

Bidding Reversals in a Multiple-Good Auction with Aggregate Reserve Price

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Abstract

We examine two-bidder sealed-bid auctions for two objects, one more valuable than the other, and unit demand. The auction has a single “aggregate” reserve price, which must be met by the combination of winning bids, and each bidder can bid on both objects without fear of winning both. A bidder’s private values for the two objects are perfectly correlated, so types are one-dimensional. We demonstrate the existence of symmetric equilibria where over some range of types, (i) bidders bid on the lesser object for the purpose of “sabotaging” their (higher) bid on the greater object, and (ii) bids for the lesser object are a *decreasing* function of valuations.

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1 Introduction

Two construction firms are bidding on a large municipal project. The project is too large for either one to finish on its own, but smaller than the combined capacities of the two firms. The city therefore breaks the project into a large and a small part, sets a maximum bid for each, and allows each firm to submit separate (“exclusive or”) bids on the two parts; the city also announces a price at which, if the entire project cannot be completed, it will be scrapped entirely.

Or, package bids are permitted in an auction for three blocks of wireless spectrum, one each covering the east coast, the west coast, and the center of the country. A California-based firm wants the west coast license, and would expand to the center of the country at the right price; a New York-based firm similarly wants the east coast license, with or without the center. A new entrant has publicly committed to paying a certain amount for all three blocks together, and will get them if the auction among the other two firms does not raise at least that much.

We consider a simple model of these situations, where the seller offers two items, one more valuable than the other, which the bidders view as substitutes. There is a single “aggregate” reserve price – the auction will be canceled if it does not raise a certain amount of total revenue. “Exclusive-or” bids are allowed, that is, bidders may bid on both items without fear of winning both. (In the spectrum auction example above, this corresponds to package bidding, since the New York firm might bid on a package containing the east coast and center licenses, as well as bidding separately on the east coast license alone.)

We demonstrate symmetric equilibria with the following features. Bidders with high valuations bid aggressively (at or near the reserve price) on the larger item, to ensure that the reserve price is met with high probability. However, at this price, they prefer to be outbid by their opponent and win the small item cheaply; so they bid on it as well, in an

attempt to sabotage their other bid. And since the preference for winning the small rather than the large item is stronger for bidders with lower valuations, bids for the small item are *decreasing* in the bidder’s type.

2 Model and Result

There are two bidders, $i = \{1, 2\}$, each with a type t_i drawn independently from the uniform distribution on $[0, 1]$. There are two items, a large one and a small one. Bidder i values the large item at $U(t_i)$ and the small item at $u(t_i)$, and can benefit from only one of them.

Assumption 1. $U(\cdot)$, $u(\cdot)$, and $U(\cdot) - u(\cdot)$ are positive, strictly increasing, and continuous.

We consider a sealed-bid, pay-as-bid auction with “exclusive-or” bids – each bidder can submit separate bids b_i and B_i on the small and large items, respectively, knowing that at most one bid from each bidder will be chosen as a winner. When no bid is submitted, a bid of zero is assumed. (A bid of zero can still win one item when the other bidder bids at least the reserve price for the other.) There is a single aggregate reserve price r : if $b_i + B_j > b_j + B_i$ and $b_i + B_j \geq r$, then i pays b_i for the small item and j pays B_j for the large one. If $b_1 + B_2 = b_2 + B_1 \geq r$, the tie is broken randomly. If $\max\{b_1 + B_2, b_2 + B_1\} < r$, neither item is sold and the game ends.

Note that without a reserve price, there would be no reason to bid on the small item, as the losing bidder would be assured of getting it for free; bids for the large item would be identical to those in a single-item auction for an object worth $U(t_i) - u(t_i)$ to a bidder with type t_i . Thus, all the interesting effects we get are due to the presence of a positive aggregate reserve price.

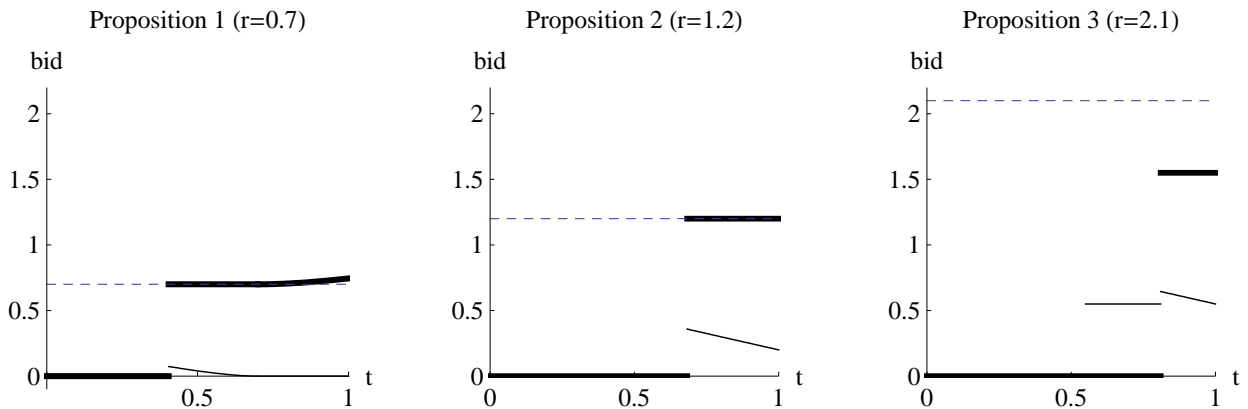
Our result is the existence of certain symmetric equilibria:

Theorem 1. For any reserve price $r \in (U(0) - u(0), U(1) + u(1))$, there exists a symmetric Bayesian Nash equilibrium where both bidders use the same bidding function $(b(t_i), B(t_i))$, and over some interval $(\underline{t}, \bar{t}) \subset [0, 1]$,

- $B(\cdot)$ and $b(\cdot)$ are both strictly positive, and both win with strictly positive probability
- $u(t_i) - b(t_i) > U(t_i) - B(t_i)$, so a bidder strictly prefers to be outbid for the large prize and win the small one instead
- $B(\cdot)$ is constant and $b(\cdot)$ is strictly decreasing

Define $X(\cdot) \equiv U(\cdot) - u(\cdot)$. The proof of Theorem 1 is constructive: the three propositions below establish such equilibria for three cases, $r \in (X(0), X(1))$, $r \in [X(1), U(1))$, and $r \in [U(1), U(1) + u(1))$. These equilibria are pictured here:

Figure 1: Three representative equilibria when $u(t) = t$ and $U(t) = 2t$. The dashed lines represent the reserve price, the heavy curves $B(t)$, and the lighter curves $b(t)$.



More generally, the equilibria are described in the following three propositions, which are proven in the appendix.

Proposition 1. *If $U(0) - u(0) < r < U(1) - u(1)$, then the following bidding strategies constitute a symmetric Bayesian Nash Equilibrium:*

$$(1) \quad b(t_i) = \begin{cases} 0 & \text{if } t_i < t^* \\ \frac{1}{1-t_i} \int_{t_i}^{t^{**}} (r - X(s)) ds & \text{if } t_i \in [t^*, t^{**}) \\ 0 & \text{if } t_i \geq t^{**} \end{cases}$$

$$B(t_i) = \begin{cases} 0 & \text{if } t_i < t^* \\ r & \text{if } t_i \in [t^*, t^{**}) \\ \frac{1}{t_i} \int_0^{t_i} \max\{r, X(s)\} ds & \text{if } t_i \geq t^{**} \end{cases}$$

where t^{**} solves $X(t^{**}) = r$ and t^* is the unique solution to $t^*U(t^*) + \int_{t^*}^{t^{**}} X(s) ds = t^{**}X(t^{**})$.

Proposition 2. *If $U(1) - u(1) \leq r < U(1)$, then the following bidding strategies constitute a symmetric Bayesian Nash Equilibrium:*

$$(2) \quad b(t_i) = \begin{cases} 0 & \text{if } t_i < t^* \\ r - \bar{X}(t_i) & \text{if } t_i \geq t^* \end{cases}$$

$$B(t_i) = \begin{cases} 0 & \text{if } t_i < t^* \\ r & \text{if } t_i \geq t^* \end{cases}$$

where $\bar{X}(t) \equiv E_{s \geq t} X(s)$ ¹ and t^* is the unique solution to $r = t^*U(t^*) + (1 - t^*)\bar{X}(t^*)$.

¹Formally, $\bar{X}(t) = \frac{1}{1-t} \int_t^1 X(s) ds$ when $t < 1$ and $\bar{X}(1) = X(1)$

Proposition 3. *If $U(1) \leq r < U(1) + u(1)$, then the following bidding strategies constitute a symmetric Bayesian Nash Equilibrium:*

$$(3) \quad b(t_i) = \begin{cases} 0 & \text{if } t_i < t^* \\ \frac{1}{2}(r - X(1)) & \text{if } t_i \in [t^*, t^{**}) \\ \frac{1}{2}(r + X(1)) - \bar{X}(t_i) & \text{if } t_i \geq t^{**} \end{cases}$$

$$B(t_i) = \begin{cases} 0 & \text{if } t_i < t^{**} \\ \frac{1}{2}(r + X(1)) & \text{if } t_i \geq t^{**} \end{cases}$$

where $t^* = 0$ if $u(0) > \frac{1}{2}(r - X(1))$, t^* solves $u(t^*) = \frac{1}{2}(r - X(1))$ otherwise, and t^{**} is the unique solution over $(t^*, 1)$ to

$$(4) \quad (1 - t^{**})(X(1) - \bar{X}(t^{**})) = (t^{**} - t^*) \left(U(t^{**}) - \frac{1}{2}(r + X(1)) \right)$$

Theorem 1 follows directly, with the interval (\underline{t}, \bar{t}) corresponding to (t^*, t^{**}) in Proposition 1, $(t^*, 1)$ in Proposition 2, and $(t^{**}, 1)$ in Proposition 3. Note that within this interval, reducing $b(t_i)$ (to zero under Propositions 1 and 2, or to $\frac{1}{2}(r - X(1))$ under Proposition 3) would not reduce a bidder's chance of winning *something*, only increase his chance of winning the large object; thus, the bid b_i can be thought of as an attempt to undermine the same bidder's larger bid B_i . Note also that in all of these equilibria, whenever the reserve price is met, the objects are allocated efficiently – the bidder with the higher type never wins the small object.

At a given reserve price, there may be more than one symmetric equilibrium. For example, when $u(t_i) = t_i$, $U(t_i) = 2t_i$, and $r \in (1, 2)$, the strategies in both Propositions 2 and 3 are equilibria.² And all symmetric equilibria need not satisfy the conditions of Theorem 1. We

²For general u and U and $r \in (U(1) - u(1), U(1))$, the strategies in Proposition 3 may or may not be an equilibrium: when $t^* = u^{-1}(\frac{1}{2}(r - X(1)))$ is high enough, bidders with high types can gain by deviating to

show in a separate working paper (Lien and Quint (2008)) that when $u(t_i) = t_i$, $U(t_i) = 2t_i$, and $r = \frac{3}{2}$, the strategies

$$(5) \quad b(t_i) = \frac{1}{2}t_i \quad \text{and} \quad B(t_i) = \begin{cases} 0 & \text{if } t_i \leq \frac{2}{3} \\ t_i + \frac{1}{2} & \text{if } t_i > \frac{2}{3} \end{cases}$$

are also an equilibrium. Both b and B here are increasing in a bidder's type, but the first two parts of Theorem 1 still hold: bidders with high types bid seriously on both items, and strictly prefer to "lose" and get the small one. For general valuation functions u and U , when $r \geq U(1)$, there is also a symmetric no-trade equilibrium, as well as a continuum of asymmetric equilibria.

3 Literature and Discussion

Equilibrium strategies in pay-as-you-bid package auctions with incomplete information are notoriously difficult to calculate. In an experimental paper, Chernomaz and Levin (2007) calculate equilibrium strategies in a first price package auction for two identical items with two local bidders (each demanding one item) and a global bidder (demanding both). To maintain tractability, they assume that the local bidders' private values are perfectly correlated. Discussions on package auctions with various designs can be found in Milgrom (2004), Cramton, Shoham, and Steinberg (2006), and Goeree and Holt (2007).

Equilibrium bids in Levin, Peck, and Ye (2007) show a similar sort of reversal to ours. In an ascending auction with common and private value components, they show that bidders may bid more aggressively following an earlier dropout than a later dropout by a competing bidder, because the earlier dropout indicates a higher common value due to the particular $(b, B) = (0, r)$. For $u(t) = t$ and $U(t) = 2t$, this deviation turns out not to be profitable, and these strategies are indeed an equilibrium.

relationship between signals and values. In contrast, there is no common value component in our model, so our reversal has nothing to do with a revised estimate of an object's value.

McAdams (2006) considers auctions for multiple identical units to bidders with multi-unit demand and private values. He shows that with risk neutrality and independent (multi-dimensional) types, all equilibria in both the uniform-price and discriminatory auction are outcome-equivalent (in both allocation and interim expected payoff) to a monotone pure-strategy equilibrium. McAdams (2007) shows that independence and risk-neutrality are both necessary for this result; with either affiliated private values or risk aversion, he gives examples where all equilibria in undominated strategies are non-monotone. Our paper differs by considering non-identical goods and unit demand (and a different auction rule), but generates non-monotone equilibria (which are not outcome-equivalent to monotone equilibria³) while maintaining independent private values and risk-neutrality.

Ausubel and Cramton (2004) consider Vickrey auctions with reserve prices, but in a setting with a single, divisible good; they find that bidding truthfully is an ex post equilibrium. Blume et al. (2008) characterize all Nash equilibria of the Vickrey auction for multiple identical (indivisible) items; they show that with any positive reserve price, truthful bidding is the unique equilibrium. In both these papers, the reserve price is set for each unit, while in ours, the reserve price is for the overall revenue of the auction; with only an aggregate reserve price, payments in the Vickrey auction are not clearly defined.⁴

³ No monotone equilibrium can match the allocations of the equilibria in this paper. For example, consider the equilibrium in Proposition 2, which allocates the objects efficiently as long as either bidder's type is at least t^* . With monotone strategies, this requires both bidders to bid $B_i(t^*) \geq r$ and $B_i(\cdot)$ strictly increasing on $(t^*, 1)$. Take any $t_i > t^*$. If $b_i(t_i) > 0$, a deviation to $(B_i, b_i) = (B_i(t_i) - \epsilon, b_i(t_i) - \epsilon)$ is profitable for small enough ϵ ; if $b_i(t_i) = 0$, a deviation to $(B_i, b_i) = (r, 0)$ is profitable.

⁴If the objects are only sold when the usual Vickrey payments exceed the reserve price, there will be cases in which the auction is ex-post inefficient given the reported types, and truth-telling may not be an equilibrium. If the auction is modified to allocate the items efficiently whenever their combined value exceeds

Papers on multi-unit auctions sometimes refer to “small,” or “local,” bidders, who want only a single item and do not compete with each other; and a “large,” or “global,” bidder, who bids on a package of all available items. In this terminology, we see our bidders as “medium-sized”; they are big enough to have overlapping interests, but the overlap is not complete. Thus, their interaction contains elements of both coordination (jointly meeting the reserve price) and competition (for the more desirable of the two prizes).

We offered two examples in the introduction to motivate the use of an aggregate reserve price. The first was a procurement setting where the two prizes were parts of a single large project. To see how that story maps to our model, suppose a construction firm with type t_i could complete the larger piece at cost $M - U(t_i)$, or the smaller at cost $m - u(t_i)$, but not both. Interpret a bid (b, B) as an offer to perform the larger piece of the project for a price $M - B$, or the smaller for price $m - b$; the reserve price r is now the difference between $M + m$ and the government’s maximum willingness to pay to complete both projects.

The second example was a spectrum auction for three licenses, with east and west coast firms each wanting the license to their home region or to their home region plus the middle of the country, and the seller has a standing offer for all three licenses from a third buyer. Interpret a bid (b_i, B_i) by firm i as an offer to pay b_i for the license for its home region, or B_i for the licenses for its home region and the middle of the country. As long as $B_i \geq b_i$ for $i \in \{1, 2\}$ (which occurs naturally in equilibrium), the seller chooses between revenues of $B_1 + b_2$, $b_1 + B_2$, and the offer from the third buyer.

An aggregate reserve price seems natural when the seller values the unsold goods as complements, and therefore wants to sell all or none of them – a motorcyclist can repair his bike after a crash or strip it down and sell the parts individually to finance his next bike,

the reserve price, using some payment rule to make up the “shortfall” when the usual Vickrey payments are below the reserve price, truth-telling would not be expected to remain an equilibrium.

or a seller must manufacture the items after the auction and faces economies of scale. FCC spectrum auction 66, which raised \$13.7 billion in 2006, effectively had an aggregate reserve of \$2.06 billion on all 1122 licenses for sale, “in order to implement a Congressional mandate to recover estimated relocation costs for government incumbent operators.”⁵ The combination of an aggregate reserve price and package bidding seems natural when the bidders, as well as the seller, value at least some of the objects as complements, and would therefore face an exposure problem if the items were auctioned separately; or, as in our model, when the objects are sufficiently different and the buyers have unit demand. In fact, we show in Lien and Quint that in the example mentioned above – $u(t_i) = t_i$, $U(t_i) = 2t_i$, $r = \frac{3}{2}$ – the auction considered in this paper, along with the equilibrium described in Equation 5, implements the optimal mechanism when the seller values the unsold goods at $v_0 = 1$ and (as in the procurement setting) is unable to sell one without the other.

Thus, in this paper, we consider an auction which seems sensible in many settings, and in one example can even implement the optimal mechanism; and demonstrate equilibria in which bidders “bid against themselves” and, even with independent private values and one-dimensional types, bid functions are decreasing over some range of valuations.

⁵Auction No. 66 Procedures Public Notice, FCC 06-47. Auction 73 (January-March 2008) also employed aggregate reserve prices: the 1099 licenses being offered were organized into five “blocks,” with an aggregate reserve price applying to each block; four of the five were met, and the auction raised \$19 billion.

Appendix. Proofs of Propositions 1-3

A.1 Proposition 1

t^* Uniquely Defined, Less Than t^{**}

t^{**} is uniquely defined since $X(\cdot)$ is assumed to be strictly increasing and continuous with $X(0) < r < X(1)$. Let $\phi(t) = tU(t) + \int_t^{t^{**}} X(s)ds$, which is continuous and strictly increasing. As $t \rightarrow 0$, $\phi(t) \rightarrow \int_0^{t^{**}} X(s)ds < t^{**}X(t^{**})$. As $t \rightarrow t^{**}$, $\phi(t) \rightarrow t^{**}U(t^{**}) > t^{**}X(t^{**})$. So $\phi(t) = t^{**}X(t^{**})$ has a unique solution which is strictly below t^{**} .

Deviations to Equilibrium Strategies of Other Types

Let $\pi(\hat{t}, t)$ be the expected payoff of a bidder with type \hat{t} who makes the equilibrium bid of a bidder with type t . For $t \geq t^{**}$, he wins the small item for free when his opponent has type $t_j > t$, and the large object for $B(t)$ otherwise, so

$$\begin{aligned} \pi(\hat{t}, t) &= (1-t)u(\hat{t}) + tU(\hat{t}) - tE(\max\{r, X(s)\} \mid s < t) \\ &= u(\hat{t}) + tX(\hat{t}) - t^{**}r - \int_{t^{**}}^t X(s)ds \\ &\quad \downarrow \\ \frac{\partial}{\partial t}\pi(\hat{t}, t) &= X(\hat{t}) - X(t) \end{aligned}$$

For $t^* \leq t < t^{**}$, he wins the small object for $b(t)$ when $t_j > t$, and the large object for r otherwise, so

$$\begin{aligned} \pi(\hat{t}, t) &= (1-t)u(\hat{t}) - (1-t)\left(\frac{t^{**}-t}{1-t}E(r - X(s) \mid s \in (t, t^{**}))\right) + tU(\hat{t}) - tr \\ &= u(\hat{t}) - (t^{**}-t)r + (t^{**}-t)E(X(s) \mid s \in (t, t^{**})) + tX(\hat{t}) - tr \\ &= u(\hat{t}) + tX(\hat{t}) - t^{**}r + \int_t^{t^{**}} X(s)ds \\ &\quad \downarrow \\ \frac{\partial}{\partial t}\pi(\hat{t}, t) &= X(\hat{t}) - X(t) \end{aligned}$$

(Note that the left- and right-limits of $\pi(\hat{t}, t)$ as $t \rightarrow t^{**}$ are identical, since equilibrium bids approach $(0, r)$ from both sides.) Finally, the difference in expected payoff between bidding as type t^* and

bidding $(0, 0)$ is

$$\begin{aligned}
\pi(\hat{t}, t^*) - \pi(\hat{t}, (0, 0)) &= (1 - t^*)u(\hat{t}) - (1 - t^*) \left(\frac{t^{**} - t^*}{1 - t^*} E(r - X(s) \mid s \in (t^*, t^{**})) \right) \\
&\quad + t^*U(\hat{t}) - t^*r - (1 - t^*)u(\hat{t}) \\
&= -(t^{**} - t^*)r + (t^{**} - t^*)E(X(s) \mid s \in (t^*, t^{**})) + t^*U(\hat{t}) - t^*r \\
&= -t^{**}r + t^*U(\hat{t}) + \int_{t^*}^{t^{**}} X(s)ds
\end{aligned}$$

which is increasing in \hat{t} and, by construction of t^* , equal to 0 at $\hat{t} = t^*$. Together, these establish that $\pi(\hat{t}, t)$ is at least weakly increasing in t for $t < \hat{t}$ and decreasing in t for $t > \hat{t}$, ruling out profitable deviations to other bidders' equilibrium bids.

Deviations to Nobody's Equilibrium Strategy

A deviation to (B, b) with $B > r$ and $b > 0$ is dominated by a deviation to either $(B - b, 0)$ or $(r, b - (B - r))$. A deviation to $(B, 0)$ with $B > B(1)$ is dominated by $(B(1), 0)$. A deviation to (r, b) with $b > b(t^*)$ is dominated by $(r, b(t^*))$. This rules out all additional deviations to (B, b) with $B \geq r$.

A deviation to (B, b) with $b \geq r > B > 0$ is dominated by $(0, b - B)$ if $b - B \geq r$, and by $(B - (b - r), r)$ otherwise. A deviation to (B, r) with $B < r$ is equivalent to a bid of $(0, r)$, since either one will always win the small object for price r ; this is dominated by bidding $(r, 0)$, since this wins either the large object at r or the small object for free. Bids of $(0, b)$ with $b > r$ are similarly dominated.

All remaining deviations only win anything when the opponent's type is $t_j \geq t^*$. Since bidding $(0, 0)$ gets the small item for free in all these situations, the only potential deviations are to bid high enough to sometimes win two items against opponent types above t^* . Since $b(t^*)$ is the highest

opponent bid on the small item, such bids only matter if

$$\begin{aligned}
B &> r - b(t^*) \\
&= \frac{1}{1-t^*} \left((1-t^*)r - (t^{**} - t^*)r + \int_{t^*}^{t^{**}} X(s)ds \right) \\
&= \frac{1}{1-t^*} \left((1-t^{**})r + \int_{t^*}^{t^{**}} X(s)ds \right) \\
&= E(\min\{X(s), X(t^{**})\} \mid s > t^*) \\
&> X(t^*)
\end{aligned}$$

Thus, this is only profitable for types $t > t^*$, since types below t^* prefer the small item for free.

Consider a deviation by type $t > t^*$ to $(0, B)$ with $B = r - b(t')$ for some $t' \in (t^*, t^{**})$, and compare it to simply playing the equilibrium strategy of type t' . This being profitable would require

$$\begin{aligned}
(t' - t^*)(U(t) - r + b(t')) + (1 - t')u(t) &> t'(U(t) - r) + (1 - t')(u(t) - b(t')) \\
t'U(t) - t'r + t'b(t') - t^*U(t) + t^*r - t^*b(t') &> t'U(t) - t'r - b(t') + t'b(t') \\
(t' - t^*)(U(t) - r + b(t')) &> t'(U(t) - r) - (1 - t')b(t') \\
t'U(t) - t'r + t'b(t') - t^*U(t) + t^*r - t^*b(t') &> t'U(t) - t'r - b(t') + t'b(t') \\
-t^*U(t) + t^*r - t^*b(t') &> -b(t') \\
-t^*U(t) + t^*r + (1 - t^*)b(t') &> 0 \\
(1 - t^*)u(t) &> t^*(U(t) - r) + (1 - t^*)(u(t) - b(t')) \\
(1 - t^*)u(t) &> t^*(U(t) - r) + (1 - t^*)(u(t) - b(t^*))
\end{aligned}$$

since $b(t^*) < b(t')$. But this means bidding like a low type must be preferable to bidding like t^* for some type $t > t^*$, which we ruled out above.

A.2 Proposition 2

t^* Uniquely Defined

Let $\psi(t) = tU(t) + (1-t)\bar{X}(t) = tU(t) + \int_t^1 (U(s) - u(s))ds = \int_0^1 (U(\max(t, s)) - \mathbf{1}_{s>t}u(s)) ds$. This is continuous, strictly increasing, approaches $\bar{X}(0) < X(1) \leq r$ as $t \rightarrow 0$, and approaches $U(1) > r$ as $t \rightarrow 1$; so $\psi(t) = r$ has a unique solution.

Deviations to Equilibrium Strategies of Other Types

First, consider a bidder with type \hat{t} who bids as if he had type $t > t^*$. He wins the big item for r when $t_j < t$, and the small item for $b_i(t)$ when $t_j > t$, for an expected payoff of

$$\begin{aligned}\pi(\hat{t}, t) &= t(U(\hat{t}) - r) + (1 - t)(u(\hat{t}) - r + \bar{X}(t)) \\ &= u(\hat{t}) - r + tX(\hat{t}) + \int_t^1 X(s)ds \\ &\quad \downarrow \\ \frac{\partial}{\partial t}\pi(\hat{t}, t) &= X(\hat{t}) - X(t)\end{aligned}$$

The difference in expected payoffs between bidding like type t^* and bidding like a low type is

$$\begin{aligned}\pi(\hat{t}, t^*) - \pi(\hat{t}, (0, 0)) &= u(\hat{t}) - r + t^*X(\hat{t}) + \int_{t^*}^1 X(s)ds - (1 - t^*)u(\hat{t}) \\ &= t^*u(\hat{t}) - r + t^*X(\hat{t}) + \int_{t^*}^1 X(s)ds \\ &= t^*U(\hat{t}) - r + \int_{t^*}^1 X(s)ds\end{aligned}$$

which is increasing in \hat{t} and, by construction of t^* , 0 at $\hat{t} = t^*$. Together, these tell us that types above t^* prefer their own equilibrium strategies to those of other types above t^* , including t^* , and prefer this to bidding like a low type; and types below t^* prefer bidding their equilibrium strategies to bidding like t^* , which they prefer to bidding like a higher type. Type t^* is indifferent between his own strategy and bidding $(0, 0)$, and strictly prefers either to bidding like a higher type. So no type can gain by bidding like a different type.

Deviations to Nobody's Equilibrium Strategies

Bids of (B, b) with $B > r$ and $b > 0$ are dominated by either $(r, b - (B - r))$ or $(B - b, 0)$. Bids of (r, b) with $b > b(t^*)$ are dominated by $(r, b(t^*))$, the equilibrium strategy of type t^* . Bids of $(B, 0)$ with $B > r$ are dominated by $(r, 0)$, since given other bidders' strategies, this still always wins the large item. Given equilibrium bids, $(r, 0)$ gives the same result as $(r, b(1))$, the equilibrium strategy of type $t = 1$. This rules out profitable deviations to (B, b) with $B \geq r$.

A deviation to (B, b) with $b \geq r > B > 0$ is dominated by $(0, b - B)$ if $b - B \geq r$, and by $(B - (b - r), r)$ otherwise. A deviation to (B, r) with $B < r$ is equivalent to a bid of $(0, r)$, since

either one will always win the small object for price r ; this is dominated by bidding $(r, 0)$, since this wins either the large object at r or the small object for free. Bids of $(0, b)$ with $b > r$ are similarly dominated. This rules out deviations which ever win against types $t_j < t^*$. Bidding $(0, 0)$ wins the small object for free against all types $t_j \geq t^*$, so the only possible deviations remaining are to sometimes win the large object rather than the small one when $t_j \geq t^*$, by bidding $B > r - b(t^*)$. We can assume without loss that these are deviations to $(B, 0)$ with $B < r$.

Such a deviation only matters if $B > r - b(t^*) = \bar{X}(t^*)$. For $t < t^*$, this deviation is not profitable – a bidder with $t < t^*$ prefers the small item for free to the large item at price $X(t^*) < \bar{X}(t^*)$. So we only need to worry about deviations by high types.

Fix $t > t^*$, and suppose a bidder with type t considers such a strategy. Bids $B \leq r - b(t^*)$ are irrelevant, since they never win; bids $B > r - b(1)$ are dominated by $B = r - b(1)$, since either one wins the big object against all opponents $t_j \geq t^*$ but the latter involves paying less. So we may assume that any potential deviation strategy $(B, 0)$ with $B < r$ satisfies $r - b(t^*) \leq B \leq r - b(1)$, and therefore, by continuity of b above t^* , $B = r - b(t')$ for some $t' \in [t^*, 1]$. The expected payoff to this strategy, then, is

$$\begin{aligned}
\pi(t, (B, 0)) &= (1 - t')u(t) + (t' - t^*)(U(t) - (r - b(t'))) \\
&= (1 - t')(u(t) - b(t')) + t'(U(t) - r) - t^*(U(t) - r) + (1 - t^*)b(t') \\
&= \pi(t, t') - t^*(U(t) - r) + (1 - t^*)(r - \bar{X}(t')) \\
&= \pi(t, t') + r - t^*U(t) - (1 - t^*)\bar{X}(t') \\
&= \pi(t, t') + t^*U(t^*) + (1 - t^*)\bar{X}(t^*) - t^*U(t) - (1 - t^*)\bar{X}(t') \\
&= \pi(t, t') - t^*(U(t) - U(t^*)) - (1 - t^*)(\bar{X}(t') - \bar{X}(t^*)) \\
&\leq \pi(t, t') \leq \pi(t, t)
\end{aligned}$$

where the first inequality is because by assumption, $t \geq t^*$, $t' \geq t^*$, and U and \bar{X} are increasing, and the second is because we've already ruled out profitable deviations to other types' equilibrium strategies.

A.3 Proposition 3

t^* and t^{**} Uniquely Defined

Since $r < U(1) + u(1)$, $\frac{1}{2}(r - X(1)) < u(1)$; so either $u(0) \geq \frac{1}{2}(r - X(1))$ or $u(t) = \frac{1}{2}(r - X(1))$ has a unique interior solution t^* .

As for t^{**} , the expression $(1 - t)(X(1) - \bar{X}(t))$ is decreasing in t , goes to 0 as $t \rightarrow 1$, and is strictly positive everywhere else. Let t^{***} solve $U(t^{***}) = \frac{1}{2}(r + X(1))$ if it has a solution above t^* , otherwise let $t^{***} = t^*$. (Since $r < U(1) + u(1)$, $\frac{1}{2}(r + X(1)) < U(1)$, so $t^{***} < 1$.) Then the expression $(t - t^*)(U(t) - \frac{1}{2}(r + X(1)))$ is 0 at $t = t^{***}$ and strictly increasing above it. This guarantees equation 4 has a unique solution on $(t^*, 1)$.

Deviations to Equilibrium Strategies of Other Types

A bidder with type \hat{t} who bids like type $t > t^{**}$ gets expected payoff

$$\begin{aligned}
 \pi(\hat{t}, t) &= (1 - t) \left(u(\hat{t}) - \frac{1}{2}(r + X(1)) + \bar{X}(t) \right) + (t - t^*) \left(U(\hat{t}) - \frac{1}{2}(r + X(1)) \right) \\
 &= (1 - t^*)u(\hat{t}) + (t - t^*)X(\hat{t}) - (1 - t^*)\frac{1}{2}(r + X(1)) + (1 - t)\bar{X}(t) \\
 &= (1 - t^*)u(\hat{t}) + (t - t^*)X(\hat{t}) - (1 - t^*)\frac{1}{2}(r + X(1)) + \int_t^1 X(s)ds \\
 &\quad \downarrow \\
 \frac{\partial}{\partial t} \pi(\hat{t}, t) &= X(\hat{t}) - X(t)
 \end{aligned}$$

so no type $\hat{t} \neq t$ will deviate to the equilibrium strategy of type $t > t^{**}$, and no type $\hat{t} > t^{**}$ will deviate to the equilibrium strategy of type t^{**} . The gain from bidding like t^{**} instead of like a type $t \in [t^*, t^{**})$ is

$$\begin{aligned}
 \pi(\hat{t}, t^{**}) - \pi(\hat{t}, t) &= (t^{**} - t^*) \left(U(\hat{t}) - \frac{1}{2}(r + X(1)) \right) + (1 - t^{**}) \left(u(\hat{t}) - \frac{1}{2}(r + X(1)) + \bar{X}(t^{**}) \right) \\
 &\quad - (1 - t^{**}) \left(u(\hat{t}) - \frac{1}{2}(r - X(1)) \right) \\
 &= (t^{**} - t^*) \left(U(\hat{t}) - \frac{1}{2}(r + X(1)) \right) \\
 &\quad + (1 - t^{**}) \left(\frac{1}{2}(r - X(1)) - \frac{1}{2}(r + X(1)) + \bar{X}(t^{**}) \right) \\
 &= (t^{**} - t^*) \left(U(\hat{t}) - \frac{1}{2}(r + X(1)) \right) - (1 - t^{**}) \left(X(1) - \bar{X}(t^{**}) \right)
 \end{aligned}$$

which is strictly increasing in \hat{t} and, by equation 4, equal to 0 at $\hat{t} = t^{**}$, so no type $\hat{t} < t^{**}$ can gain from this deviation, and no type $\hat{t} > t^{**}$ can benefit from bidding like a type $t \in [t^*, t^{**})$.

Finally, bidding like a type $t \in [t^*, t^{**})$ gives a payoff of $(1 - t^{**}) (u(\hat{t}) - \frac{1}{2}(r - X(1)))$, which by construction is positive for $\hat{t} > t^*$ and negative for $\hat{t} < t^*$; so low types can't gain by impersonating medium types (or, by transitivity, high types), and medium types can't gain by impersonating low types (nor can high types by transitivity).

Deviations to Nobody's Equilibrium Strategy

Deviations to $B \geq r$ are unprofitable because by assumption $r \geq U(1)$. (When $X(1) \leq r < U(1)$, the strategies in Proposition 3 may still be an equilibrium, but this deviation must be explicitly checked; if $u(\cdot)$ is steep enough near 0, the equilibrium may break down.) Note that the rule for choosing a winner implies increasing B cannot increase the chance your other bid wins. Deviations to $b \geq r$ are similarly unprofitable.

Deviations to (b, B) with $B \in (\frac{1}{2}(r + X(1)), r)$ can be ruled out because if $b > 0$, reducing both b and B until either $B = \frac{1}{2}(r + X(1))$ or $b = 0$ does strictly better, and if $b = 0$, then reducing B to $\frac{1}{2}(r + X(1))$ does strictly better. The latter case – bidding $B = \frac{1}{2}(r + X(1))$ and $b = 0$ – is payoff-equivalent to making the equilibrium bids of type $t = 1$, since the low bid $b(1)$ never wins in equilibrium. The former case – bidding $B = \frac{1}{2}(r + X(1))$ and $b > 0$ – is either the equilibrium bid of some type $t \in [t^*, t^{**})$, or else dominated by bidding like type t^* . We've already ruled out profitable deviations to equilibrium bids of other types, so these are not profitable either.

Any other deviation wins nothing against opponent types $t_j < t^{**}$, and bidding like a type $t \in [t^*, t^{**})$ already gets the highest possible payoff against opponents $t_j \geq t^{**}$, so there are no other profitable deviations.

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