $v_i(s)$ be time s continuation values for player i on the equilibrium path of subgame $((\theta_i, a'_j), t')$. Using the induction assumption (for player i at time t''), we have that $v_i(t''') \geqslant \theta_i(T - t''')$. For player i to get this much, it must be that player j (at t''' or later) eventually changes his high demand of a'_j. Let *AllCosts* Denote all switching costs spent (by both players) on the equilibrium path of $((\theta_i, a'_j), t''')$. Because we know that player j will eventually change his high demand, we can write $AllCosts = \varepsilon_j + OtherCosts$. Let also a'_i and a''_i denote player i's equilibrium play at $((\theta_i, a'_j), t'')$ and at $((\theta_i, \theta_j), t'')$, respectively.

Now, note that

$$v_i(t''') + v_j(t''') \leqslant (T - t''') - AllCosts = (T - t''') - \varepsilon_j - OtherCosts \tag{13}$$

[24] While, in principle, there could be other equilibria of the subgame $((\theta_i, a_j), next_j(t^*))$ in which player j sets $a'_j = a_j > \theta_j$ (because of indifference), it is easy to show that such equilibria of the subgame cannot be part of an equilibrium of the whole game. Since player j must switch in equilibrium at $((\theta_i - \eta, a_j), next_j(t^*))$ for any $\eta > 0$, if he did not switch at $((\theta_i, a_j), next_j(t^*))$, player i would have no best response earlier in the game.

and thus

$$\begin{aligned} v_j(t') &= (t''-t')\pi_j(\theta_i, a'_j) + (t'''-t'')\pi_j(a'_i, a'_j) + v_j(t''') \\ &\leqslant 0 + (t'''-t'')\pi_j(a'_i, a'_j) + (T-t''') - v_i(t''') - \varepsilon_j - \mathit{OtherCosts} \\ &\leqslant (t'''-t'')\pi_j(a'_i, a'_j) + (T-t''') - \theta_i(T-t''') - \varepsilon_j - \mathit{OtherCosts} \\ &= (t'''-t'')\pi_j(a'_i, a'_j) + \theta_j(T-t''') - \varepsilon_j - \mathit{OtherCosts}, \end{aligned} \tag{14}$$

where the first inequality arises from Eq. (13) and the second from the induction assumption. The last equality just uses the fact that $\theta_i + \theta_j = 1$.

Now, we can finally check that this continuation value of $v_j(t')$ is lower than what j would get by playing θ_j at t'. By playing a'_j player j gets

$$v_j(t') \leqslant (t'''-t'')\pi_j(a'_i, a'_j) + \theta_j(T-t''') - \varepsilon_j - \mathit{OtherCosts} \tag{15}$$

while by playing θ_j player j gets

$$\theta_j(T-t') - \varepsilon_j = \theta_j(t'''-t') + \theta_j(T-t''') - \varepsilon_j. \tag{16}$$

The latter is greater as long as

$$\theta_j(t'''-t') \geqslant (t'''-t'')\pi_j(a'_i, a'_j) - \mathit{OtherCosts}. \tag{17}$$

Now, if $\pi_j(a'_i, a'_j) = 0$ we are done. If $\pi_j(a'_i, a'_j) > 0$, this implies that player i reduced his demand in period t'' (from θ_i to a'_i), and thus $\mathit{OtherCosts} \geqslant \varepsilon_i$. For a sufficiently fine grid (e.g. $t''' - t'' < \varphi(g) < \varepsilon_i$), the inequality of Eq. (17) holds, which finishes the proof. □

Proposition 5. *Let the fineness of the grid satisfy* $\varphi(g) < \frac{1}{2}\min(\varepsilon_i, \varepsilon_j)$. *On the equilibrium path of the flow-payoff game each player i demands* $a_i^* \in \left(\theta_i \frac{T-2\varphi(g)}{T}, 1-\theta_j \frac{T-2\varphi(g)}{T}\right)$ *the first time he plays and never switches thereafter.*

Proof. Without loss of generality, let $\mathit{next}_i(0) < \mathit{next}_j(0)$. Let $t_1 = \mathit{next}_j(0)$ and $t_2 = \mathit{next}_i(\mathit{next}_j(0))$. As the first decisions do not involve switching costs, player i's value of the game v_i satisfies $v_i \geqslant \theta_i(T-t_1)$ and player j's value of the game v_j satisfies $v_j \geqslant \theta_j(T-t_2)$. Note that $v_i + v_j \geqslant \theta_i(T-t_1) + \theta_j(T-t_2) \geqslant T - t_2 > T - 2\varphi(g)$. It is now easy to see that there are no switches on equilibrium path: a switch implies that $v_i + v_j < T - \min(\varepsilon_i, \varepsilon_j)$, which is a contradiction. Without switching, it is clear that demands will be compatible starting at $\mathit{next}_j(0) = t_1$ (the first time at which both players have demands). Thus, each player's demand a_i^* must satisfy $(T-t_1)a_i^* = v_i$. This establishes the proof. Note that the bounds on demands are tighter once we know which player moves first. The lower bound for the first mover is θ_i, making the upper bound for the second mover $1-\theta_i$. □

At this point, all we need is to take limits of the result in Proposition 5 in order to finish the proof of Theorem 4. □

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