

Bidding Reversals in a Multiple-Good Auction with Aggregate Reserve Price

Online Appendix

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1 Environment

We are in the environment of Lien and Quint (2011). There are two objects for sale, one large and one small. There are two bidders, with types t_i which are independent and drawn from the uniform distribution on $[0, 1]$. A bidder with type t_i values the big object at $U(t_i)$ and the small object at $u(t_i)$, and gets no additional benefit (beyond $U(t_i)$) from getting both. U , u , and $U - u$ are all strictly increasing and continuous. (See Lien and Quint for motivation of the setup.)

2 Proof of Equilibria Discussed In Paper

As in the paper, let $X(t) \equiv U(t) - u(t)$; and let $\bar{X}(t) = \frac{1}{1-t} \int_t^1 X(s)ds$ for $t < 1$ and $\bar{X}(1) = X(1)$.

Proposition 1. *If $U(0) - u(0) < r < U(1) - u(1)$, then the following bidding strategies constitute a symmetric Bayesian Nash Equilibrium:*

$$(b(t_i), B(t_i)) = \begin{cases} (0, 0) & \text{if } t_i < t^* \\ \left(\frac{1}{1-t_i} \int_{t_i}^{t^{**}} (r - X(s))ds, 0 \right) & \text{if } t_i \in [t^*, t^{**}) \\ \left(0, \frac{1}{t_i} \int_0^{t_i} \max\{r, X(s)\}ds \right) & \text{if } t_i \geq t^{**} \end{cases}$$

where t^{**} solves $X(t^{**}) = r$ and t^* is the unique solution to $t^*U(t^*) + \int_{t^*}^{t^{**}} X(s)ds = t^{**}X(t^{**})$.

Proof. $t^* < t^{**}$ uniquely defined. t^{**} is uniquely defined since $X(\cdot)$ is assumed to be strictly increasing and continuous with $X(0) < r < X(1)$. Let $\phi(t) = tU(t) + \int_t^{t^{**}} X(s)ds$, which is continuous and strictly increasing. $\phi(0) = \int_0^{t^{**}} X(s)ds < t^{**}X(t^{**})$ and $\phi(t^{**}) = t^{**}U(t^{**}) > t^{**}X(t^{**})$; so $\phi(t) = t^{**}X(t^{**})$ has a unique solution, which is strictly below t^{**} .

Deviations to other types' equilibrium strategies. Let $\pi(\hat{t}, t)$ be the expected payoff of a bidder with type \hat{t} who makes the equilibrium bid of a bidder with type t . For $t \geq t^{**}$, he wins the small item for free when his opponent has type $t_j > t$, and the large object for $B(t)$ otherwise, so

$$\begin{aligned} \pi(\hat{t}, t) &= (1-t)u(\hat{t}) + tU(\hat{t}) - tE(\max\{r, X(s)\} \mid s < t) \\ &= u(\hat{t}) + tX(\hat{t}) - t^{**}r - \int_{t^{**}}^t X(s)ds \\ &\quad \downarrow \\ \frac{\partial}{\partial \hat{t}} \pi(\hat{t}, t) &= X(\hat{t}) - X(t) \end{aligned}$$

For $t^* \leq t < t^{**}$, he wins the small object for $b(t)$ when $t_j > t$, and the large object for r otherwise, so

$$\begin{aligned} \pi(\hat{t}, t) &= (1-t)u(\hat{t}) - (1-t) \left(\frac{t^{**}-t}{1-t} E(r - X(s) \mid s \in (t, t^{**})) \right) + tU(\hat{t}) - tr \\ &= u(\hat{t}) - (t^{**}-t)r + (t^{**}-t)E(X(s) \mid s \in (t, t^{**})) + tX(\hat{t}) - tr \\ &= u(\hat{t}) + tX(\hat{t}) - t^{**}r + \int_t^{t^{**}} X(s)ds \\ &\quad \downarrow \\ \frac{\partial}{\partial \hat{t}} \pi(\hat{t}, t) &= X(\hat{t}) - X(t) \end{aligned}$$

(Note that the left- and right-limits of $\pi(\hat{t}, t)$ as $t \rightarrow t^{**}$ are identical, since equilibrium bids approach $(0, r)$ from both sides.) Finally, the difference in expected payoff between bidding as type t^* and bidding $(0, 0)$ is

$$\begin{aligned}
\pi(\hat{t}, t^*) - \pi(\hat{t}, (0, 0)) &= (1 - t^*)u(\hat{t}) - (1 - t^*) \left(\frac{t^{**} - t^*}{1 - t^*} E(r - X(s) \mid s \in (t^*, t^{**})) \right) \\
&\quad + t^*U(\hat{t}) - t^*r - (1 - t^*)u(\hat{t}) \\
&= -(t^{**} - t^*)r + (t^{**} - t^*)E(X(s) \mid s \in (t^*, t^{**})) + t^*U(\hat{t}) - t^*r \\
&= -t^*r + t^*U(\hat{t}) + \int_{t^*}^{t^{**}} X(s)ds
\end{aligned}$$

which is increasing in \hat{t} and, by construction of t^* , equal to 0 at $\hat{t} = t^*$. Together, these establish that $\pi(\hat{t}, t)$ is at least weakly increasing in t for $t < \hat{t}$ and decreasing in t for $t > \hat{t}$, ruling out profitable deviations to other bidders' equilibrium bids.

Other deviations. A deviation to (B, b) with $B > r$ and $b > 0$ is dominated by a deviation to either $(B - b, 0)$ or $(r, b - (B - r))$. A deviation to $(B, 0)$ with $B > B(1)$ is dominated by $(B(1), 0)$. A deviation to (r, b) with $b > b(t^*)$ is dominated by $(r, b(t^*))$. This rules out all additional deviations to (B, b) with $B \geq r$.

A deviation to (B, b) with $b \geq r > B > 0$ is dominated by $(0, b - B)$ if $b - B \geq r$, and by $(B - (b - r), r)$ otherwise. A deviation to (B, r) with $B < r$ is equivalent to a bid of $(0, r)$, since either one will always win the small object for price r ; this is dominated by bidding $(r, 0)$, since this wins either the large object at r or the small object for free. Bids of $(0, b)$ with $b > r$ are similarly dominated.

All remaining deviations only win anything when the opponent's type is $t_j \geq t^*$. Since bidding $(0, 0)$ gets the small item for free in all these situations, the only potential deviations are to bid high enough to sometimes win two items against opponent types above t^* . Since $b(t^*)$ is the highest opponent bid on the small item, such bids only matter if

$$\begin{aligned}
B &> r - b(t^*) \\
&= \frac{1}{1 - t^*} \left((1 - t^*)r - (t^{**} - t^*)r + \int_{t^*}^{t^{**}} X(s)ds \right) \\
&= \frac{1}{1 - t^*} \left((1 - t^{**})r + \int_{t^*}^{t^{**}} X(s)ds \right) \\
&= E(\min\{X(s), X(t^{**})\} \mid s > t^*) \\
&> X(t^*)
\end{aligned}$$

Thus, this is only profitable for types $t > t^*$, since types below t^* prefer the small item for free.

Consider a deviation by type $t > t^*$ to $(0, B)$ with $B = r - b(t')$ for some $t' \in (t^*, t^{**})$, and compare it to simply playing the equilibrium strategy of type t' . This being profitable would require

$$\begin{aligned}
(t' - t^*)(U(t) - r + b(t')) + (1 - t')u(t) &> t'(U(t) - r) + (1 - t')(u(t) - b(t')) \\
t'U(t) - t'r + t'b(t') - t^*U(t) + t^*r - t^*b(t') &> t'U(t) - t'r - b(t') + t'b(t') \\
(t' - t^*)(U(t) - r + b(t')) &> t'(U(t) - r) - (1 - t')b(t') \\
t'U(t) - t'r + t'b(t') - t^*U(t) + t^*r - t^*b(t') &> t'U(t) - t'r - b(t') + t'b(t') \\
-t^*U(t) + t^*r - t^*b(t') &> -b(t') \\
-t^*U(t) + t^*r + (1 - t^*)b(t') &> 0 \\
(1 - t^*)u(t) &> t^*(U(t) - r) + (1 - t^*)(u(t) - b(t')) \\
(1 - t^*)u(t) &> t^*(U(t) - r) + (1 - t^*)(u(t) - b(t^*))
\end{aligned}$$

since $b(t^*) < b(t')$. But this means bidding like a low type must be preferable to bidding like t^* for some type $t > t^*$, which we ruled out above. \square

Proposition 2. *If $U(1) - u(1) \leq r < U(1)$, then the following bidding strategies constitute a symmetric Bayesian Nash Equilibrium:*

$$(b(t_i), B(t_i)) = \begin{cases} (0, 0) & \text{if } t_i < t^* \\ (r - \bar{X}(t_i), r) & \text{if } t_i \geq t^* \end{cases}$$

where t^* is the unique solution to $r = t^*U(t^*) + (1 - t^*)\bar{X}(t^*)$.

Proof. t^* uniquely defined. Let $\psi(t) = tU(t) + (1 - t)\bar{X}(t) = tU(t) + \int_t^1 (U(s) - u(s))ds = \int_0^1 (U(\max(t, s)) - \mathbf{1}_{s>t}u(s)) ds$. This is continuous, strictly increasing, approaches $\bar{X}(0) < X(1) \leq r$ as $t \rightarrow 0$, and approaches $U(1) > r$ as $t \rightarrow 1$; so $\psi(t) = r$ has a unique solution.

Deviations to other types' equilibrium strategies. First, consider a bidder with type \hat{t} who bids as if he had type $t > t^*$. He wins the big item for r when $t_j < t$, and the small item for $b_i(t)$ when $t_j > t$, for an expected payoff of

$$\begin{aligned} \pi(\hat{t}, t) &= t(U(\hat{t}) - r) + (1 - t)(u(\hat{t}) - r + \bar{X}(t)) \\ &= u(\hat{t}) - r + tX(\hat{t}) + \int_t^1 X(s)ds \\ &\quad \downarrow \\ \frac{\partial}{\partial t}\pi(\hat{t}, t) &= X(\hat{t}) - X(t) \end{aligned}$$

The difference in expected payoffs between bidding like type t^* and bidding like a low type is

$$\begin{aligned} \pi(\hat{t}, t^*) - \pi(\hat{t}, (0, 0)) &= u(\hat{t}) - r + t^*X(\hat{t}) + \int_{t^*}^1 X(s)ds - (1 - t^*)u(\hat{t}) \\ &= t^*u(\hat{t}) - r + t^*X(\hat{t}) + \int_{t^*}^1 X(s)ds \\ &= t^*U(\hat{t}) - r + \int_{t^*}^1 X(s)ds \end{aligned}$$

which is increasing in \hat{t} and, by construction of t^* , 0 at $\hat{t} = t^*$. Together, these tell us that types above t^* prefer their own equilibrium strategies to those of other types above t^* , including t^* , and prefer this to bidding like a low type; and types below t^* prefer bidding their equilibrium strategies to bidding like t^* , which they prefer to bidding like a higher type. Type t^* is indifferent between his own strategy and bidding $(0, 0)$, and strictly prefers either to bidding like a higher type. So no type can gain by bidding like a different type.

Other deviations. Bids of (B, b) with $B > r$ and $b > 0$ are dominated by either $(r, b - (B - r))$ or $(B - b, 0)$. Bids of (r, b) with $b > b(t^*)$ are dominated by $(r, b(t^*))$, the equilibrium strategy of type t^* . Bids of $(B, 0)$ with $B > r$ are dominated by $(r, 0)$, since given other bidders' strategies, this still always wins the large item. Given equilibrium bids, $(r, 0)$ gives the same result as $(r, b(1))$, the equilibrium strategy of type $t = 1$. This rules out profitable deviations to (B, b) with $B \geq r$.

A deviation to (B, b) with $b \geq r > B > 0$ is dominated by $(0, b - B)$ if $b - B \geq r$, and by $(B - (b - r), r)$ otherwise. A deviation to (B, r) with $B < r$ is equivalent to a bid of $(0, r)$, since either one will always win the small object for price r ; this is dominated by bidding $(r, 0)$, since this wins either the large object at r or the small object for free. Bids of $(0, b)$ with $b > r$ are similarly dominated. This rules out deviations which ever win against types $t_j < t^*$. Bidding $(0, 0)$ wins the small object for free against all types $t_j \geq t^*$, so the only possible deviations remaining are to sometimes win the large object rather than the small one when $t_j \geq t^*$, by bidding $B > r - b(t^*)$. We can assume without loss that these are deviations to $(B, 0)$ with $B < r$.

Such a deviation only matters if $B > r - b(t^*) = \bar{X}(t^*)$. For $t < t^*$, this deviation is not profitable – a bidder with $t < t^*$ prefers the small item for free to the large item at price $X(t^*) < \bar{X}(t^*)$. So we only need to worry about deviations by high types.

Fix $t > t^*$, and suppose a bidder with type t considers such a strategy. Bids $B \leq r - b(t^*)$ are irrelevant, since they never win; bids $B > r - b(1)$ are dominated by $B = r - b(1)$, since either one wins the big object against all opponents $t_j \geq t^*$ but the latter involves paying less. So we may assume that any potential deviation strategy $(B, 0)$ with $B < r$ satisfies $r - b(t^*) \leq B \leq r - b(1)$, and therefore, by continuity of b above t^* , $B = r - b(t')$ for some $t' \in [t^*, 1]$. The expected payoff to this strategy, then, is

$$\begin{aligned}
\pi(t, (B, 0)) &= (1 - t')u(t) + (t' - t^*)(U(t) - (r - b(t'))) \\
&= (1 - t')(u(t) - b(t')) + t'(U(t) - r) - t^*(U(t) - r) + (1 - t^*)b(t') \\
&= \pi(t, t') - t^*(U(t) - r) + (1 - t^*)(r - \bar{X}(t')) \\
&= \pi(t, t') + r - t^*U(t) - (1 - t^*)\bar{X}(t') \\
&= \pi(t, t') + t^*U(t^*) + (1 - t^*)\bar{X}(t^*) - t^*U(t) - (1 - t^*)\bar{X}(t') \\
&= \pi(t, t') - t^*(U(t) - U(t^*)) - (1 - t^*)(\bar{X}(t') - \bar{X}(t^*)) \\
&\leq \pi(t, t') \leq \pi(t, t)
\end{aligned}$$

where the first inequality is because by assumption, $t \geq t^*$, $t' \geq t^*$, and U and \bar{X} are increasing, and the second is because we've already ruled out profitable deviations to other types' equilibrium strategies. \square

Proposition 3. *If $U(1) \leq r < U(1) + u(1)$, then the following bidding strategies constitute a symmetric Bayesian Nash Equilibrium:*

$$(b(t_i), B(t_i)) = \begin{cases} (0, 0) & \text{if } t_i < t^* \\ (\frac{1}{2}(r - X(1)), 0) & \text{if } t_i \in [t^*, t^{**}] \\ (\frac{r+X(1)}{2} - \bar{X}(t_i), \frac{r+X(1)}{2}) & \text{if } t_i \geq t^{**} \end{cases}$$

where $t^* = 0$ if $u(0) > \frac{1}{2}(r - X(1))$, t^* solves $u(t^*) = \frac{1}{2}(r - X(1))$ otherwise, and t^{**} is the unique solution over $(t^*, 1)$ to $(1 - t^{**})(X(1) - \bar{X}(t^{**})) = (t^{**} - t^*)(U(t^{**}) - \frac{1}{2}(r + X(1)))$.

Proof. t^* and t^{**} uniquely defined. Since $r < U(1) + u(1)$, $\frac{1}{2}(r - X(1)) < u(1)$; so either $u(0) \geq \frac{1}{2}(r - X(1))$ or $u(t) = \frac{1}{2}(r - X(1))$ has a unique interior solution t^* .

As for t^{**} , the expression $(1 - t)(X(1) - \bar{X}(t))$ is decreasing in t , goes to 0 as $t \rightarrow 1$, and is strictly positive everywhere else. Let t^{***} solve $U(t^{***}) = \frac{1}{2}(r + X(1))$ if it has a solution above t^* , otherwise let $t^{***} = t^*$. (Since $r < U(1) + u(1)$, $\frac{1}{2}(r + X(1)) < U(1)$, so $t^{***} < 1$.) Then the expression $(t - t^*)(U(t) - \frac{1}{2}(r + X(1)))$ is 0 at $t = t^{***}$ and strictly increasing above it. This guarantees equation ?? has a unique solution on $(t^*, 1)$.

Deviations to other types' equilibrium strategies. A bidder with type \hat{t} who bids like type $t > t^{**}$ gets expected payoff

$$\begin{aligned}
\pi(\hat{t}, t) &= (1 - t)(u(\hat{t}) - \frac{1}{2}(r + X(1)) + \bar{X}(t)) + (t - t^*)(U(\hat{t}) - \frac{1}{2}(r + X(1))) \\
&= (1 - t^*)u(\hat{t}) + (t - t^*)X(\hat{t}) - (1 - t^*)\frac{1}{2}(r + X(1)) + (1 - t)\bar{X}(t) \\
&= (1 - t^*)u(\hat{t}) + (t - t^*)X(\hat{t}) - (1 - t^*)\frac{1}{2}(r + X(1)) + \int_t^1 X(s)ds \\
&\quad \downarrow \\
\frac{\partial}{\partial t}\pi(\hat{t}, t) &= X(\hat{t}) - X(t)
\end{aligned}$$

so no type $\hat{t} \neq t$ will deviate to the equilibrium strategy of type $t > t^{**}$, and no type $\hat{t} > t^{**}$ will deviate to the equilibrium strategy of type t^{**} . The gain from bidding like t^{**} instead of like a type

$t \in [t^*, t^{**})$ is

$$\begin{aligned}
\pi(\hat{t}, t^{**}) - \pi(\hat{t}, t) &= (t^{**} - t^*) \left(U(\hat{t}) - \frac{1}{2}(r + X(1)) \right) + (1 - t^{**}) \left(u(\hat{t}) - \frac{1}{2}(r + X(1)) + \bar{X}(t^{**}) \right) \\
&\quad - (1 - t^{**}) \left(u(\hat{t}) - \frac{1}{2}(r - X(1)) \right) \\
&= (t^{**} - t^*) \left(U(\hat{t}) - \frac{1}{2}(r + X(1)) \right) \\
&\quad + (1 - t^{**}) \left(\frac{1}{2}(r - X(1)) - \frac{1}{2}(r + X(1)) + \bar{X}(t^{**}) \right) \\
&= (t^{**} - t^*) \left(U(\hat{t}) - \frac{1}{2}(r + X(1)) \right) - (1 - t^{**}) \left(X(1) - \bar{X}(t^{**}) \right)
\end{aligned}$$

which is strictly increasing in \hat{t} and, by equation ??, equal to 0 at $\hat{t} = t^{**}$, so no type $\hat{t} < t^{**}$ can gain from this deviation, and no type $\hat{t} > t^{**}$ can benefit from bidding like a type $t \in [t^*, t^{**})$.

Finally, bidding like a type $t \in [t^*, t^{**})$ gives a payoff of $(1 - t^{**}) \left(u(\hat{t}) - \frac{1}{2}(r - X(1)) \right)$, which by construction is positive for $\hat{t} > t^*$ and negative for $\hat{t} < t^*$; so low types can't gain by impersonating medium types (or, by transitivity, high types), and medium types can't gain by impersonating low types (nor can high types by transitivity).

Other deviations. Deviations to $B \geq r$ are unprofitable because by assumption $r \geq U(1)$. (When $X(1) \leq r < U(1)$, the strategies in Proposition 3 may still be an equilibrium, but this deviation must be explicitly checked; if $u(\cdot)$ is steep enough near 0, the equilibrium may break down.) Note that the rule for choosing a winner implies increasing B cannot increase the chance your other bid wins. Deviations to $b \geq r$ are similarly unprofitable.

Deviations to (b, B) with $B \in (\frac{1}{2}(r + X(1)), r)$ can be ruled out because if $b > 0$, reducing both b and B until either $B = \frac{1}{2}(r + X(1))$ or $b = 0$ does strictly better, and if $b = 0$, then reducing B to $\frac{1}{2}(r + X(1))$ does strictly better. The latter case – bidding $B = \frac{1}{2}(r + X(1))$ and $b = 0$ – is payoff-equivalent to making the equilibrium bids of type $t = 1$, since the low bid $b(1)$ never wins in equilibrium. The former case – bidding $B = \frac{1}{2}(r + X(1))$ and $b > 0$ – is either the equilibrium bid of some type $t \in [t^*, t^{**})$, or else dominated by bidding like type t^* . We've already ruled out profitable deviations to equilibrium bids of other types, so these are not profitable either.

Any other deviation wins nothing against opponent types $t_j < t^{**}$, and bidding like a type $t \in [t^*, t^{**})$ already gets the highest possible payoff against opponents $t_j \geq t^{**}$, so there are no other profitable deviations. \square

3 Optimal Mechanism for This Environment

We characterize the optimal mechanism under two additional assumptions:

Assumption 1. *The auctioneer cannot sell one object and keep the other.*

This is motivated by the procurement example, where the two “objects” are contracts to complete two parts of a larger project – the government would not want to award a contract to finish half the project without someone committing to build the other half. (We can also characterize the optimal mechanism when the seller can sell one object and withhold the other, but at the cost of more notation and complexity; the simpler case will be sufficient for the example we want.)

Assumption 2. *The random variables $u(t_i)$, $U(t_i)$, and $X(t_i) = U(t_i) - u(t_i)$ are all regular, in the sense of Myerson (1981): letting F_u , F_U , and F_X be their respective cumulative probability distributions and f_u , f_U , and f_X their corresponding densities, $s - \frac{1 - F_u(s)}{f_u(s)}$, $s - \frac{1 - F_U(s)}{f_U(s)}$, and $s - \frac{1 - F_X(s)}{f_X(s)}$ are all nondecreasing.*

The analysis is nearly identical to that in Myerson. Restrict attention to direct-revelation mechanisms. Let v_0 be the seller's benefit from selling nothing. Let $p_i(t)$ be the probability that given reported types $t = (t_1, t_2)$, bidder i gets the large object (and, consequently, j gets the small), and $p_0(t) = 1 - p_1(t) - p_2(t)$ the probability the objects go unsold. Let $Q_i(t_i) = E_{t_j} p_i(t_i, t_j)$ be bidder i 's probability of winning the big object, and $q_i(t_i) = E_{t_j} p_j(t_i, t_j)$ his probability of winning the small object.

Let $V_i(t_i)$ denote the expected payoff to bidder i with type t_i ; the envelope theorem gives

$$V_i(t_i) = V_i(0) + \int_0^{t_i} [q_i(s)u'(s) + Q_i(s)U'(s)] ds \quad (1)$$

which is a necessary condition for incentive compatibility. The monotonicity requirement is now for both Q_i and $q_i + Q_i$ to be nondecreasing. Since equation 1 implies V_i is increasing, individual rationality requires $V_i(0) \geq 0$; together, these conditions are sufficient for a mechanism to be feasible.

The monotonicity requirement as given here may be stronger than necessary, but it will not bind under Assumption 2. To see why it (along with the envelope condition) is sufficient for incentive-compatibility, note that the expected payment by a bidder reporting type \hat{t}_i must be $q_i(\hat{t}_i)u(\hat{t}_i) + Q_i(\hat{t}_i)U(\hat{t}_i) - V_i(\hat{t}_i)$. For a bidder with true type t_i , the expected gain from reporting type $\hat{t}_i > t_i$ is therefore

$$\begin{aligned} \pi_i(\hat{t}_i, t_i) - \pi_i(t_i, t_i) &= q_i(\hat{t}_i)u(t_i) + Q_i(\hat{t}_i)U(t_i) \\ &\quad - \left(q_i(\hat{t}_i)u(\hat{t}_i) + Q_i(\hat{t}_i)U(\hat{t}_i) - V_i(0) - \int_0^{\hat{t}_i} [q_i(s)u'(s) + Q_i(s)U'(s)] ds \right) \\ &\quad - V_i(0) - \int_0^{t_i} [q_i(s)u'(s) + Q_i(s)U'(s)] ds \\ &= q_i(\hat{t}_i) (u(t_i) - u(\hat{t}_i)) + Q_i(\hat{t}_i) (U(t_i) - U(\hat{t}_i)) \\ &\quad + \int_{t_i}^{\hat{t}_i} [q_i(s)u'(s) + Q_i(s)U'(s)] ds \\ &= \int_{t_i}^{\hat{t}_i} [(q_i(s) - q_i(\hat{t}))u'(s) + (Q_i(s) - Q_i(\hat{t}))U'(s)] ds \\ &= - \int_{t_i}^{\hat{t}_i} [q_i(\hat{t}) + Q_i(\hat{t}) - q_i(s) - Q_i(s)] u'(s) ds - \int_{t_i}^{\hat{t}_i} [Q_i(\hat{t}) - Q_i(s)] X'(s) ds \end{aligned}$$

u' and X' are both positive; if $q_i + Q_i$ and Q_i are both nondecreasing, both integrands are everywhere nonnegative, so the entire expression is nonpositive. A symmetric argument holds for $\hat{t}_i < t_i$.

The ex-ante expected payoff to bidder i is

$$\begin{aligned} \pi_i &\equiv E_{t_i} V_i(t_i) = \int_0^1 \left(V_i(0) + \int_0^{t_i} [q_i(s)u'(s) + Q_i(s)U'(s)] ds \right) dt_i \\ &= V_i(0) + \int_0^1 \int_s^1 [q_i(s)u'(s) + Q_i(s)U'(s)] dt_i ds \\ &= V_i(0) + \int_0^1 [q_i(s)u'(s) + Q_i(s)U'(s)] (1 - s) ds \end{aligned}$$

The expected payment from a bidder with type t_i can be written as $EP(t_i) = q_i(t_i)u(t_i) +$

$Q_i(t_i)U(t_i) - V_i(t_i)$, so the seller's expected payoff is

$$\begin{aligned}
\Pi &= E_t \{EP(t_1) + EP(t_2) + p_0(t)v_0\} \\
&= E_{t_1, t_2} \{p_1(t)(U(t_1) + u(t_2)) + p_2(t)(U(t_2) + u(t_1)) + p_0(t)v_0\} \\
&\quad - V_1(0) - \int_0^1 (q_1(t_1)u'(t_1) + Q_1(t_1)U'(t_1))(1 - t_1)dt_1 \\
&\quad - V_2(0) - \int_0^1 (q_2(t_2)u'(t_2) + Q_2(t_2)U'(t_2))(1 - t_2)dt_2 \\
&= E_{t_1, t_2} \{p_1(t)(U(t_1) + u(t_2)) + p_2(t)(U(t_2) + u(t_1)) + p_0(t)v_0\} \\
&\quad - V_1(0) - \int_0^1 \int_0^1 (p_2(t_1, t_2)u'(t_1) + p_1(t_1, t_2)U'(t_1))(1 - t_1)dt_2dt_1 \\
&\quad - V_2(0) - \int_0^1 \int_0^1 (p_1(t_1, t_2)u'(t_2) + p_2(t_1, t_2)U'(t_2))(1 - t_2)dt_1dt_2 \\
&= -V_1(0) - V_2(0) + E_{t_1, t_2} \{p_1(t) [U(t_1) - (1 - t_1)U'(t_1) + u(t_2) - (1 - t_2)u'(t_2)] \\
&\quad + p_2(t) [U(t_2) - (1 - t_2)U'(t_2) + u(t_1) - (1 - t_1)u'(t_1)] + p_0(t)v_0\}
\end{aligned}$$

Let $R(t) = U(t) - (1 - t)U'(t)$ and $r(t) = u(t) - (1 - t)u'(t)$; then this is equal to

$$\Pi = -V_1(0) - V_2(0) + E_t \{p_1(t) [R(t_1) + r(t_2)] + p_2(t) [R(t_2) + r(t_1)] + p_0(t)v_0\}$$

Provided it satisfies monotonicity, this is maximized by setting $V_1(0) = V_2(0) = 0$ and $p_i(t) = 1$ whenever $R(t_i) + r(t_j) > \max\{R(t_j) + r(t_i), v_0\}$.

Next, we show that under Assumption 2, this allocation rule is feasible. $t_i \sim U[0, 1]$, so $F_u(s) = \Pr(u(t) < s) = \Pr(t < u^{-1}(s)) = u^{-1}(s)$; differentiating,

$$f_u(s) = \frac{\partial}{\partial s} u^{-1}(s) = \frac{1}{u'(u^{-1}(s))}$$

and so

$$s - \frac{1 - F_u(s)}{f_u(s)} = s - (1 - u^{-1}(s)) u'(u^{-1}(s)) = u(t) - (1 - t)u'(t)$$

where $t \equiv u^{-1}(s)$ (which is strictly increasing in s). So if F_u is regular, $r(t)$ is nondecreasing in t . Similarly, F_U regular implies $R(t)$ is nondecreasing, and F_X regular implies $X(t) - (1 - t)X'(t) = R(t) - r(t)$ is nondecreasing.

These lead to a clean characterization of the optimal mechanism:

Theorem 1. *Suppose the seller cannot sell one object without selling the other. Let $t^1 = \max\{t_1, t_2\}$ and $t^2 = \min\{t_1, t_2\}$. Then the optimal mechanism is:*

- *If $R(t^1) + r(t^2) \geq v_0$, give the big object to the bidder with type t^1 , the small object to the bidder with type t^2*
- *If $R(t^1) + r(t^2) < v_0$, keep both objects*
- *Charge the transfers implied by this allocation rule, $V_i(0) = 0$, and the envelope theorem*

The envelope condition has already been imposed, as has $V_i(0) \geq 0$. Under this allocation rule, since R and r are increasing,

$$q(t_i) + Q(t_i) = \Pr(\max\{R(t_i) + r(t_j), r(t_i) + R(t_j)\} \geq v_0)$$

is increasing; and since R and $R - r$ are increasing,

$$Q(t_i) = \Pr(R(t_i) + r(t_j) > v_0 \text{ and } R(t_i) - r(t_i) > R(t_j) - r(t_j))$$

is as well. So the three sufficient conditions are satisfied, and the allocation rule is therefore implementable.

(We can use the same logic to characterize the optimal mechanism when the seller is able to sell one object without the other. For $i, j \in \{0, 1, 2\}$, let $p_{ij}(t)$ be the probability, given reported types t , then the big item is allocated to bidder i and the small to bidder j (where “bidder 0” indicates the seller keeping the object). Letting v_1 , v_2 , and v_{12} be the seller’s reservation value for the big item, the small item, and both together, the integrand in the simplified expression for expected payoff would become

$$\begin{aligned} \Pi = & E_{t_1, t_2} \{ p_{12}(t) [R(t_1) + r(t_2)] + p_{21}(t) [R(t_2) + r(t_1)] + p_{10}(t) [R(t_1) + v_2] \\ & + p_{20}(t) [R(t_2) + v_2] + p_{01}(t) [r(t_1) + v_1] + p_{02}(t) [r(t_2) + v_1] + p_{00}(t) [v_{12}] \} \end{aligned}$$

which leads to a characterization of the optimal allocation rule based on which of the seven terms in square brackets is largest at each t .)

4 An Example

Next, we show a particular example where the optimal mechanism can be implemented using the auction considered in Lien and Quint: a pay-as-bid auction with exclusive-or bids and an aggregate reserve price.

Example 1. *Assume the seller can only sell the two objects together. Let $u(t) = t$, $U(t) = 2t$, and $v_0 = 1$. Consider a pay-as-bid auction with exclusive-or bidding and an aggregate reserve price of $\frac{3}{2}$. One symmetric equilibrium in this auction is*

$$b(t_i) = \frac{1}{2}t_i \quad \text{and} \quad B(t_i) = \begin{cases} 0 & \text{if } t_i \leq \frac{2}{3} \\ t_i + \frac{1}{2} & \text{if } t_i > \frac{2}{3} \end{cases}$$

and this equilibrium implements the optimal mechanism.

First we will solve explicitly for the optimal mechanism (allocation rule and expected payoffs to each bidder type) in this example. We then show that these strategies are indeed an equilibrium, and give the same allocation rule and expected payoffs as the optimal mechanism.

4.1 Optimal Mechanism when $u(t) = t$, $U(t) = 2t$, and $v_0 = 1$

When $u(t) = t$ and $U(t) = 2t$, $R(t) = 2t - (1 - t)2 = 4t - 2$ and $r(t) = t - (1 - t) = 2t - 1$, so

$$R(t_1) + r(t_2) = 4t_1 - 2 + 2t_2 - 1 = 4t_1 + 2t_2 - 3$$

The optimal auction, then, allocates the big object to the bidder with the higher type, and the small object to the bidder with the lower type, whenever $4t^1 + 2t^2 - 3 \geq v_0 = 1$, or when $t^1 + \frac{1}{2}t^2 \geq 1$.

Given this allocation rule, there are two relevant regions of types. Types $t_i \leq \frac{2}{3}$ never win the big object, since $t_j < t_i \rightarrow t_i + \frac{1}{2}t_j < \frac{2}{3} + \frac{1}{3} = 1$. They win the big object whenever $t_j + \frac{1}{2}t_i \geq 1$, which is when $t_j \geq 1 - \frac{1}{2}t_i$. For $t_i \leq \frac{2}{3}$, then, $Q_i(t_i) = 0$ and $q_i(t_i) = \Pr(t_j \geq 1 - \frac{1}{2}t_i) = \frac{1}{2}t_i$.

Since $U'(t) = 2$ and $u'(t) = 1$, the envelope theorem states that

$$V(t_i) = V(0) + \int_0^{t_i} [2Q_i(s) + q_i(s)] ds$$

For $t_i \leq \frac{2}{3}$, this is

$$V(t_i) = \int_0^{t_i} \frac{1}{2} s ds = \frac{1}{4} t_i^2$$

Note that $V\left(\frac{2}{3}\right) = \frac{1}{4} \left(\frac{2}{3}\right)^2 = \frac{1}{9}$.

Types $t_i > \frac{2}{3}$ can win either object. Specifically, they win the small object whenever $t_j > t_i$, since that means $t_j + \frac{1}{2}t_i > 1$. And they win the big object whenever $t_j < t_i$ but $t_i + \frac{1}{2}t_j \geq 1$, which is when $t_j \geq 2 - 2t_i$. So $Q_i(t_i) = t_i - (2 - 2t_i) = 3t_i - 2$, and $q_i(t_i) = 1 - t_i$. For $t_i > \frac{2}{3}$, then,

$$\begin{aligned} V(t_i) &= V\left(\frac{2}{3}\right) + \int_{\frac{2}{3}}^{t_i} [2(3s - 2) + (1 - s)] ds = \frac{1}{9} + \int_{\frac{2}{3}}^{t_i} [5s - 3] ds \\ &= \frac{1}{9} + \frac{5}{2} t_i^2 - \frac{5}{2} \left(\frac{2}{3}\right)^2 - 3 \left(t_i - \frac{2}{3}\right) = \frac{5}{2} t_i^2 - 3t_i + 1 \end{aligned}$$

4.2 Strategies in Example 1 Give This Outcome

Allocation

Bidders with types $t_i \leq \frac{2}{3}$ bid $(\frac{1}{2}t_i, 0)$, and bidders with types $t_i > \frac{2}{3}$ bid $(\frac{1}{2}t_i, t_i + \frac{1}{2})$. With a global reserve of $\frac{3}{2}$, then, this means that (i) when t_1 and t_2 are both below $\frac{2}{3}$, the reserve is not met; (ii) when t_1 and t_2 are both above $\frac{2}{3}$, the reserve is met, and $B(t^1) + b(t^2) > b(t^1) + B(t^2)$, so the bidder with the higher type gets the big object; and (iii) when $t^1 > \frac{2}{3} > t^2$, the reserve is met (by $B(t^1) + b(t^2)$) if and only if $t^1 + \frac{1}{2} + \frac{1}{2}t^2 \geq \frac{3}{2}$, or $t^1 + \frac{1}{2}t^2 \geq 1$. So these strategies implement the same allocation as the optimal mechanism.

Payoffs

Given the allocation rule, by following this strategy, bidder i with type $t_i \leq \frac{2}{3}$ never wins the big object, and wins the small object whenever $t_j + \frac{1}{2}t_i \geq 1$, or $t_j \geq 1 - \frac{1}{2}t_i$, which is has probability $\frac{1}{2}t_i$; so his expected payoff is $\frac{1}{2}t_i (t_i - \frac{1}{2}t_i) = \frac{1}{4}t_i^2$.

For $t_i > \frac{2}{3}$, he wins the big object (and pays $t_i + \frac{1}{2}$) when $t_j < t_i$ and $t_i + \frac{1}{2}t_j \geq 1$, or $t_j \in [2 - 2t_i, t_i]$, which has probability $3t_i - 2$; and wins the small object (for $\frac{1}{2}t_i$) when $t_j > t_i$, with probability $1 - t_i$. His expected payoff is then

$$\begin{aligned} (3t_i - 2) \left(2t_i - \left(t_i + \frac{1}{2} \right) \right) + (1 - t_i) \left(t_i - \frac{1}{2}t_i \right) &= (3t_i - 2) \left(t_i - \frac{1}{2} \right) + \frac{1}{2}t_i(1 - t_i) \\ &= 3t_i^2 - 2t_i - \frac{3}{2}t_i + 1 + \frac{1}{2}t_i - \frac{1}{2}t_i^2 = \frac{5}{2}t_i^2 - 3t_i + 1 \end{aligned}$$

4.3 Strategies in Example 1 Are An Equilibrium

When bidder $j \neq i$ plays the strategies described above, bidder i 's expected payoffs are continuous in his bids. Since undominated bids are taken from a subset of a compact space (say, $[0, 1] \times [0, 2]$), a maximizer exists. Thus, we need only show that this maximizer cannot achieve an expected payoff higher than the payoffs calculated above. We consider three cases: when the maximizer (b, B) satisfies $B = 0$, satisfies $b = 0$, or satisfies $b > 0$ and $B > 0$. (Since $B_j(t_j) = t_j + \frac{1}{2} \leq \frac{3}{2}$, a bid of $(0, 0)$ earns 0 expected payoff.)

Bidding on Just The Small Object

A bidder who bids only on the small object will win it whenever $b + t_j + \frac{1}{2} \geq \frac{3}{2}$, or whenever $t_j \geq 1 - b$, which has probability b ; so expected payoff is $b(t_i - b)$, which is maximized at $b = \frac{1}{2}t_i$. For $t_i \leq \frac{2}{3}$, this is the equilibrium strategy; for $t_i > \frac{2}{3}$, this gives expected payoff $\frac{1}{4}t_i^2$, which is less than the equilibrium payoff because $\frac{5}{2}t_i^2 - 3t_i + 1 - \frac{1}{4}t_i^2 = \frac{9}{4}t_i^2 - 3t_i + 1 = (\frac{3}{2}t_i - 1)^2$.

Bidding on Just The Large Object

A bidder who bids only on the large object will win it whenever $B + \frac{1}{2}t_j \geq \frac{3}{2}$, or whenever $t_j \geq 3 - 2B$, which has probability 0 when $B \leq 1$ and probability $1 - (3 - 2B) = 2B - 2$ otherwise. Assuming $B \geq 1$, expected payoff is $(2B - 2)(2t_i - B)$. When $t_i \leq \frac{1}{2}$, this is never profitable. When $t_i > \frac{1}{2}$, this is maximized at $B = t_i + \frac{1}{2}$, giving expected payoff $2(t_i - \frac{1}{2})^2$. If $t_i \leq \frac{2}{3}$, this is at most $\frac{2}{6^2} = \frac{1}{18}$; but if $t_i \geq \frac{1}{2}$, the equilibrium payoff is $\frac{1}{4}t_i^2 \geq \frac{1}{16}$. For $t_i > \frac{2}{3}$, the gain from deviating is $2t_i^2 - 2t_i + \frac{1}{2} - (\frac{5}{2}t_i^2 - 3t_i + 1) = -\frac{1}{2}t_i^2 + t_i - \frac{1}{2} = -\frac{1}{2}(1 - t_i)^2 < 0$, so this is never a profitable deviation.

Bidding on Both Objects

A bid that never wins is payoff-equivalent to no bid; so we only have to consider further deviations such where the profit-maximizing bid $(b, B) > 0$ and both b and B win with strictly positive probability.

First, suppose $B - b < \frac{5}{6}$. Since $t_j > \frac{2}{3}$ implies $B_j - b_j = \frac{1}{2}t_j + \frac{1}{2} > \frac{5}{6}$, bidder i never wins the big object when $t_j > \frac{2}{3}$. i wins the big object when $t_j \leq \frac{2}{3}$ and $B + b_j = B + \frac{1}{2}t_j \geq \frac{3}{2}$, or $t_j \geq 3 - 2B$, so i wins the big object if and only if $t_j \in [3 - 2B, \frac{2}{3}]$. For this to be nonempty requires $B > \frac{7}{6}$, which means $b > \frac{2}{6}$, which means that whenever $t_j > \frac{2}{3}$, $B_j + b > \frac{2}{3} + \frac{1}{2} + \frac{1}{3} = \frac{3}{2}$, so i wins the small object whenever $t_j > \frac{2}{3}$. A small enough decrease in b would not change this probability, but would reduce what i pays for the small object, so $B - b < \frac{5}{6}$ is never optimal. So without loss, we can add the additional constraint $B - b \geq \frac{5}{6}$ to the maximization problem.

Now, consider $B - b \geq \frac{5}{6}$. Bidder i wins the big object when $B + b_j \geq \frac{3}{2}$ and $B + b_j \geq B_j + b$. The former is $B + \frac{1}{2}t_j \geq \frac{3}{2}$, or $t_j \geq 3 - 2B$. The latter holds whenever $t_j \leq \frac{2}{3}$, as well as when $B - b \geq B_j - b_j = \frac{1}{2}t_j + \frac{1}{2}$, or $t_j \leq 2(B - b) - 1$. When $B - b \geq \frac{5}{6}$, $2(B - b) - 1 \geq \frac{2}{3}$, so $B + b_j \geq B_j + b$ if and only if $t_j \leq 2(B - b) - 1$. So bidder i wins the big object if and only if $t_j \in [3 - 2B, 2B - 2b - 1]$. By assumption, this interval is nonempty, so $3 - 2B < 2B - 2b - 1$, or $4B - 2b - 4 > 0$. In addition, bidder i wins the small object whenever $t_j > 2B - 2b - 1$ and $B_j + b \geq \frac{3}{2}$, or $t_j + \frac{1}{2} + b \geq \frac{3}{2}$, or $t_j \geq 1 - b$. We're already assuming that $4B - 2b - 4 > 0$, which means $2B - b - 2 > 0$, or $2B - 2b - 1 > 1 - b$; so i wins the small object when $t_j \geq 2B - 2b - 1$.

Locally, then, expected payoffs are

$$\begin{aligned} \pi(t_i, B, b) &= (2B - 2b - 1 - (3 - 2B))(2t_i - B) + (1 - (2B - 2b - 1))(t_i - b) \\ &= (4B - 2b - 4)(2t_i - B) + (2 - 2B + 2b)(t_i - b) \end{aligned}$$

$$\begin{aligned} \frac{\partial \pi}{\partial B} &= 4(2t_i - B) - (4B - 2b - 4) - 2(t_i - b) \\ &= 6t_i - 8B + 4b + 4 \end{aligned}$$

$$\begin{aligned} \frac{\partial \pi}{\partial b} &= -2(2t_i - B) + 2(t_i - b) - (2 - 2B + 2b) \\ &= -2t_i + 4B - 4b - 2 \end{aligned}$$

Both first-order conditions hold only at $B = t_i + \frac{1}{2}$ and $b = \frac{1}{2}t_i$, and π is strictly concave, so this is the unique maximizer.

For $t_i \geq \frac{2}{3}$, the maximizer satisfies $B - b \geq \frac{5}{6}$, so this is the “constrained” maximizer. (For $t_i > \frac{2}{3}$, this is the equilibrium strategy. For $t_i = \frac{2}{3}$, this gives the same expected payoff as the equilibrium strategy $(b, B) = (\frac{1}{3}, 0)$, since at $(\frac{1}{3}, \frac{7}{6})$, the big bid never wins.) For $t_i < \frac{2}{3}$, however, the “unconstrained” maximizer has $B - b < \frac{5}{6}$; since the objective function is strictly concave, this means the constrained optimum satisfies $B - b = \frac{5}{6}$. For $t_i < \frac{2}{3}$, then, we can impose $B - b = \frac{5}{6}$ and solve for B .

$$\begin{aligned}\pi(t_i, B, b) &= (2(B - b) - 1 - (3 - 2B))(2t_i - B) + (2 - 2(B - b))(t_i - (B - \frac{5}{6})) \\ &= (\frac{5}{3} - 1 - (3 - 2B))(2t_i - B) + (2 - \frac{5}{3})(t_i - (B - \frac{5}{6})) \\ &= (2B - \frac{7}{3})(2t_i - B) + \frac{1}{3}(t_i - B + \frac{5}{6})\end{aligned}$$

$$\begin{aligned}\frac{\partial \pi}{\partial B} &= 2(2t_i - B) - (2B - \frac{7}{3}) - \frac{1}{3} \\ &= 4t_i - 4B + 2\end{aligned}$$

So expected payoff is increasing in B until $B = t_i + \frac{1}{2}$, decreasing after that. But for $t_i < \frac{2}{3}$, $t_i + \frac{1}{2} < \frac{5}{6}$, which would make b negative; so the maximizer is $B = \frac{5}{6}$ and $b = 0$, which we already dismissed as a profitable deviation.

References

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