Bidding Reversals in a Multiple-Good Auction with Aggregate Reserve Price

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Abstract

In an auction for two heterogeneous goods, we show an "aggregate" reserve price leads to equilibria where bidders bid on one item to sabotage their own bid on the other, and bids are *decreasing* in valuations over a certain range.

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1 Introduction

The literature on multi-unit auctions typically assumes that the auctioneer's valuation for unsold goods is additive, or, in the case of the much-studied FCC spectrum auctions, zero. Just as buyers might perceive goods as complements, however, there are a number of reasons why a seller might value them as complements:

- If a procurement auction is being used to solicit low bids for two parts of a single construction project, awarding one contract without the other may be pointless – there's no value in awarding a cheap contract to build a road going to a bridge, if the bridge itself cannot be built affordably.
- A seller might take pre-orders for goods which he will manufacture after the auction; if there are economies of scale in production, he may only want to sell any goods if he can sell at least a certain number.
- An outside buyer might be interested in all of the objects together, so the seller would only want to "break up the set" if a certain amount of money can be raised.
- The seller might only value the unsold goods if he keeps all of them. After a crash, a motorcyclist might be unsure whether to get his damaged bike fixed, or whether to sell off the undamaged parts separately to finance a new bike; the last thing he would want would be to sell off some parts but not others.

Multi-unit auctions also raise the possibility of having reserve prices that apply to collections of goods rather than individual goods – *aggregate* reserve prices which bids on several objects must add up to, or else *none* of those objects is awarded. An aggregate reserve price would allow a government to only award either contract if both projects could be completed at a certain cost; a manufacturer to only sell goods if the revenue were enough to finance a run of production at the efficient scale; and the motorcyclist to only sell off anything if all the undamaged parts on his bike together could raise a certain amount of money. Thus, when objects for sale are complements to the seller, aggregate reserve prices seem potentially appealing.

Aggregate reserves have been employed in multi-unit auctions, if sparingly. Two of the largest FCC spectrum auctions to date employed aggregate reserve prices of some sort. Auction 66, held in 2006, had an effective aggregate reserve of \$2.06 billion applying to all 1122 licenses for sale, "in order to implement a Congressional mandate to recover estimated relocation costs for government incumbent operators" already using the bandwidth for sale.[1]

[1] Auction No. 66 Procedures Public Notice, FCC 06-47.

The reserve was easily exceeded, and the auction raised \$13.7 billion. Auction 73, held in early 2008, offered 1099 licenses organized into five "blocks," with one aggregate reserve price applying to each block; four of the five were met, and \$19 billion was raised.

Aggregate reserve prices, however, introduce an interesting wrinkle. Bidding involves both competition (for the "more desirable" objects) and coordination (bidders attempting to jointly meet the reserve price so the auction is not cancelled). In this paper, we show that even a very simple model (two objects, risk-neutral bidders, one-dimensional types) with aggregate reserve prices introduces threshold effects leading to unusual bidding behavior: bidders effectively bid against themselves, and do so in a way that leads to non-monotonic equilibrium strategies.

2 Result and Discussion

There are two bidders, $i = \{1, 2\}$, each with a type t_i drawn independently from the uniform distribution on $[0, 1]$. There are two items, a large one and a small one. Bidder i values the large item at $U(t_i)$ and the small item at $u(t_i)$, and can benefit from only one of them.[2]

Assumption 1. *$U(\cdot)$, $u(\cdot)$, and $U(\cdot) - u(\cdot)$ are positive, strictly increasing, and continuous.*

We consider a sealed-bid, pay-as-bid auction with "exclusive-or" bids – each bidder can submit separate bids b_i and B_i on the small and large items, respectively, knowing that at most one bid from each bidder will be chosen as a winner. When no bid is submitted, a bid of zero is assumed. (A bid of zero can still win one item when the other bidder bids at least the reserve price for the other.) There is a single aggregate reserve price r: if $b_i + B_j > b_j + B_i$ and $b_i + B_j \geq r$, then i pays b_i for the small item and j pays B_j for the large one. If $b_1 + B_2 = b_2 + B_1 \geq r$, the tie is broken randomly. If $\max\{b_1 + B_2, b_2 + B_1\} < r$, neither item is sold and the game ends.

Note that without a reserve price, there would be no reason to bid on the small item, as the losing bidder would be assured of getting it for free; bids for the large item would be identical to those in a single-item auction for an object worth $U(t_i) - u(t_i)$ to a bidder with type t_i. Thus, the effects we demonstrate below are due to the presence of a positive aggregate reserve price.

Our result is the existence of certain symmetric equilibria:

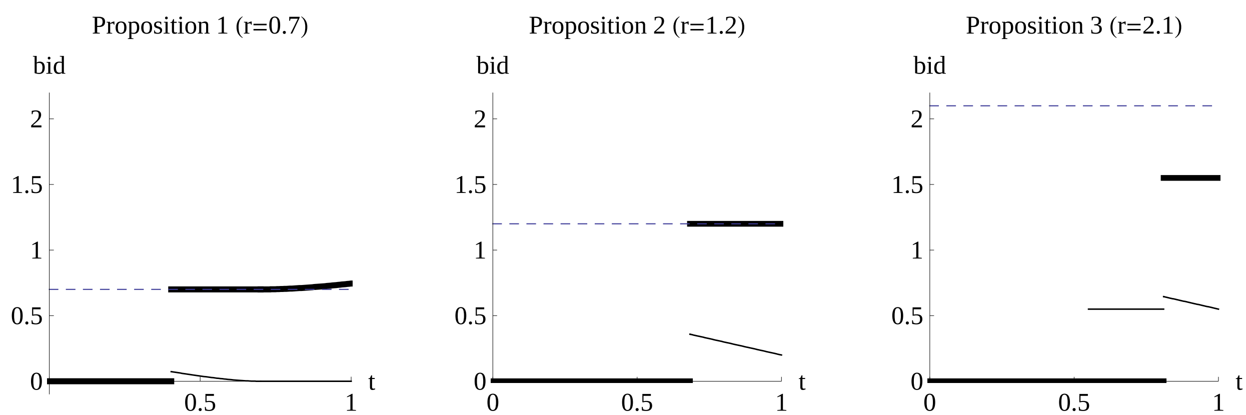
[2]Given the lack of restrictions on U and u, $t_i \sim \mathrm{U}[0, 1]$ is just a normalization: if $t_i \sim F$, replace t_i with $\tilde{t}_i = F(t_i) \sim \mathrm{U}[0, 1]$, and u and U with $\tilde{u}(\tilde{t}_i) = u(F^{-1}(\tilde{t}_i))$ and $\tilde{U}(\tilde{t}_i) = U(F^{-1}(\tilde{t}_i))$. Also note that our model nests the "pure chopstick game" considered by Szentes and Rosenthal (2003): suppose there are three identical objects, and let u denote the value of a single one and U the value of a pair.

Theorem 1. *For any reserve price $r \in (U(0) - u(0),\ U(1) + u(1))$, there exists a symmetric Bayesian Nash equilibrium where both bidders use the same bidding function $(b(t_i), B(t_i))$, and over some interval $\left(\underline{t}, \bar{t}\right) \subset [0,1]$,*

- *$B(\cdot)$ and $b(\cdot)$ are both strictly positive, and both win with strictly positive probability*
- *$u(t_i) - b(t_i) > U(t_i) - B(t_i)$, so a bidder strictly prefers to be outbid for the large prize and win the small one instead*
- *$B(\cdot)$ is constant and $b(\cdot)$ is strictly* **decreasing**

The proof of Theorem 1 is constructive: three propositions in the appendix establish such equilibria for three cases, $r \in (U(0) - u(0), U(1) - u(1))$, $r \in [U(1) - u(1), U(1))$, and $r \in [U(1), U(1) + u(1))$. Examples of each of these equilibria are pictured here:

Figure 1: Three representative equilibria when $u(t) = t$ and $U(t) = 2t$. The dashed lines represent the reserve price, the heavy curves $B(t)$, and the lighter curves $b(t)$.



Note what is driving the non-monotonicity in all of these equilibria. In the first two cases, due to the discontinuity created by the reserve price, a positive mass of (relatively) high-type bidders bid exactly the reserve price for the more valuable object, to ensure it is met; but at this price, since $U(t_i) - r < u(t_i)$, they prefer to *lose* the competition for the more valuable object and win the lesser object cheaply. Thus, each bidder "bids against himself," bidding on the lesser object to help his opponent outbid him for the greater. Since the incentive to do this is decreasing in a bidder's type, bids on the lesser object are decreasing over that range. (In the region where b is decreasing, lowering it slightly does not reduce a bidder's chance of winning *something*, only increases his chance of winning the large rather than the small object; so the bid b_i can really be thought of purely as a bidder's attempt to undermine his own, larger bid B_i.) In the third equilibrium (Proposition 3), the intuition is the same, but the high-value bidders bid some focal bid less than r. Also note that in all of these equilibria, whenever the reserve price is met, the objects are allocated efficiently – the bidder with the higher type never wins the small object.

The equilibrium satisfying Theorem 1 need not be unique: for example, when $u(t) = t$, $U(t) = 2t$, and $r \in (1, 2)$, the strategies in both Propositions 2 and 3 are equilibria.[3] And all symmetric equilibria need not satisfy the conditions of Theorem 1.[4] For any u and U, when $r \geq U(1)$, there is also a symmetric no-trade equilibrium, as well as a continuum of asymmetric equilibria.

To see how our model fits the stories in the introduction, consider two examples. First, a large construction project is divided into two parts. A firm with type t_i could complete the larger part at cost $M - U(t_i)$, or the smaller at cost $m - u(t_i)$, but not both. Interpret a bid (b, B) as an offer to perform the larger for price $M - B$ or the smaller at price $m - b$. The reserve price r is now the difference between $M + m$ and the government's maximum willingness to pay to complete both projects. Second, consider a spectrum auction for three licenses, one each for the eastern, central, and western parts of the country. An east coast and a west coast firm would each earn profits $u(\cdot)$ from the license to their home region, and $U(\cdot)$ from the licenses to their home region plus the center of the country; the seller has a standing offer of r for all three licenses from a third buyer. Interpret a firm's bid (b, B) as a bid of b on the firm's home region and B on a package containing the home region and the center; the seller chooses between revenues of $B_1 + b_2$, $b_1 + B_2$, and the offer from the third buyer, which therefore serves as an aggregate reserve price.

Papers on multi-unit auctions sometimes refer to "small," or "local," bidders, who want only a single item and do not compete with each other; and a "large," or "global," bidder, who views the items as complements. In this terminology, we see our bidders as "medium-sized"; they are big enough to have overlapping interests, but the overlap is not complete. Thus, their interaction contains elements of both coordination (jointly meeting the reserve price) and competition (for the more desirable of the two prizes), leading to the equilibrium features highlighted in Theorem 1.

[3] While they are when $u(t) = t$ and $U(t) = 2t$, the strategies in Proposition 3 are not an equilibrium for all u and U when $r \in (U(1) - u(1), U(1))$.

[4] In an online appendix at http://www.ssc.wisc.edu/~dquint/papers/bidding-reversals-appendix.pdf we show that when $u(t) = t$, $U(t) = 2t$, and $r = \frac{3}{2}$, the strategies $b(t_i) = \frac{1}{2} t_i$ and $B(t_i) = \left(t_i + \frac{1}{2}\right) \mathbf{1}\{t_i > \frac{2}{3}\}$ are also an equilibrium (and in fact implements the optimal mechanism under certain additional assumptions). We do not know whether a symmetric, monotone equilibrium always exists, although we conjecture that it does *not*.

3 Related Literature

Equilibrium strategies in pay-as-you-bid package auctions with incomplete information are notoriously difficult to calculate. In an experimental paper, Chernomaz and Levin (2009) calculate equilibrium strategies in a first price package auction for two identical items with two local bidders (each demanding one item) and a global bidder (demanding both). To maintain tractability, they assume that the local bidders' private values are perfectly correlated. Discussions on package auctions with various designs can be found in Milgrom (2004), Cramton, Shoham, and Steinberg (2006), and Goeree and Holt (2010). Ausubel and Cramton (2004) and Blume et al. (2008) consider equilibrium bidding in Vickrey auctions with reserve prices; unlike in our paper, reserve prices there apply only to individual items.[5]

Equilibrium bids in Levin, Peck, and Ye (2007) and McAdams (2007) show reversals similar to ours. Levin, Peck and Ye model a single-object ascending auction with common and private value components; bidders may bid more aggressively following an earlier dropout by a competing bidder, because the earlier dropout signals a higher common value due to the particular relationship between signals and values. McAdams studies uniform-price auctions for multiple identical objects; he gives examples with risk-averse bidders or affiliated private values in which all equilibria in undominated strategies are non-monotone. We consider a different setting than these authors, but find non-monotone equilibria in a model with pure private values, risk-neutral bidders, and independent types.

[5] It is not obvious how the payment rule of the VCG mechanism should be modified to respect an aggregate reserve price; with such a change, truthful bidding would no longer be a dominant strategy. See Day and Milgrom (2009) and Ausubel and Baranov (2010) for analysis of a closely-related problem.

Appendix. Equilibrium Strategies.

Let $X(t) \equiv U(t) - u(t)$; and let $\overline{X}(t) = \frac{1}{1-t}\int_t^1 X(s)ds$ for $t < 1$ and $\overline{X}(1) = X(1)$.

Proposition 1. *If $U(0) - u(0) < r < U(1) - u(1)$, then the following bidding strategies constitute a symmetric Bayesian Nash Equilibrium:*

$$(b(t_i),\ \ B(t_i)) \ \ = \ \ \begin{cases} (0,\ \ 0) & if \quad t_i < t^* \\ \left(\frac{1}{1-t_i}\int_{t_i}^{t^{**}}(r - X(s))ds,\ \ 0\right) & if \quad t_i \in [t^*, t^{**}) \\ \left(0,\ \ \frac{1}{t_i}\int_0^{t_i}\max\{r, X(s)\}ds\right) & if \quad t_i \geq t^{**} \end{cases}$$

where t^{**} solves $X(t^{**}) = r$ and t^* is the unique solution to $t^*U(t^*) + \int_{t^*}^{t^{**}} X(s)ds = t^{**}X(t^{**})$.

Proposition 2. *If $U(1) - u(1) \leq r < U(1)$, then the following bidding strategies constitute a symmetric Bayesian Nash Equilibrium:*

$$(b(t_i),\ \ B(t_i)) \ \ = \ \ \begin{cases} (0,\ \ 0) & if \quad t_i < t^* \\ \left(r - \overline{X}(t_i),\ \ r\right) & if \quad t_i \geq t^* \end{cases}$$

where t^* is the unique solution to $r = t^*U(t^*) + (1 - t^*)\overline{X}(t^*)$.

Proposition 3. *If $U(1) \leq r < U(1) + u(1)$, then the following bidding strategies constitute a symmetric Bayesian Nash Equilibrium:*

$$(b(t_i),\ \ B(t_i)) \ \ = \ \ \begin{cases} (0,\ \ 0) & if \quad t_i < t^* \\ \left(\frac{r - X(1)}{2},\ \ 0\right) & if \quad t_i \in [t^*, t^{**}) \\ \left(\frac{r + X(1)}{2} - \overline{X}(t_i),\ \ \frac{r + X(1)}{2}\right) & if \quad t_i \geq t^{**} \end{cases}$$

where $t^* = 0$ if $u(0) > \frac{1}{2}(r - X(1))$, t^* solves $u(t^*) = \frac{1}{2}(r - X(1))$ otherwise, and t^{**} is the unique solution over $(t^*, 1)$ to $(1 - t^{**})\left(X(1) - \overline{X}(t^{**})\right) = (t^{**} - t^*)\left(U(t^{**}) - \frac{1}{2}(r + X(1))\right)$.

Theorem 1 follows directly, with the interval $\left(\underline{t}, \overline{t}\right)$ corresponding to (t^*, t^{**}) in Proposition 1, $(t^*, 1)$ in Proposition 2, and $(t^{**}, 1)$ in Proposition 3. Proofs that the strategies above constitute equilibria are lengthy but mechanical, and are included in the online appendix.[6]

[6] See http://www.ssc.wisc.edu/~dquint/papers/bidding-reversals-appendix.pdf

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