

# Identification and Inference in Ascending Auctions with Correlated Private Values

Andrés Aradillas-López, Amit Gandhi, Daniel Quint\*

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## Abstract

We introduce and apply a new nonparametric approach to identification and inference on data from ascending auctions. We exploit variation in the number of bidders across auctions to nonparametrically identify and estimate useful bounds on seller profit and other measures of economic interest within a general model of correlated private values. Applying our methods to much-studied U.S. Forest Service timber auctions, we find evidence of correlation among values after controlling for relevant auction covariates; this correlation causes expected profit and the profit-maximizing reserve price to be substantially lower than conventional (IPV) analysis would suggest.

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# 1 Introduction

Applying insights from auction theory to real world settings requires knowledge of the primitives of the game being played by bidders. In the case of a private value ascending (English) auction, the main primitive of interest is the latent distribution of bidder valuations. Once this distribution is known, a seller can forecast expected revenue and assess the effects of a change in auction design.

Ascending auctions, however, present a unique empirical challenge. Since each auction ends when every bidder but one is unwilling to raise his bid, bidding behavior never reveals how high the winner would have been willing to go. Even in the idealized environment of a “button auction,” where each losing bidder’s private valuation is exactly revealed from his bidding behavior, the joint distribution of valuations among *all* bidders cannot be nonparametrically identified, even when bidders are symmetric.<sup>1</sup>

The standard approach in the literature has therefore been to impose stronger structural assumptions to achieve identification. This is most commonly done by assuming that bidders’ valuations, in addition to being symmetric, are independent (IPV). Nonparametric identification of the distribution of valuations from bid data in this setting was established by Athey and Haile (2002) and Haile and Tamer (2003). Independence is also a standard identifying restriction used in the empirical literature on first-price auctions (see, for example, Guerre, Perrigne, and Vuong (2000) and Guerre, Perrigne, and Vuong (2009)).

Of course, the IPV model is misspecified if there is dependence among bidder valuations. Whether values are independent, and if not, whether this misspecification is innocuous or economically meaningful, are empirical questions. The goal of our paper is to analyze the seller’s problem in a nonparametric model that allows for correlation (and nests IPV as a special case), to see empirically whether values are correlated and if so, to understand what implications this correlation has.

One approach to dealing with correlation is to make parametric or semiparametric assumptions about the joint distribution of valuations (see, for example, Hong and Shum (2003) and Roberts (2009)). We instead focus on the narrower question of analyzing the seller’s problem (rather than recovering the full joint distribution), but are able to remain fully nonparametric. We show that the seller’s profit maximization problem (as a function of reserve price) depends only on the marginal distributions of the two highest valuations; and that these two marginal distributions can be partially identified from bid data without parametric assumptions by exploiting variation in the number of bidders. This allows us to identify bounds on the seller’s profit function; we are the first to do so nonparametrically in a general correlated setting.

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<sup>1</sup>Athey and Haile (2002) Theorem 4.

In addition to generality, there are two other advantages to our approach. First, the data requirements are minimal: we need only observe the number of bidders and the transaction price in each auction. (Observation of auction covariates, and of multiple bids from each auction, may allow us to tighten the inference, but are not strictly necessary to proceed.) And second, even though the seller’s expected profit function is only partially identified, the bounds we characterize have a closed-form expression and are themselves point-identified; as a result, we can adapt standard econometric methods for inference, as well as allow for a rich collection of observable covariates.

We apply our approach to the much-analyzed data from U.S. Forest Service timber auctions. The ascending auctions in this data have been exclusively analyzed in the past under the IPV assumption. Even conditional on the rich vector of available covariates (whose presence is often used to defend the IPV assumption), we find evidence of positive dependence among valuations. Comparing our results side-by-side with traditional IPV analysis, we find substantially lower expected profit and optimal reserve prices. These findings may help to explain the seemingly conservative reserve prices used by the Forest Service, as discussed in Paarsch (1997) (using different data) and Haile and Tamer (2003).

Our paper proceeds as follows. In section 2, we discuss the most closely-related literature. In section 3, we present our model and theoretical results. In section 4, we apply our model to timber auction data, and perform both auction- and portfolio-level counterfactual analysis. Section 5 discusses extensions and limitations of our approach; section 6 concludes. Appendix A contains further extensions and supporting results referenced in the text; all proofs are in Appendix B.

## 2 Related Literature

The literature on identification in ascending auctions is fairly sparse. Paarsch (1997) models ascending auctions as “button” auctions – bidders hold down a button to remain active as the price rises continuously, releasing the button to drop out, and thus each losing bidder’s willingness to pay is learned exactly from his bid – and estimates a parametric model of independent private values. Donald and Paarsch (1996) discuss identification and estimation of parametric IPV models. Hong and Shum (2003) estimate a parametric, affiliated values model.<sup>2</sup> Athey and Haile (2002) show that in button auctions, an IPV model is nonparametrically identified from transaction prices alone. Haile and Tamer (2003) depart from the button auction model and introduce an “incomplete” model of bidding in English auctions, making two behavioral assumptions about bidders which essentially

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<sup>2</sup>Roberts (2009) estimates a model of IPV with unobserved heterogeneity, but assumes that sellers set reserve prices in a way that is strictly monotonic in the unobserved characteristic, allowing it to be backed out from the data.

amount to weakly-undominated strategies; within the IPV setting, they show that this leads to bounds on the distribution of valuations.

Athey and Haile (2002) also give a non-identification result: a general private-values model is not identified from ascending-auction bid data even if all bids are observed. As a result, there has been no work on ascending auctions in a nonparametric, correlated-values setting. We are the first to get positive results in such a setting, which we do by narrowing the focus to the seller’s problem (rather than the full joint distribution of valuations).

The literature on first-price auctions offers approaches to identification of models with correlation or unobserved heterogeneity, but these techniques are not applicable to ascending auctions. For example, Li, Perrigne, and Vuong (2002) extend the nonparametric identification technique introduced by Guerre, Perrigne, and Vuong (2000) to affiliated private values; Li, Perrigne, and Vuong (2000), Krasnokutskaya (2009), and Hu, McAdams, and Shum (2009) use a measurement error approach to identify models of first-price auctions with unobserved heterogeneity. Both of these strategies rely on multiple “informative” bids from each auction; thus, they cannot be applied to ascending auctions, where only one bid can be tightly linked to a bidder’s valuation.

## 3 Theory

### 3.1 Environment and Bidding Behavior

Here, we introduce our theoretical framework, and in particular, three assumptions about the environment and bidding behavior which we will maintain throughout the paper.

**Assumption 1** *Bidders have private values, and the joint distribution of values is symmetric.*<sup>3</sup>

Assumption 1 is very standard in the literature, and is best justified within the context of a particular application.

**Assumption 2** *The transaction price in an auction is the greater of the reserve price and the second-highest bidder’s willingness to pay.*

Rather than an assumption on primitives, Assumption 2 is an assumption on equilibrium play, but it can be motivated in a number of ways. It is the exact outcome that would result in a button auction (discussed above), a stylized version of an ascending auction. It would also hold in the incomplete model of open-outcry ascending auctions of Haile and Tamer (2003), if the bid increment

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<sup>3</sup>That is, if  $\mathbf{F}(\cdot, \dots, \cdot)$  is the joint CDF of  $n$  bidders’ valuations, then  $\mathbf{F}(v_1, \dots, v_n) = \mathbf{F}(v_{\sigma(1)}, \dots, v_{\sigma(n)})$ , where  $\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$  is any permutation. This is equivalent to valuations being exchangeable random variables.

is small and bidders do not use “jump bids” at the end of an auction (which can be verified directly in the data). This assumption can also be motivated pragmatically: in some datasets, only the transaction price is recorded, so relying on multiple bids is impossible.<sup>4</sup> (In Appendix A.1, we show how our results can be modified to rely instead on the bidding assumptions of Haile and Tamer (2003) and observation of the top two bids in each auction.)

Finally, we impose a restriction that the correlation among bidder values be non-negative. As we show, this assumption is general enough to nest three standard models of correlated private values. Let  $N$  denote the number of bidders in an auction,  $n$  a realization of  $N$ , and  $(V_1, V_2, \dots, V_n)$  the private values of the bidders in an  $n$ -bidder auction.<sup>5</sup>

**Assumption 3** *Fixing the number of bidders  $n$ , the joint distribution of valuations is such that for any  $v$ ,  $\Pr(V_i < v)$  is weakly increasing in the number of bidders  $j \neq i$  with  $V_j < v$ .*

**Lemma 1** *Assumption 3 is satisfied by any model of (i) symmetric, affiliated private values; (ii) symmetric, conditionally independent private values; or (iii) symmetric, independent private values with unobserved heterogeneity.*

Affiliation is the standard formulation of positive dependence in the theoretical auction literature, going back to Milgrom and Weber (1982). The latter two models, which are used in the empirical literature, differ in the following way. In either one, there is a parameter or set of parameters  $\theta$  which is unobserved to the econometrician, and bidder valuations are *i.i.d.* draws from a distribution which varies with these parameters. In the CIPV model, the bidders are assumed to learn only their valuations, not the value of  $\theta$ , while in the IPV with unobserved heterogeneity model, the bidders learn both  $\theta$  and their valuations. Thus, in the former model, bidders perceive their valuations as correlated, while in the latter, conditional on their information, bidders perceive values as independent. These have very different implications in first-price auctions and entry games, since equilibrium play depends on a bidder’s beliefs about his opponents’ valuations. In our setting, however, bidding is effectively in dominant strategies, and so the distinction is immaterial; we need not even specify what, beyond his own private value, each bidder knows. Note also that our model

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<sup>4</sup>For example, the data used by Asker (2010) to study the behavior of a collusive bidding ring contained detailed data about activities within the ring, but only the transaction prices of the auctionhouse auctions themselves.

<sup>5</sup>To be proper,  $N$  should refer to the number of *potential* bidders – the number of buyers who learned their valuations and would have won the auction given a high enough private value – so that we can interpret the winning bidder as the “highest of  $N$ ”. Observability of this number is problematic in some settings, although (as we discuss below) not in our application. See Song (2004) and An, Hu, and Shum (2010) for empirical approaches to auctions when the number of potential bidders is not observable.

nesses both of these without any restriction on the dimensionality of the unobserved variables or the way in which they effect the distribution of valuations.

## 3.2 Identification

Next, we show how observation of transaction prices for auctions with different numbers of bidders is sufficient to identify the seller’s expected profit function, and thus solve the problem of determining the optimal reserve price. For a given value  $n$  of  $N$ , let  $V_{n:n} \geq V_{n-1:n} \geq \dots \geq V_{1:n}$  denote the order statistics of bidder values, and  $F_{k:n}(\cdot)$  the cumulative distribution function of  $V_{k:n}$ . Assumption 2 implies that in the absence of binding reserve prices, the distribution  $F_{n-1:n}$  is identified from the transaction prices of a series of  $n$ -bidder auctions; thus, our analysis will begin with the assumption that this distribution is known for each  $n$ . All the results below also apply conditional on the realization of a set of auction covariates  $X$ , if we began with nonparametric estimates of  $F_{n-1:n}(\cdot | X)$  – this is what we will do in our application.

Let  $v_0$  denote the value of the unsold good to the seller (or the seller’s cost, or opportunity cost, of parting with the object); let  $\pi_n(r)$  denote the seller’s expected profit in an  $n$ -bidder auction with reserve price  $r$ , and  $u_n(r)$  each bidder’s ex-ante expected payoff from participating in the auction.

**Lemma 2** *In an ascending auction with  $n$  bidders and reserve price  $r$ ,*

$$\pi_n(r) = \int_0^\infty \max\{r, v\} dF_{n-1:n}(v) - v_0 - F_{n:n}(r)(r - v_0) \quad (1)$$

$$u_n(r) = \frac{1}{n} \left[ \int_0^\infty \max\{r, v\} dF_{n:n}(v) - \int_0^\infty \max\{r, v\} dF_{n-1:n}(v) \right] \quad (2)$$

and  $\pi_n$  and  $u_n$  therefore depend only on  $F_{n-1:n}$  and  $F_{n:n}$ .

While “identification” in auctions typically aims to recover the entire joint distribution of bidder valuations, which is impossible in a model as general as ours, this result frees us to focus only on identifying the two marginal distributions  $F_{n-1:n}$  and  $F_{n:n}$ .<sup>6,7</sup> Since  $F_{n-1:n}$  is directly revealed by transaction price, what remains to be identified is  $F_{n:n}$ . Because nothing directly linked to  $V_{n:n}$  is

<sup>6</sup>In a different but analogous setting, Athey and Haile (2007) point out that identification of the *joint* distribution of  $V_{n-1:n}$  and  $V_{n:n}$  is sufficient for “evaluation of rent extraction by the seller, the effects of introducing a reserve price, and the outcomes under a number of alternative selling mechanisms.” Lemma 2 allows us to study the seller’s problem in ascending auctions by focusing only on the two marginal distributions, not the joint.

<sup>7</sup>(1) also gives some informal intuition as to why correlation among values tends to reduce the seller’s expected gain from imposing a reserve price.  $\pi_n$  is decreasing in  $F_{n:n}(r)$ , and  $F_{n:n}(r)$  can be expressed as  $F_{n-1:n}(r) \Pr(V_{n:n} < r | V_{n-1:n} < r)$ . The more-highly-correlated are bidder valuations, the higher will be  $\Pr(V_{n:n} < r | V_{n-1:n} < r)$ ; and so for a given set of observables ( $F_{n-1:n}$ ), the lower will be expected profit at any reserve price  $r > v_0$ .

observed in ascending auctions – the auction ends as soon as the second-to-last bidder drops out – we will use variation in  $n$  to identify bounds on  $F_{n:n}$ , which will lead via (1) to bounds on  $\pi_n$ .

To show why this is necessary, we first observe that without variation in  $n$ , the range of values of  $F_{n:n}$  consistent with a particular distribution of transaction prices is too wide to be empirically useful. To see this, for  $n \geq 2$ , define a function  $\phi_n : [0, 1] \rightarrow [0, 1]$  implicitly by

$$v = \int_0^{\phi_n(v)} n(n-1)s^{n-2}(1-s)ds \quad (3)$$

Note that  $\phi_n$  is the inverse of the function mapping  $x$  to  $nx^{n-1} - (n-1)x^n$ , which takes any distribution function to the distribution of the second-highest of  $n$  independent draws from it.

**Lemma 3** *For any  $n$  and  $v$ , under Assumptions 1 and 3,  $F_{n:n}(v) \in [(\phi_n(F_{n-1:n}(v)))^n, F_{n-1:n}(v)]$ , and both bounds are tight.<sup>8</sup>*

The interval  $[(\phi_n(F_{n-1:n}(v)))^n, F_{n-1:n}(v)]$ , however, is fairly wide. We illustrate this with a numerical example, which we will return to again to illustrate later results. For the example, we will assume that  $F_{n-1:n}$  is known exactly – that is, we will ignore the problem of estimation, to focus on the effect of uncertainty about  $F_{n:n}$  even given certainty about  $F_{n-1:n}$ .

**Example 1** *Consider a symmetric, IPV model with unobserved heterogeneity. There are 3 bidders. An unobserved “demand shifter”  $\theta$  takes two values,  $\theta \in \{H, L\}$ , with equal probability. Conditional on a realization of  $\theta$ , bidder valuations are i.i.d. log-normal:  $\ln(V_i) \sim N(2.5, 0.5)$  when  $\theta = H$  and  $\ln(V_i) \sim N(2.0, 0.5)$  when  $\theta = L$ . The seller’s reservation value is  $v_0 = 5$ .*

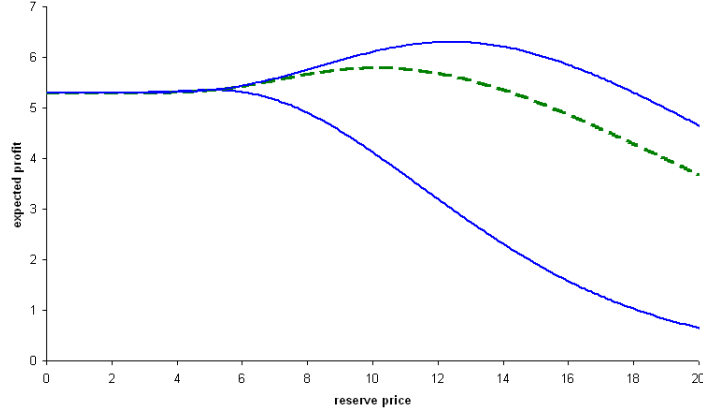
*Figure 1 below shows the true expected profit function  $\pi_3(\cdot)$  (the dashed line), and the tightest bounds that can be put on  $\pi_3(\cdot)$  using only  $F_{2:3}$  (the solid lines).*

*Actual expected profit is maximized at a reserve price of  $r = 10.1$ ; but the range of reserve prices which could be optimal given the distribution  $F_{2:3}$  is at least  $[5.0, 12.4]$ ; and over that range, the upper bound is as much as 2.1 times the lower bound.*

The upper bound on profit in Figure 1 is consistent with a model of independent private values, and the lower bound is consistent with perfectly-correlated values. Thus, what Lemma 3 and Figure 1 are saying is that, observing only a series of transaction prices from  $n$ -bidder auctions, there is no way to tell the degree of correlation among bidder values from auction to auction, which could range from none (the best-case scenario for the seller) to perfect (the worst-case scenario); and that the difference between these possibilities, in terms of the reserve price a seller should set and the

<sup>8</sup>This is proven in Quint (2008) for symmetric, affiliated values, but not for the more general case given here.

Figure 1: Bounds on  $\pi_3(r)$  using only  $F_{2:3}$  (from Lemmas 2 and 3)



revenue he should expect to receive, are substantial. To tighten our bounds on  $\pi_n$ , then, we will pool information across auctions with different numbers of bidders.<sup>9</sup>

### 3.3 Using Variation in the Number of Bidders

To see how variation in  $N$  allows us to tighten the bounds on  $F_{n:n}$ , we begin with the following observation. If  $n$  bidders are chosen at random out of  $n + 1$ , the highest valuation of the sample will be either the highest of the original group (with probability  $\frac{n}{n+1}$ ), or the second-highest (with probability  $\frac{1}{n+1}$ ). Thus, if we let  $F_{k:m}^n$  denote the distribution of the  $k^{\text{th}}$  order statistic out of  $m$  bidder valuations taken at random from an auction with  $n$  bidders ( $n \geq m \geq k$ ), then<sup>10</sup>

$$F_{n:n}^{n+1}(v) = \frac{1}{n+1}F_{n:n+1}(v) + \frac{n}{n+1}F_{n+1:n+1}(v) \quad (4)$$

Our ability to use this relation, however, hinges on the relationship between  $F_{n:n}$  and  $F_{n:n}^{n+1}$  – that is, on whether, and how, the bidders in an  $n$ -bidder auction differ from the bidders in an  $n + 1$ -bidder auction. Let  $\mathbf{F}_m^n$  be the joint distribution of  $m$  randomly-chosen bidders in an  $n$ -bidder auction.

**Definition 1** *Valuations are independent of  $N$  if  $\mathbf{F}_m^n = \mathbf{F}_m^{n'}$  for any  $(n, n', m)$ .*

Valuations are independent of  $N$  if a group of bidders in an  $n$ -bidder auction is statistically indistinguishable from a similarly-sized group of bidders in an  $n'$ -bidder auction. Independence

<sup>9</sup>Other (losing) bids could potentially tighten the upper bound on  $\pi_n$ , but not the lower bound. In Appendix A.2, we show how this would be done, and discuss why we choose to ignore losing bids in our analysis.

<sup>10</sup>(4) is a special case of Eq. 9 from Athey and Haile (2002).

between valuations and  $N$  has been assumed in several recent studies of first-price IPV auctions. Guerre, Perrigne, and Vuong (2000) make this assumption to bolster the sample size for nonparametric estimation. Haile, Hong, and Shum (2003) make the same assumption to test between common and private values. (They extend their results to allow for endogenous participation based on standard models of entry, or to use an instrument for the number of bidders.) Guerre, Perrigne, and Vuong (2009) use the same assumption to identify the coefficient of risk aversion in a model with risk-averse bidders; Gillen (2009) uses the same assumption to identify the distribution of bidders' level of strategic sophistication in a "level- $k$ " behavioral model. The dominance-solvability of ascending auctions makes risk preferences and strategic sophistication irrelevant in our environment; thus, we use the same identifying assumption to identify correlation, leading to much more useful bounds on  $\pi_n$  than those shown above.

**Lemma 4** *Fix  $n$ . For any  $\bar{n} > n$  and any  $v$ , if valuations are independent of  $N$ , then*

$$F_{n:n}(v) = \sum_{m=n+1}^{\bar{n}} \frac{n}{(m-1)m} F_{m-1:m}(v) + \frac{n}{\bar{n}} F_{\bar{n}:\bar{n}}(v)$$

Since  $\{F_{m-1:m}\}$  are known, we need only bound  $F_{\bar{n}:\bar{n}}$  (using Lemma 3) to get bounds on  $F_{n:n}$ ; and plugging these bounds into (1) will give bounds on expected profit.<sup>11</sup>

**Theorem 1** *Fix  $n$  and  $\bar{n} > n$ . Define two functions  $\underline{\pi}_n$  and  $\bar{\pi}_n : \mathfrak{R}^+ \rightarrow \mathfrak{R}$  by*

$$\underline{\pi}_n(r) \equiv \int_0^\infty \max\{r, v\} dF_{n-1:n}(v) - v_0 - \left( \sum_{m=n+1}^{\bar{n}} \frac{n}{(m-1)m} F_{m-1:m}(r) + \frac{n}{\bar{n}} F_{\bar{n}-1:\bar{n}}(r) \right) (r - v_0)$$

$$\bar{\pi}_n(r) \equiv \int_0^\infty \max\{r, v\} dF_{n-1:n}(v) - v_0 - \left( \sum_{m=n+1}^{\bar{n}} \frac{n}{(m-1)m} F_{m-1:m}(r) + \frac{n}{\bar{n}} (\phi_{\bar{n}}(F_{\bar{n}-1:\bar{n}}(r)))^{\bar{n}} \right) (r - v_0)$$

*If valuations are independent of  $N$ , then  $\pi_n(r) \in [\underline{\pi}_n(r), \bar{\pi}_n(r)]$  for every  $r \geq v_0$ .*

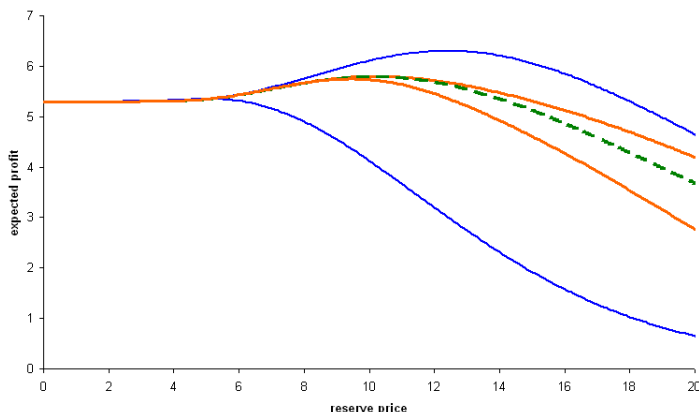
*Further, let  $\underline{\pi}_n^* = \max_{r'} \underline{\pi}_n(r')$ , and let  $\underline{r}_n$  and  $\bar{r}_n$  be the min and the max, respectively, of the set  $\{r \geq v_0 : \bar{\pi}_n(r) \geq \underline{\pi}_n^*\}$ . If valuations are independent of  $N$ , then  $\arg \max_r \pi_n(r) \in [\underline{r}_n, \bar{r}_n]$ .*

To see what these bounds accomplish, we return to our example from before.

**Example 2** *Consider again the model of Example 1, but now suppose that the number of bidders varies "exogenously" (independently of  $\theta$  and valuations) between 3 and 12. Again suppose that each distribution  $F_{m-1:m}$  is learned exactly. Again, the true value of  $\pi_3(r)$  is the dashed line, and the solid blue lines are the bounds calculated only from auctions with 3 bidders. The solid red lines are the bounds calculated from Theorem 1, using  $\bar{n} = 12$ .*

<sup>11</sup>We focus here on the seller's problem; we could calculate analogous bounds on  $u_n(r)$  by plugging the bounds on  $F_{n:n}$  into (2).

Figure 2: Bounds on  $\pi_3(r)$  from Theorem 1, using  $\{F_{n-1:n}\}_{n=3}^{12}$



Note that the bounds from Theorem 1 are much tighter. Recall  $\pi_3$  is maximized at  $r = 10.1$ . With  $\bar{n} = 12$ , Theorem 1 pins down the optimal reserve price to within the interval  $[8.9, 11.6]$ ; and over that range of possibly-optimal reserve prices, the upper and lower bounds differ by no more than 4%.

If valuations are independent of  $N$ , then for each  $r$ , the interval  $[\underline{\pi}_n(r), \bar{\pi}_n(r)]$  shrinks as  $\bar{n}$  increases, and collapses to the true value of  $\pi_n(r)$  as  $\bar{n} \rightarrow \infty$ .

### 3.4 When Valuations Are Not Independent of $N$

The assumption that valuations are independent of  $N$  is essentially the assumption that bidders' decisions to participate in an auction are independent of the value of the object for sale. In some applications, this may be unrealistic. If, even conditional on observable covariates, the objects for sale are heterogeneous, we might expect auctions for more valuable objects to attract more bidders, leading to a positive association between bidder valuations and the number of bidders in an auction. Next, we show that partial identification results hold in this case as well. We will rely on a weaker identifying assumption, essentially stochastic monotonicity – higher  $N$  must be associated with higher valuations in a particular sense.

**Definition 2** *Valuations are stochastically increasing in  $N$  if  $n > n'$  implies  $F_{m:m}^n \succ_{FOSD} F_{m:m}^{n'}$  for any  $m \leq n'$ .*

That is, valuations are stochastically increasing in  $N$  if a group of bidders sampled from a larger auction tend to have higher valuations than a group of bidders sampled from a smaller auction, where “higher” is measured as a first-order stochastic dominance ranking of the highest

order statistic. In Appendix A.3, we discuss three standard models of endogenous participation in auctions – those of Levin and Smith (1996), Samuelson (1985), and Marmer, Shneyerov, and Xu (2010) – and show conditions under which equilibrium play in each model would lead to valuations stochastically increasing in  $N$ .

When valuations are not independent of  $N$  but rather stochastically increasing, one-sided analogs of Lemma 4 and Theorem 1 hold:

**Lemma 4'** *Fix  $n$ . For any  $\bar{n} > n$  and any  $v$ , if valuations are stochastically increasing in  $N$ , then  $F_{n:n}(v) \geq \sum_{m=n+1}^{\bar{n}} \frac{n}{(m-1)m} F_{m-1:m}(v) + \frac{n}{\bar{n}} F_{\bar{n}:\bar{n}}(v)$ .*

**Theorem 1'** *Fix  $n$  and  $\bar{n} > n$ . If valuations are stochastically increasing in  $N$ , then for  $r \geq v_0$ ,  $\pi_n(r) \leq \bar{\pi}_n(r)$  (where  $\bar{\pi}_n(r)$  is as defined in Theorem 1).*

One interpretation of Theorem 1' is as a “robustness check” on Theorem 1. Theorem 1 gives two-sided bounds on  $\pi_n(r)$  when valuations are independent of  $N$ ; Theorem 1' says that the upper bound is still valid if valuations are instead stochastically increasing in  $N$ .

### 3.5 Estimation and Inference

We conclude the theory section by showing how to translate the bounds  $[\underline{\pi}_n(r), \bar{\pi}_n(r)]$  into an asymptotically valid confidence interval for  $\pi_n(r)$  from a sample of auction data which may contain auction-level covariates. For a given auction in the data indexed by  $i$ , let  $N_i$  refer to the number of bidders,  $B_i$  the transaction price, and  $X_i$  a vector of covariates. Let  $X_i^c$  denote the subset of continuous covariates,  $X_i^d$  the subset of discrete covariates, and  $z = \dim(X^c)$  the number of continuous covariates. Take a given reserve price  $r$  and a given realization of  $X_i$ , labeled  $x = (x^c, x^d)$ . Define

$$T_{n-1:n}(r|x) \equiv \int_0^\infty \max\{r, v\} dF_{n-1:n}(v|x) = E[\max\{r, B\} | X = x, N = n]$$

which is the first term in (1). Let  $K$  be a  $z$ -dimensional kernel, and  $h_L \rightarrow 0$  a decreasing bandwidth sequence. For any  $n$  and any  $x = (x^c, x^d)$ , define the sample analogs

$$\begin{aligned} \hat{T}_{n-1:n}(r|x) &= \frac{\sum_{i=1}^L \max\{r, B_i\} \cdot K\left(\frac{X_i^c - x^c}{h_L}\right) \cdot \mathbf{1}\{X_i^d = x^d\} \cdot \mathbf{1}\{N_i = n\}}{\sum_{i=1}^L K\left(\frac{X_i^c - x^c}{h_L}\right) \cdot \mathbf{1}\{X_i^d = x^d\} \cdot \mathbf{1}\{N_i = n\}} \\ \hat{F}_{n-1:n}(r|x) &= \frac{\sum_{i=1}^L \mathbf{1}\{B_i \leq r\} \cdot K\left(\frac{X_i^c - x^c}{h_L}\right) \cdot \mathbf{1}\{X_i^d = x^d\} \cdot \mathbf{1}\{N_i = n\}}{\sum_{i=1}^L K\left(\frac{X_i^c - x^c}{h_L}\right) \cdot \mathbf{1}\{X_i^d = x^d\} \cdot \mathbf{1}\{N_i = n\}} \end{aligned} \tag{5}$$

and

$$\begin{aligned}\widehat{\pi}_n(r|x) &= \widehat{T}_{n-1:n}(r|x) - v_0 - \left( \sum_{m=n+1}^{\bar{n}} \frac{n}{(m-1)m} \widehat{F}_{m-1:m}(r|x) + \frac{n}{\bar{n}} \left( \phi_{\bar{n}} \left( \widehat{F}_{\bar{n}-1:\bar{n}}(r|x) \right) \right)^{\bar{n}} \right) (r - v_0) \\ \underline{\widehat{\pi}}_n(r|x) &= \widehat{T}_{n-1:n}(r|x) - v_0 - \left( \sum_{m=n+1}^{\bar{n}} \frac{n}{(m-1)m} \widehat{F}_{m-1:m}(r|x) + \frac{n}{\bar{n}} \widehat{F}_{\bar{n}-1:\bar{n}}(r|x) \right) (r - v_0)\end{aligned}\tag{6}$$

Appendix A.4 establishes asymptotic properties of these estimators under standard assumptions; the asymptotics hold uniformly over any compact range of values  $(r, x)$  on which  $F_{m-1:m}(r|x)$  is bounded away from 0 and 1. In particular, suppressing the dependence of everything on  $r$  and  $x$ ,

$$\sqrt{L \cdot h_L^z} \begin{pmatrix} \widehat{\pi}_n - \pi_n \\ \underline{\widehat{\pi}}_n - \bar{\pi}_n \end{pmatrix} \xrightarrow{d} \mathcal{N} \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_n^2 & \rho_n \cdot \underline{\sigma}_n \cdot \bar{\sigma}_n \\ \rho_n \cdot \underline{\sigma}_n \cdot \bar{\sigma}_n & \bar{\sigma}_n^2 \end{pmatrix} \right)$$

uniformly over any such range. Expressions for  $\bar{\sigma}_n(r|x)$ ,  $\underline{\sigma}_n(r|x)$ , and  $\rho_n(r|x)$  and their estimators are given in Appendix A.4.

If valuations are independent of  $N$  (so Theorem 1 holds), we can use these asymptotic properties of  $\underline{\widehat{\pi}}_n(r|x)$  and  $\widehat{\pi}_n(r|x)$  to construct a confidence interval for  $\pi_n(r|x)$ . In Appendix A.4 we show how to choose critical values  $c_\alpha^\ell(r|x)$  and  $c_\alpha^u(r|x)$  so that the confidence interval

$$\left[ \underline{\widehat{\pi}}_n(r|x) - c_\alpha^\ell(r|x) \cdot \frac{\widehat{\sigma}_n(r|x)}{\sqrt{L \cdot h_L^z}}, \widehat{\pi}_n(r|x) + c_\alpha^u(r|x) \cdot \frac{\widehat{\sigma}_n(r|x)}{\sqrt{L \cdot h_L^z}} \right]$$

contains the true value of  $\pi_n(r|x)$  with asymptotic probability  $1 - \alpha$  uniformly over a prespecified range of  $(r, x)$ . To achieve this, we adapt the methodology in Imbens and Manski (2004) and Stoye (2009) to allow for  $\bar{\pi}_n(r|x) - \underline{\pi}_n(r|x)$  possibly being arbitrarily close to zero, since  $\bar{\pi}_n = \underline{\pi}_n$  at  $r = v_0$ .

If we assume instead that valuations are stochastically increasing in  $N$  (so Theorem 1' holds), we can instead construct a one-sided (upper) confidence interval for  $\pi_n(r|x)$  based on the asymptotic distribution of  $\widehat{\pi}_n(r|x)$  alone.

Finally, if bidders are assumed to have *independent* private values (IPV), then  $F_{n:n}(r|x) = \phi_n(F_{n-1:n}(r|x))^n$ , so the sample analog for  $\pi_n(r|x)$  is

$$\widehat{\pi}_n^{IPV}(r|x) = \widehat{T}_{n-1:n}(r|x) - v_0 - \left( \phi_n(\widehat{F}_{n-1:n}(r|x)) \right)^n (r - v_0)\tag{7}$$

Inference for this case is also studied in Appendix A.4.

In the Supplementary Appendix, we extend these auction-level inference results to aggregate profits at the ‘‘portfolio’’ level. This allows us to study the counterfactual effects of various reserve price *policies* applied to the mix of auctions in the data, as we do in our application.

## 4 Application to Timber Auctions

### 4.1 Institution and Data

Next, we apply the techniques developed in the previous section to data from timber auctions held by the United States Forest Service. We will examine whether, controlling for observable covariates, correlation remains among bidder valuations, and the implications that this has for reserve price policy. Greater correlation in values will tend to favor lower reserve prices. It is generally recognized that reserve prices during the period we study were low enough to not bind,<sup>12</sup> and whether they should be raised to extract greater rents has been a subject of ongoing debate within the Forest Service.

A number of other empirical papers have also studied Forest Service auctions. Paarsch (1997) estimates his model on data from British Columbian timber auctions; he concludes that reserve prices should be three times higher than those in use. Baldwin, Marshall, and Richard (1997) provide much institutional background on US timber auctions; their focus is to test for collusion. Haile (2001) considers the effects of resale on valuations. Haile, Hong, and Shum (2003) test for common values against private values, using the fact that the Forest Service runs both first-price and ascending auctions. Lu and Perrigne (2008) uses this same variation to estimate risk aversion among bidders. Athey and Levin (2001), Athey, Levin, and Seira (2011), and Haile and Tamer (2003) analyze USFS data to study mechanism design issues. Of these papers, only Athey, Levin, and Seira (2011) allow for unobserved heterogeneity, which they find to be important; they estimate a parametric model on the data from first-price auctions.

USFS timber auctions are used to allocate the right to harvest timber from a tract of public land. Auctions are conducted in two rounds. Before an auction is announced, the Forest Service conducts a “cruise” of the tract, and publishes detailed information on the tract for potential bidders, including an appraisal value. In the first (qualifying) round, bidders submit sealed bids, which must exceed the appraisal value to qualify. All qualifying bidders then advance to the second round, which is an open-outcry ascending auction beginning at the highest of the first-round bids.

Data covering all USFS timber auctions held between 1978 and 1996 was made available to us by Phil Haile. Following Haile and Tamer (2003), we focus on a subset of auctions which are most likely to satisfy the assumption of private values. We consider only scaled sales, where bids are per unit of timber actually harvested, and therefore common-value uncertainty about the total amount of timber

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<sup>12</sup>Campo, Guerre, Perrigne, and Vuong (2002) write, “It is well known that this reserve price does not act as a screening device to participating,” and perform analysis that confirms that “the possible screening effect of the reserve price is negligible” (p. 33). See also Haile (2001), Froeb and McAfee (1988), and Haile and Tamer (2003).

on the tract should not affect valuations. Further, we consider only sales in Region 6 (encompassing mostly Oregon), where bidders typically do not conduct their own pre-auction cruises, to further minimize private information about common-value uncertainty;<sup>13</sup> and we consider only sales whose contracts expire within a year, to minimize the effect of resale on valuations. Finally, we consider only auctions held between 1982 and 1990, as the USFS reserve price policy within this period was stable. What is left is a sample of 2,036 auctions with  $N$  between 2 and 11, in which the private values assumption is thought best to hold.<sup>14</sup>

In addition to the number of bidders and each bidder’s highest bid in the oral round of the auction, our data contains detailed covariate information about each auction from the government’s cruise report. Our analysis controls for four auction covariates which have been emphasized in the previous literature as being relevant demand shifters: the density of timber (timber volume over acres in the tract, which we label  $X^1$ ); the government’s appraisal value of the timber (which we label  $X^2$ ); the estimated profit from manufacturing the timber (sales value minus manufacturing cost,  $X^3$ ); and the species concentration (HHI computed as a function of the volume of various species present,  $X^4$ ). Bids and monetary covariates ( $X^2$  and  $X^3$ ) are all measured in 1983 dollars. We let  $X = (X^1, X^2, X^3, X^4)$  refer to the vector of covariates, and  $X_i = (X_i^1, X_i^2, X_i^3, X_i^4)$  the data corresponding to the  $i^{th}$  auction. Table 1 gives some summary statistics of the data.

Note that the highest bid (transaction price) and second-highest bid have very similar distributions. In fact, the median ratio of second-highest bid to transaction price is 0.994, which gives us confidence in applying Assumption 2 to this data.

Another feature of the data is that the two covariates that are most relevant in predicting transaction price –  $X^2$  and  $X^3$  – are nearly uncorrelated of the number of bidders.<sup>15</sup> If bidders are not selecting based on observables, it may suggest they are also not selecting based on unobservables, and therefore independence between valuations and  $N$  may be plausible. In a separate paper, Aradillas-López, Gandhi, and Quint (2011), we introduce and apply a formal test of independence between valuations and  $N$ , and fail to reject it in these auctions.<sup>16</sup> Thus in the analysis that follows, we

<sup>13</sup>Although potential bidders are *allowed* to conduct their own cruises, this is rarely done in region 6.

<sup>14</sup>Like Haile and Tamer (2003), we drop auctions with one bidder from the analysis, because without binding reserve prices, they give us no information about the distribution of valuations.  $F_{1:1}$  can still be identified from  $\{F_{m-1:m}\}_{m \geq 2}$  via Lemma 4 or 4', and  $\max_r (r - v_0)(1 - F_{1:1}(r))$  can then be solved to find the optimal reserve price when  $N = 1$ . We also drop observations with  $N = 12$ , as this appears to be top-coding for “12 or more” and our methods depend on accurate measurement of  $N$ .

<sup>15</sup>The correlation between  $N$  and  $\ln(X^2)$  is  $-0.03$ , and the correlation between  $N$  and  $\ln(X^3)$  is  $0.01$ .

<sup>16</sup>Roberts and Sweeting (2011) find evidence of selective entry in California auctions. The apparent reason for this difference is that bidders in California conduct their own cruises, so the cost of participation is much greater than in region 6, where they do not.

Table 1: Summary Statistics of Auctions Used In Analysis

<b>variable</b>	<b>mean</b>	<b>median</b>	<b>std dev</b>	<b>min</b>	<b>max</b>
number of bidders ( $N$ )	5.10	5	2.47	2	11
transaction price	129.93	114.48	131.44	5.34	4,263.77
second-highest bid	128.11	112.73	131.12	5.29	4,263.06
timber density ( $X^1$ )	22.35	12.16	27.92	0	475
appraisal price ( $X^2$ )	68.44	56.30	57.54	3.18	825.00
estimated profit ( $X^3$ )	223.64	217.16	59.29	6.24	1,175.65
species concentration ( $X^4$ )	0.678	0.635	0.229	0.232	1.000
timber volume	925.59	522.50	1,227.92	15	13,100
number of observations: 2,036					

Transaction price, second-highest bid, appraisal price, and estimated profit are in dollars per thousand board-feet; timber volume is in thousands of board-feet.

present results which rely on independence between valuations and  $N$ , which gives two sided bounds on the counterfactuals of interest. As noted earlier, if valuations were stochastically increasing in  $N$  rather than independent, the upper bound on expected profit would still be valid (Theorem 1'), and one-sided inference could be conducted based on just that bound.

## 4.2 Expected Profit at the Individual Auction Level

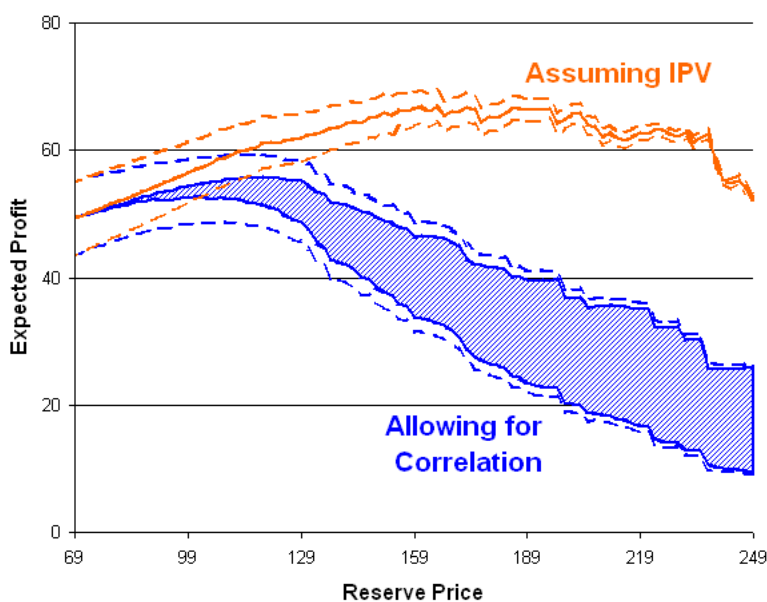
In this section, we estimate the bounds and confidence interval on  $\pi_n(r)$ , as a function of  $r$ , for a representative value of  $X$  (the sample mean) and  $N$ .

Knowledge of  $\pi_n(\cdot)$  allows the seller to set a reserve price to maximize expected profit. It is important to note that since we are not assuming independent private values, the profit-maximizing reserve price depends on the realization of  $N$ . (In IPV models, the solution to  $\max_r \pi_n(r|x)$  does not depend on  $n$ , and this subtlety can therefore be ignored.) In our setting, the two-round nature of the auctions would allow the Forest Service to select a reserve price for the oral round after receiving the first-round bids, at which point  $N$  is known. In applications where this is not possible, a seller would choose  $r$  to maximize  $\sum_n p(n)\pi_n(r|x)$ , where  $p(n)$  is the prior probability of  $n$  bidders participating.

As a natural benchmark, we compare our estimated bounds on  $\pi_n(r|x)$  to expected profit calculated under the assumption of IPV (conditional on  $X$ ).<sup>17</sup>

Throughout the analysis, we assume that the seller's valuation matches the appraisal value, i.e.,  $v_0 = X_i^2$ . Figure 3 shows our estimate of the bounds on expected profit  $\pi_n(r|x)$  (the shaded blue region), as well as a 95% confidence interval, plotted against reserve price, for  $n = 4$  and  $x$  its average realization in the data. We use  $\bar{n} = 11$  and  $v_0 = \overline{X^2} = 68.44$ . For comparison, we plot the point estimate and 95% confidence interval for expected profit under IPV as well (in red).

Figure 3: Expected profit for  $n = 4$  and  $x = \bar{X}$ , assuming  $v_0 = \overline{X^2} = 68.44$



For this case, the point estimates of our bounds suggest that the optimal reserve price lies between 87 and 136, and gives expected profit between 52.7 and 56.0, compared to profits of 49.3 from setting  $r = v_0$ . (When  $r = v_0$ , the last term in (1) vanishes, and our upper and lower bounds are the same, and coincide with the IPV calculation.) IPV analysis, on the other hand, gives an optimal reserve of 189 (although profits under IPV are fairly flat for reserve prices from 155 to 195). At this reserve price, the IPV calculation yields expected profit of 67.3, but our calculation suggests the actual expected profit at that reserve price would be between 23.6 and 39.7. Thus, at the price that IPV analysis suggests is optimal, our bounds indicate profits are substantially lower than if reserve price were simply set equal to appraisal value.

<sup>17</sup>IPV profits are defined as

$$\pi_n^{IPV}(r|x) = \int_0^\infty \max\{r, v\} dF_{n-1:n}(v|x) - v_0 - (\phi_n(F_{n-1:n}(r|x)))^n (r - v_0)$$

and estimated via (7).

In Appendix A.5 (figures 5, 6 and 7), we show similar figures for several other values of  $N$  (from 2 to 7) and  $X$  (the elementwise 10<sup>th</sup>, 25<sup>th</sup>, 75<sup>th</sup>, and 90<sup>th</sup> percentiles of the data). While the details vary from figure to figure, three main observations are consistent:

1. In the most relevant range – the range of reserve prices which are potentially optimal – the interval  $[\underline{\pi}_n(r|x), \bar{\pi}_n(r|x)]$  is narrow enough to contain useful information.
2. For many values of  $r$ ,  $N$  and  $X$ , both the point estimate and 95% confidence interval for expected profit under IPV are outside our 95% confidence interval for  $\pi_n(r|x)$ ; thus, in an informal sense, our bounds are narrow enough to “reject” IPV.
3. Our bounds consistently suggest both lower expected profits and lower optimal reserve prices than IPV. This effect is strongest for low  $N$ , but consistent in sign across  $N$  and  $X$ .

### 4.3 Counterfactual Policy Analysis

Next, we show how these auction-level results aggregate up to the “portfolio level” given the mix of auctions in the data. This will allow us to examine the effect on profit of a counterfactual change in reserve price *policy*. We can think of a reserve price policy as a mapping from  $X$  and  $N$  (the details of a particular auction) to a reserve price  $r(X, N)$ . Letting  $r_0$  and  $r_1$  denote two alternative policies, we will construct point estimates and confidence intervals for the expected profit under each policy,  $E_{X,N}[\pi_N(r_i(X, N)|X)]$ , and the difference in expected profits,  $E_{X,N}[\pi_N(r_1(X, N)|X) - \pi_N(r_0(X, N)|X)]$ .<sup>18</sup> We can also calculate the expected fraction of auctions under each policy at which the reserve price will not be met (leaving the tract unsold),  $E_{X,N}[F_{N:N}(r_i(X, N)|X)]$ . We also consider the analogous measures calculated under the assumption of IPV. The asymptotic properties of these measures are characterized in the Supplementary Appendix, and follow from the properties of  $\hat{\pi}_n(r|x)$  and  $\hat{\underline{\pi}}_n(r|x)$  shown in Appendix A.4. The kernels and bandwidths we use are also described in Appendix A.4.

The “baseline policy” we use for comparison is to set reserve price equal to appraisal value, as the USFS was doing in the period our data is from. Given the distribution of volume and other covariates (and numbers of bidders) in the data, this gives expected baseline profits of \$57,062 per auction, with a 95% confidence interval of [53,178 , 60,945].

For the first counterfactual, we compare this baseline policy to three alternatives, which are constrained profit-maximization under three different estimates of the profit function: IPV, and the

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<sup>18</sup>These expectations are taken with respect to the distribution of  $(X, N)$  in the data, conditional on  $(X, N) \in \mathcal{S}$ , where  $\mathcal{S}$  is a compact set on which (roughly) the estimated joint density of  $(X, N)$  is bounded away from 0. (See Appendix A.4 for details.)

“best-case” and “worst-case” scenarios consistent with our estimated bounds. The constraint comes from a mandate that the Forest Service sell at least 85% of timber tracts; we assume for simplicity that this constraint must be applied at the individual auction level. Thus, for the first policy, the reserve price for each auction is set to the lesser of two numbers: the maximizer of estimated expected profits under the assumption of IPV, and the reserve price which IPV analysis predicts would give a 15% chance of a failed sale (no bidder meeting the reserve price). For the second policy, each reserve price is the lesser of the maximizer of the best-case estimate  $\widehat{\pi}_n(r|x)$ , and the reserve price at which the best-case (lower-bound) estimate of the probability of a failed sale is 15%; for the third policy, each reserve price is the lesser of the maximizer of  $\widehat{\underline{\pi}}_n(r|x)$  and the reserve price at which the worst-case (upper-bound) estimate of the probability of a failed sale is 15%. The results are shown in Table 2. To get a sense of how high reserve prices are under each of these policies, the table also shows the median ratio of reserve price to appraisal value under each policy.

Due to the correlation among values, there is significantly less upside available from setting the reserve price optimally than IPV analysis would suggest; and the policy that appears optimal under IPV is significantly too aggressive in reality. The first policy – maximizing expected profit from each auction under the IPV assumption, subject to the 85% constraint – appears under IPV analysis to offer an 8% increase in profits. Our method, however, shows that this policy would violate the 85% constraint substantially – we would expect between 69% and 78% of auctions to be successful – and offers at best a 1.5% increase, and at worst an 8% *decrease*, in expected profits. The latter two policies – constrained profit maximization, based on either the upper or lower bound on  $\pi_n$  estimated under our method – offer moderate profit increases (at most 4%), but with little or no downside risk relative to the baseline policy. This echoes the takeaway from the auction-level analysis above: we find both expected profit and optimal reserve to be substantially lower than under IPV.

Our second counterfactual is based on a policy change actually implemented by the USFS. During the period our data is from, appraisal value (and therefore reserve price) were based on estimates of accounting profits; in the 1990s, the USFS moved to a “market-based” approach, using the outcomes of past auctions. This was implemented through the Transaction Evidence Appraisal (TEA) method, described in detail in Athey, Crampton, and Ingraham (2003). Under TEA, the appraisal value (and reserve price) would be the predicted value of the winning bid in a given auction based on historical data, discounted by some “rollback factor” to encourage competition. USFS policy requires rollback rates to be no more than 30% of the predicted transaction price; Athey, Crampton, and Ingraham (2003) report discount rates actually employed in various locations ranging from 5% to 30%, with the vast majority between 10% and 30%.

Table 2: Expected Results for Baseline and Three Alternative Reserve Price Policies

**Baseline Policy ( $r =$  appraisal value)**

	Our Results		Assuming IPV	
	Bound Estimates	95% Confidence interval	Point Estimates	95% Confidence interval
Profits	57,062	[53,178 , 60,945]	57,062	[53,178 , 60,945]

**Constrained “IPV Profit” Maximization**

Probability of no sale	[22.2% , 30.8%]	[21.7% , 31.4%]	14.3%	[14.1% , 14.5%]
Profits	[52,399 , 57,947]	[49,518 , 61,093]	61,643	[57,692 , 65,593]
Change in profits	[-4,664 , 884]	[-5,673 , 1,677]	4,581	[3,788 , 5,374]

**Constrained “Best-Case Profit” Maximization**

Probability of no sale	[12.9% , 17.7%]	[12.7% , 18.1%]	8.2%	[7.9% , 8.4%]
Profits	[57,341 , 59,519]	[54,172 , 62,809]	60,164	[56,214 , 64,113]
Change in profits	[280 , 2,457]	[-276 , 2,959]	3,101	[2,533 , 3,669]

**Constrained “Worst-Case Profit” Maximization**

Probability of no sale	[9.4% , 11.1%]	[9.1% , 11.4%]	5.7%	[5.4% , 6.1%]
Profits	[58,576 , 58,783]	[55,297 , 62,069]	58,977	[55,061 , 62,893]
Change in profits	[1,513 , 1,720]	[1,189 , 2,048]	1,914	[1,519 , 2,309]

Median value of  $\frac{\text{reserve price}}{\text{appraisal value}}$  for the four policies are 1.00, 1.87, 1.62, and 1.27, respectively.

We simulate the effect of the TEA approach by nonparametrically estimating the expected transaction price in a given auction as a function of the covariates  $X_i$ , and then applying various discount rates to that result. Table 3 presents our results for this policy exercise.

What we find is that relative to the baseline reserve price policy, a shift to the TEA policy with a 0% or 10% discount rate looks profitable under IPV analysis, but under our analysis offers much greater downside than upside. Even a TEA policy with a discount rate of 20% would likely fall short of the target 85% sales rate, and could potentially yield lower profits than the existing policy. A switch to a TEA policy with a discount rate of 30%, however, would likely meet the 85% successful sale target, and increase profits between 2.6% and 3.5% relative to the baseline policy.

This seems to agree with the Forest Service’s experience actually implementing TEA in region 6 (the region our data is from). According to a U.S. GAO analysis of timber sales under TEA between 1992 and 1996 (Meissner (1997)), the average discount rate used in region 6 was 31.1%, while the average rollback rates used in the six other regions where TEA was implemented were between 7% and 20%. This is consistent with our prediction that under discount rates lower than 30%, the fraction of successful sales in region 6 would fall below the 85% target. (IPV analysis, on the other hand, would have suggested that any discount rate above 10% would meet the 85% target.)

## 5 Extensions and Limitations

Two important extensions to our primary model have already been discussed. Section 3.4 shows how to relax the assumption of independence between valuations and  $N$  (in favor of stochastic monotonicity); and Appendix A.1 shows how to modify our results to rely on the bidding assumptions of Haile and Tamer (2003) instead of Assumption 2. Note also that our analysis is robust to risk-averse bidders: since bidding is in dominant strategies, our results are all perfectly valid under risk aversion (and our techniques would not detect risk aversion).

Here, we consider other extensions and limitations of our approach.

### 5.1 Asymmetric Bidders

Our approach relies on ex-ante symmetry among bidders. However, under an additional assumption, our model can accommodate asymmetric bidders. Suppose, as in Athey, Levin, and Seira (2011), that each bidder is either a large sawmill or a small logging firm, with mills having private values drawn from a different distribution than loggers. Ex ante, view each bidder as having the same, independent probability of being a mill, so that the marginal distribution of each bidder’s valuation is a weighted average of the “mill distribution” and the “logger distribution.” The fact that each bidder’s identity

Table 3: Expected Results for TEA Policy with Various Discount Rates

**0% discount rate**

	Our Results		Assuming IPV	
	Bound Estimates	95% Confidence interval	Point Estimates	95% Confidence interval
Probability of no sale	[26.8% , 36.1%]	[26.1% , 36.8%]	20.5%	[19.9% , 21.2%]
Profits	[50,574 , 57,428]	[47,602 , 60,627]	60,023	[56,160 , 63,887]
Change in profits	[-6,489 , 366]	[-7,706 , 1,196]	2,961	[1,944 , 3,978]

**10% discount rate**

Probability of no sale	[21.4% , 29.0%]	[20.8% , 29.6%]	15.6%	[15.0% , 16.1%]
Profits	[52,943 , 57,903]	[49,939 , 61,115]	60,035	[56,154 , 63,916]
Change in profits	[-4,119 , 840]	[-5,043 , 1,536]	2,973	[2,116 , 3,830]

**20% discount rate**

Probability of no sale	[16.0% , 20.8%]	[15.6% , 21.3%]	10.9%	[10.4% , 11.3%]
Profits	[56,627 , 58,785]	[53,422 , 62,057]	60,124	[56,199 , 64,048]
Change in profits	[-435 , 1,722]	[-1,149 , 2,279]	3,062	[2,350 , 3,773]

**30% discount rate**

Probability of no sale	[11.7% , 13.8%]	[11.3% , 14.2%]	7.6%	[7.3% , 7.9%]
Profits	[58,560 , 59,045]	[55,267 , 62,342]	59,657	[55,725 , 63,589]
Change in profits	[1,498 , 1,983]	[949 , 2,435]	2,595	[2,029 , 3,161]

Median value of  $\frac{\text{reserve price}}{\text{appraisal value}}$  for the four policies are 2.05, 1.84, 1.64, and 1.43, respectively.

is realized, and known to his opponents, by the time bidding occurs would be important in a first-price auction, since equilibrium bids depend on beliefs about opponent valuations; but it is irrelevant in an ascending auction, since bidding is in dominant strategies. Thus, under the assumption that the “type” of each bidder is an independent random event (and independent of  $N$ ), our model extends to asymmetric bidders. If instead, say, the propensity of bidders to be mills rather than loggers were positively associated with  $N$ , and the “mill distribution” yielded stochastically higher valuations than the “logger distribution,” then valuations would be stochastically increasing in  $N$  and our weaker results (Theorem 1') would hold.

## 5.2 Other Auction Formats

The benefit of our “limited” approach to identification is that the distributions  $F_{n-1:n}$  and  $F_{n:n}$  can be identified from bid data with few structural assumptions, and are sufficient to calculate the optimal reserve price in an ascending auction. The cost of this approach, however, is that these marginal distributions are *not* sufficient to evaluate the effect of switching to a different auction format such as a first-price auction.<sup>19</sup> However, if bidders are risk-neutral, then with symmetric, affiliated private values, Milgrom and Weber (1982) show that an ascending auction revenue-dominates a first-price auction with the same reserve price; so the question we are able to address (the optimal ascending auction) is the “right” one.

If values are correlated, knowledge of  $F_{n-1:n}$  and  $F_{n:n}$  is also not sufficient to calculate the optimal mechanism, à la Cremer and McLean (1988), which requires detailed information about the joint distribution to specify appropriate “side bets”. However, the distributions  $F_{n-1:n}$  and  $F_{n:n}$  are sufficient to put bounds on how much an ascending auction falls short of the optimal mechanism, that is, how much money is being “left on the table” by using an ascending auction. The best any mechanism could achieve is efficiency plus full extraction, which would give profits of  $\int_{v_0}^{\infty} (v - v_0) dF_{n:n}(v)$ ; using the bounds on  $F_{n:n}$  to bound the difference between this integral and (1) would show the maximum that could potentially be gained by learning the full joint distribution of valuations and designing the optimal mechanism.

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<sup>19</sup>This is because equilibrium bid strategies in a first-price auction depend on the distribution of a bidder’s opponents’ valuations *conditional on* his own. The exception is if we believe the true model to be IPV with unobserved heterogeneity, in which case all “standard” auction formats are revenue-equivalent given a choice of reserve price.

## 6 Conclusion

Our paper presents techniques for identification and estimation of a general model of ascending auctions with correlated private values. Empirically, we find that even after controlling for a rich set of auction covariates, residual correlation persists among bidder valuations, and has significant implications for seller profit and optimal behavior.

A key step of our analysis was to narrow our goal to identifying the minimal features necessary to answer the relevant economic questions (rather than trying to recover all unobserved primitives). By doing so, we are able to nonparametrically identify bounds on seller profit within a very general model. Moreover, since we directly estimate expected profit rather than more abstract primitives from which it is derived, our technique lends itself to a straightforward approach to inference. While the techniques presented here are tightly tailored to ascending auctions, we feel this general approach of tying the goal of identification to the relevant economic problem may have more widespread applicability.

## A Appendix

### A.1 Extending Our Bounds to the Incomplete Model of Haile and Tamer

Consider each bidder's bid to be the highest bid he makes during an auction. Let  $B_{n:n} \geq B_{n-1:n} \geq \dots \geq B_{1:n}$  be the order statistics of bids in an  $n$ -bidder auction, and  $G_{k:n}$  the distribution of  $B_{k:n}$ . The bidding assumptions of Haile and Tamer (2003) are that bidders never bid more than their valuations, and that no bidder allows an opponent to win at a price he would have been willing to outbid. This means that  $V_{k:n} \geq B_{k:n}$  for every  $k$ , and  $V_{n-1:n} \leq B_{n:n} + \Delta$ , where  $\Delta$  is the minimum bid increment at the end of the auction. Together, these imply that

$$F_{n-1:n}(v) \in [G_{n:n}^\Delta(v), G_{n-1:n}(v)] \quad (8)$$

where  $G_{n:n}^\Delta$  is the distribution of  $B_{n:n} + \Delta$ . As long as the top two bids and the minimum bid increment are recorded in the data, the distributions  $G_{n-1:n}$ ,  $G_{n:n}$  and  $G_{n:n}^\Delta$  are identified.

If we continue to assume that counterfactual auctions satisfy Assumption 2, then our bounds become

$$\underline{\pi}_n(r) \equiv \int_0^\infty \max\{r, v\} dG_{n-1:n}(v) - v_0 - \left( \sum_{m=n+1}^{\bar{n}} \frac{n}{(m-1)m} G_{m-1:m}(r) + \frac{n}{\bar{n}} G_{\bar{n}:\bar{n}}(r) \right) (r - v_0)$$

$$\bar{\pi}_n(r) \equiv \int_0^\infty \max\{r, v\} dG_{n:n}^\Delta(v) - v_0 - \left( \sum_{m=n+1}^{\bar{n}} \frac{n}{(m-1)m} G_{m:m}^\Delta(r) + \frac{n}{\bar{n}} (\phi_{\bar{n}}(G_{\bar{n}:\bar{n}}^\Delta(r)))^{\bar{n}} \right) (r - v_0)$$

These come from plugging bounds on both  $F_{n-1:n}$  and  $F_{n:n}$  into  $\pi_n(r) = \int_0^\infty \max\{r, v\} dF_{n-1:n}(v) - v_0 - F_{n:n}(r)(r - v_0)$ . Note that since  $\max\{r, V_{n-1:n}\}$  is increasing in  $V_{n-1:n}$ , its expected value is increasing in the distribution of  $V_{n-1:n}$  (in the first-order stochastic dominance sense), so plugging in a pointwise upper bound for  $F_{n-1:n}(\cdot)$  gives a lower bound on  $\pi_n$  and vice versa; and as before,  $\pi_n(r)$  is decreasing in  $F_{n:n}(r)$ . (Also note that it is the fact that  $\pi_n(r)$  is additively separable into a term depending only on  $F_{n-1:n}$  and another depending only on  $F_{n:n}$  that allows us to plug in bounds on each marginal without considering their *joint* distribution.)

### A.2 Using More Bids to Tighten the Bounds

The bounds above rely only on observation of the distributions  $G_{m-1:m}$ ,  $G_{m:m}$ , and  $G_{m:m}^\Delta$  for various  $m$ . However, if other losing bids are observed, their distributions  $G_{k:n}$  could be used to tighten the lower bound on  $F_{n:n}$ , and therefore the upper bound on  $\pi_n$ . For example, here is how the losing bid distributions  $G_{n-2:n}$  and  $G_{n-3:n}$  would yield new, possibly tighter lower bounds on  $F_{n:n}$ :

**Proposition A1** Under Assumptions 1, 3, and the bidding assumptions of Haile and Tamer (2003),  $F_{n:n}(v) \geq z_1$  and  $F_{n:n}(v) \geq z_2$ , where  $z_1$  and  $z_2$  are the unique solutions to

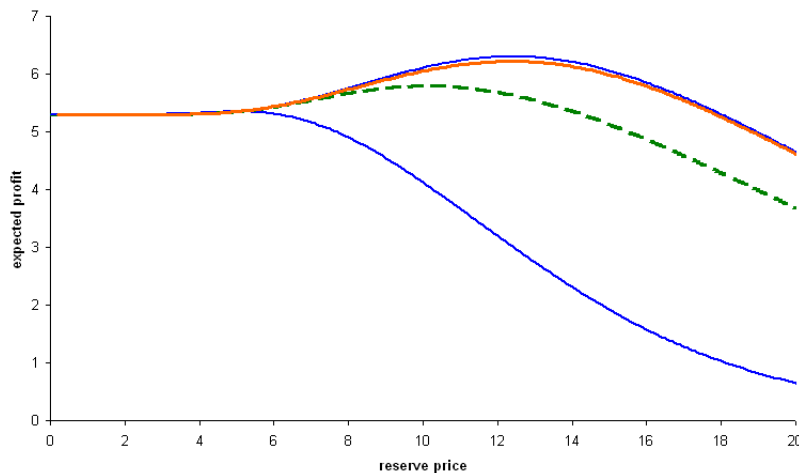
$$\frac{z_1}{(G_{n:n}^\Delta(v) - z_1)^2} = \frac{n-1}{2n} \frac{1}{G_{n-2:n}(v) - G_{n:n}^\Delta(v)}$$

$$\frac{(z_2)^2}{(G_{n:n}^\Delta(v) - z_2)^3} = \frac{(n-1)(n-2)}{6n^2} \frac{1}{G_{n-3:n}(v) - G_{n:n}^\Delta(v) - \frac{n-1}{2n} \frac{1}{z_2} (G_{n:n}^\Delta(v) - z_2)^2}$$

Since both left-hand sides are strictly increasing in the variable  $z_1$  or  $z_2$ , and both right-hand sides either constant or decreasing, these define unique values for  $z_1$  and  $z_2$ . Each additional losing bid would similarly provide a new lower bound on  $F_{n:n}$ .

To see the effect such bounds would have, we return to the example discussed in the text. Figure 4 below shows actual expected profit (dashed line); the upper and lower bounds on expected profit that come from exact knowledge of the distribution  $F_{2,3}$  (solid blue lines); and the upper bound on expected profit calculated using the bound  $z_1$  from Proposition A1 (solid red line), if the lowest bidder in a three-bidder auction bid all the way up to his valuation ( $B_{1:3} = V_{1:3}$  so  $G_{1:3} = F_{1:3}$ ).

Figure 4: Tightening the bound on  $\pi_3(r)$  using all bids



As we can see, the incorporation of the additional losing bid gives only a very slight improvement to the upper bound on  $\pi_3$  calculated only from  $F_{2,3}$ . And this is under the “best-case” assumption that the third-highest bidder bids his true valuation; if the third-highest bidder routinely bid strictly lower than his value, the upper bound would be even higher, and would not improve even on the existing bound. (The improvement from using  $F_{2,4}$  when  $N = 4$  is similarly small.) In addition, while losing bids could potentially tighten the upper bound, they cannot improve the lower bound

on  $\pi_n$  at all.<sup>20</sup> Thus, in our application, we choose to ignore the additional information contained in losing bids, and use only the transaction prices of past auctions.<sup>21</sup>

### A.3 Sufficient Conditions for $V$ Stochastically Increasing in $N$

Consider an environment of independent private values with one-dimensional unobserved heterogeneity.  $\theta \in \mathfrak{R}$  varies across auctions (and is not observed by the analyst), valuations are *i.i.d.*  $\sim F(\cdot | \theta)$ , and  $\theta > \theta'$  implies  $F(\cdot | \theta) \succeq_{FOSD} F(\cdot | \theta')$ .

Within this environment, consider first the entry model of Levin and Smith (1996).  $\bar{n}$  potential bidders observe  $\theta$  but not their own valuations, then decide simultaneously whether to pay a cost  $c$  to participate in the auction. (Levin and Smith interpret  $c$  as the cost of investigating the object and learning one's valuation.) For each realization of  $\theta$ , bidders play a symmetric, mixed-strategy equilibrium in the entry game.

**Proposition A2** *If the equilibrium probability of entry is increasing in  $\theta$ , then the entry game of Levin and Smith (1994) generates valuations stochastically increasing in  $N$ . A sufficient condition is for expected bidder surplus in an  $n$ -bidder auction to be weakly increasing in  $\theta$  for each  $n$ .*

Second, consider the same environment, but under the model of Samuelson (1985).  $\bar{n}$  potential bidders each observe both  $\theta$  and their own valuation before deciding whether to enter at cost  $c$ . For each realization of  $\theta$ , bidders play a symmetric, cutoff-strategy equilibrium in the entry game.

**Proposition A3** *If valuations and  $\theta$  are related via the Monotone Likelihood Ratio Property – that is, if  $\frac{f(v|\theta)}{f(v|\theta')}$  is increasing in  $v$  for  $\theta > \theta'$  – then the entry game of Samuelson (1985) generates valuations stochastically increasing in  $N$ .*

Finally, consider a different environment, the “affiliated model of entry” studied by Marmer, Shneyerov, and Xu (2010). Potential bidders each get a signal  $S_i$  related to their (unobserved)

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<sup>20</sup>The lower bound on  $\pi_n$  depends on the upper bound on  $F_{n:n}$ . Regardless of the distribution of losing bids in the data, the bidding assumptions of Haile and Tamer (2003) are consistent with *every* bidder's valuation in each auction being equal to the transaction price in that auction, in which case  $F_{n:n} = G_{n:n}$ .

<sup>21</sup>There are two further argument against incorporating losing bids in the analysis. First, if we did, then the lower bound on  $F_{n:n}$  would be the pointwise maximum of multiple distinct bounds, leading to a bias in estimation. Haile and Tamer (2003) confront and deal with this issue, but it complicates the analysis. And second, due to the difference terms in the denominators of the expressions in Proposition A1, we worry these bounds could be unstable on moderately-sized samples.

valuation  $V_i$  before deciding whether to enter. Valuations and signals  $(V_1, V_2, \dots, V_{\bar{n}}, S_1, S_2, \dots, S_{\bar{n}})$  are affiliated, and their distribution is symmetric in the obvious way.<sup>22</sup>

**Proposition A4** *If a symmetric equilibrium exists in cutoff strategies, then the entry game of Marmer, Shneyerov, and Xu (2010) generates valuations stochastically increasing in  $N$ .*<sup>23</sup>

## A.4 Estimation and Inference

We describe here the asymptotic properties of the expected profit estimators used in the empirical results section. The asymptotic theory for the policy counterfactuals (the inference on “portfolio-level” expected profits) is contained in the Supplementary Appendix.

Recall that we let  $X = (X^c, X^d)$ , where  $X^c \in \mathbb{R}^z$  and  $X^d$  are the continuous and discrete covariates, respectively (and similarly  $x = (x^c, x^d)$ ). We will maintain the assumption that  $(B_i, N_i, X_i)_{i=1}^L$  is an *i.i.d.* sample of a random process  $(B, N, X)$ .<sup>24</sup> By assumption, conditional on  $X_i = x$  and  $N_i = n$  for a given  $(x, n) \in \text{Supp}(X, N)$ , we have  $B_i \sim F_{n-1:n}(\cdot|x)$ . Let  $p_N$  denote the probability function of  $N$ ,  $f_X$  the unconditional density of  $X$ , and  $f_{X|N}$  the conditional density of  $X$  given  $N$ ; and let  $q_{X,N}(x, n) = f_{X|N}(x|n) \cdot p_N(n)$ .

For a given kernel function  $K : \mathbb{R}^z \rightarrow \mathbb{R}$  and bandwidth sequence  $h_L \rightarrow 0$ , in section 3.5, we defined estimators  $\widehat{\pi}_n(r|x)$  and  $\widehat{\underline{\pi}}_n(r|x)$  for the bounds  $\bar{\pi}_n(r|x)$  and  $\underline{\pi}_n(r|x)$  given in Theorem 1, based on nonparametric estimates of  $F_{n-1:n}(r|x)$  and  $T_{n-1:n}(r|x) = E(\max\{r, B_i\} | N_i = n, X_i = x)$ . These estimators are described in (6).

### A.4.1 Asymptotic Normality

**Assumption 4** *1. The observed data  $(B_i, N_i, X_i)_{i=1}^L \equiv (U_i)_{i=1}^L$  is an *i.i.d.* sample from a distribution belonging to a family  $\mathcal{F}$  satisfying the following property. For some pair of sets  $\mathcal{X} \subseteq \text{Supp}(X)$ ,  $\mathcal{N} \subseteq \text{Supp}(N)$  with  $\mathcal{X} \times \mathcal{N} \subseteq \text{Supp}(X, N)$  and some range of values  $\mathcal{R} \subseteq \mathbb{R}^+$ , the following is true for every distribution  $F \in \mathcal{F}$  and almost every  $(x, n, r) \in \mathcal{X} \times \mathcal{N} \times \mathcal{R}$  (measured w.r.t.  $F$ ):*

(i)  $F_{n-1:n}(\cdot|x)$  is continuous

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<sup>22</sup> $f(V_1, V_2, \dots, V_{\bar{n}}, S_1, S_2, \dots, S_{\bar{n}}) = f(V_{\sigma(1)}, V_{\sigma(2)}, \dots, V_{\sigma(\bar{n})}, S_{\sigma(1)}, S_{\sigma(2)}, \dots, S_{\sigma(\bar{n})})$ , where  $f$  is the joint distribution of values and signals and  $\sigma : \{1, 2, \dots, \bar{n}\} \rightarrow \{1, 2, \dots, \bar{n}\}$  is any permutation.

<sup>23</sup>Marmer et. al. assume  $V_i$  and  $S_i$  are affiliated, but  $(V_i, S_i)$  is independent of  $(V_j, S_j)$  for  $i \neq j$ , which guarantees existence of a unique symmetric cutoff-strategy equilibrium. Here, we allow general affiliation among  $(V_1, V_2, \dots, V_{\bar{n}}, S_1, S_2, \dots, S_{\bar{n}})$ ; our result holds for any symmetric cutoff equilibrium, provided one exists (which is not guaranteed).

<sup>24</sup>Our methods can potentially be adapted to allow for certain forms of dependence across observations in our sample.

- (ii)  $\exists \underline{q}, \bar{q}$  such that  $0 < \underline{q} < q_{X,N}(x, n) < \bar{q} < \infty$
- (iii)  $\exists \underline{F}, \bar{F}$  such that  $0 < \underline{F} < F_{n-1:n}(r|x) < \bar{F} < 1$
- (iv)  $f_X(x), f_{X|N}(x|n), F_{n-1:n}(r|x)$ , and  $T_{n-1:n}(r|x)$  are twice differentiable with respect to  $x^c$  (the continuous elements in  $x$ ) with bounded derivatives

2. The kernel  $K$  satisfies  $\int K(\xi)d\xi = 1$ , it is nonnegative, has compact support, and is Lipschitz-continuous, bounded, and symmetric around zero.
3. The bandwidth sequence  $h_L \rightarrow 0$  is such that  $\exists \bar{\delta} > 0$  for which  $L^{1-\bar{\delta}}h_L^z \rightarrow \infty$ , and  $L^{1+\bar{\delta}}h_L^{z+4} \rightarrow 0$

The smoothness and regularity restrictions described in Assumption 4 are fairly standard in nonparametric models. The same is true for the restrictions imposed on the kernel and bandwidth. Assuming that  $F_{n-1:n}(r|x)$  is bounded away from zero and one ensures that the mapping  $\phi_n(F_{n-1:n}(r|x))$  is smooth and differentiable for every  $(x, n, r)$  in the range of values of interest<sup>25</sup>. As before, group  $U_i \equiv (B_i, X_i, N_i)$  and let<sup>26</sup>

$$\begin{aligned}
\underline{\psi}(r, U_i|x, n) &= \frac{(\max\{r, B_i\} - T_{n-1:n}(r|x))}{q_{X,N}(x, n)} \mathbb{1}\{N_i = n\} \\
&\quad - (r - v_0) \cdot \left( \sum_{m=n+1}^{\bar{n}} \frac{n}{(m-1)m} \frac{(\mathbb{1}\{B_i \leq r\} - F_{m-1:m}(r|x))}{q_{X,N}(x, m)} \mathbb{1}\{N_i = m\} \right. \\
&\quad \left. + \frac{n}{\bar{n}} \frac{(\mathbb{1}\{B_i \leq r\} - F_{\bar{n}-1:\bar{n}}(r|x))}{q_{X,N}(x, \bar{n})} \mathbb{1}\{N_i = \bar{n}\} \right) \\
\bar{\psi}(r, U_i|x, n) &= \frac{(\max\{r, B_i\} - T_{n-1:n}(r|x))}{q_{X,N}(x, n)} \mathbb{1}\{N_i = n\} \\
&\quad - (r - v_0) \cdot \left( \sum_{m=n+1}^{\bar{n}} \frac{n}{(m-1)m} \frac{(\mathbb{1}\{B_i \leq r\} - F_{m-1:m}(r|x))}{q_{X,N}(x, m)} \mathbb{1}\{N_i = m\} \right. \\
&\quad \left. + n \cdot (\phi_{\bar{n}}(F_{\bar{n}-1:\bar{n}}(r|x)))^{\bar{n}-1} \cdot \phi'_{\bar{n}}(F_{\bar{n}-1:\bar{n}}(r|x)) \cdot \frac{(\mathbb{1}\{B_i \leq r\} - F_{\bar{n}-1:\bar{n}}(r|x))}{q_{X,N}(x, \bar{n})} \mathbb{1}\{N_i = \bar{n}\} \right)
\end{aligned} \tag{9}$$

Let  $\mu_K^2 \equiv \int K^2(\xi)d\xi$ , and let

$$\begin{aligned}
\sigma_n^2(r|x) &= E[\underline{\psi}(r, U_i|x, n)^2 | X_i = x] f_X(x) \mu_K^2 \\
\bar{\sigma}_n^2(r|x) &= E[\bar{\psi}(r, U_i|x, n)^2 | X_i = x] f_X(x) \mu_K^2 \\
\rho_n(r|x) &= \frac{1}{\sqrt{\sigma_n^2(r|x) \cdot \bar{\sigma}_n^2(r|x)}} E[\underline{\psi}(r, U_i|x, n) \cdot \bar{\psi}(r, U_i|x, n) | X_i = x] f_X(x) \mu_K^2
\end{aligned}$$

<sup>25</sup>In an econometric context, Menzel and Morganti (2011) describe the irregularity issues that arise in ascending price auctions when the reserve price is allowed to be arbitrarily close to the boundary of the support of valuations.

<sup>26</sup>If  $0 < v < 1$ , then  $\phi'_n(v) = \{n(n-1)\phi_n(v)^{n-2}(1-\phi_n(v))\}^{-1}$ .

Suppose  $\{n, n+1, \dots, \bar{n}\} \in \mathcal{N}$ . If Assumption 4 holds, we can show (see, e.g., Andrews (1994) and Andrews (1995)) that uniformly over  $(x, r) \in \mathcal{X} \times \mathcal{R}$  and over the space of distributions  $\mathcal{F}$ ,

$$\begin{pmatrix} \sqrt{Lh_L^z}(\widehat{\underline{\pi}}_n(r|x) - \underline{\pi}_n(r|x)) \\ \sqrt{Lh_L^z}(\widehat{\overline{\pi}}_n(r|x) - \overline{\pi}_n(r|x)) \end{pmatrix} \xrightarrow{d} \mathcal{N} \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \underline{\sigma}_n^2(r|x) & \rho_n(r|x)\underline{\sigma}_n(r|x)\overline{\sigma}_n(r|x) \\ \rho_n(r|x)\underline{\sigma}_n(r|x)\overline{\sigma}_n(r|x) & \overline{\sigma}_n^2(r|x) \end{pmatrix} \right) \quad (10)$$

Consistent estimators for  $\underline{\sigma}_n^2(r|x)$ ,  $\overline{\sigma}_n^2(r|x)$ , and  $\rho_n(r|x)$  can be found by first defining  $\widehat{\underline{\psi}}(r, U_i|x, n)$  and  $\widehat{\overline{\psi}}(r, U_i|x, n)$  by replacing  $F_{n-1:n}$  and  $T_{n-1:n}$  with their estimates (defined in section 3.5) and  $q_{X,N}(x, n)$  with  $\widehat{q}_{X,N}(x, n) = \frac{1}{Lh_L^z} \sum_{i=1}^L K\left(\frac{X_i^c - x^c}{h_L}\right) \cdot \mathbb{1}\{X_i^d = x^d\} \mathbb{1}\{N_i = n\}$  in the definitions of  $\underline{\psi}(r, U_i|x, n)$  and  $\overline{\psi}(r, U_i|x, n)$ ; and then defining

$$\begin{aligned} \widehat{\underline{\sigma}}_n^2(r|x) &\equiv \frac{1}{Lh_L^z} \sum_{i=1}^L \left\{ \widehat{\underline{\psi}}(r, U_i|x, n) \cdot \mathbb{1}\{X_i^d = x^d\} \cdot K\left(\frac{X_i^c - x^c}{h_L}\right) \right\}^2 \\ \widehat{\overline{\sigma}}_n^2(r|x) &\equiv \frac{1}{Lh_L^z} \sum_{i=1}^L \left\{ \widehat{\overline{\psi}}(r, U_i|x, n) \cdot \mathbb{1}\{X_i^d = x^d\} \cdot K\left(\frac{X_i^c - x^c}{h_L}\right) \right\}^2 \\ \widehat{\rho}_n(r|x) &\equiv \frac{1}{\sqrt{\widehat{\underline{\sigma}}_n^2(r|x) \cdot \widehat{\overline{\sigma}}_n^2(r|x)}} \cdot \frac{1}{Lh_L^z} \sum_{i=1}^L \widehat{\underline{\psi}}(r, U_i|x, n) \cdot \widehat{\overline{\psi}}(r, U_i|x, n) \cdot \mathbb{1}\{X_i^d = x^d\} \cdot \left\{ K\left(\frac{X_i^c - x^c}{h_L}\right) \right\}^2 \end{aligned}$$

#### A.4.2 Constructing a Confidence Interval for $\pi_n$

Under the conditions of Theorem 1, we have  $\underline{\pi}_n(r|x) \leq \pi_n(r|x) \leq \overline{\pi}_n(r|x)$ . Combining this with the asymptotic properties of our estimators for  $\underline{\pi}_n(r|x)$  and  $\overline{\pi}_n(r|x)$  we can construct a confidence interval for  $\pi_n(r|x)$  which achieves a prespecified asymptotic coverage probability while allowing for  $\overline{\pi}_n(r|x) - \underline{\pi}_n(r|x) \equiv \Lambda_n(r|x)$  to be arbitrarily close to zero over  $\mathcal{X} \times \mathcal{N} \times \mathcal{R}$ , the range of values described in Assumption 4. To accomplish this, we simply apply the insights and results from Imbens and Manski (2004) and Stoye (2009) to our setting.

Since  $K \geq 0$ , it is not hard to verify<sup>27</sup> that  $\widehat{\underline{\pi}}_n(r|x) \leq \widehat{\overline{\pi}}_n(r|x)$  (and therefore  $\widehat{\Lambda}_n(r|x) \equiv \widehat{\overline{\pi}}_n(r|x) - \widehat{\underline{\pi}}_n(r|x) \geq 0$ ) for any  $r, x$  and  $n$ . Combining this with the asymptotic normality result in (10), we can show that a feature analogous to Assumption 3 in Stoye (2009) holds in our setting. Namely, we can show that there exists a sequence  $a_L \rightarrow 0$  such that  $a_L \sqrt{Lh_L^z} \rightarrow \infty$  and  $\sqrt{Lh_L^z} |\widehat{\Lambda}_n(r|x) - \Lambda_{n_L}(r|x)| \xrightarrow{P} 0$  for all sequences  $\{F_L\}$  of distributions in  $\mathcal{F}$  for which the corresponding sequence  $\{\Lambda_{n_L}(r|x)\}$  satisfies  $\Lambda_{n_L}(r|x) \leq a_L$ . This can be shown following analogous steps to Lemma 3 in Stoye (2009). Thus, in our setting,  $\widehat{\Lambda}_n(r|x)$  satisfies the type of “superefficient” requirements described in Stoye (2009). This feature facilitates the construction of confidence intervals that achieved the desired coverage probability asymptotically, while allowing for the length of the identified interval  $[\underline{\pi}_n(r|x), \overline{\pi}_n(r|x)]$  to become arbitrarily close to zero over  $\mathcal{X} \times \mathcal{N} \times \mathcal{R}$ . In

<sup>27</sup>Note that, for any  $0 \leq u \leq v \leq 1$  and  $n \geq 2$ ,  $(\phi_n(u))^n \leq (\phi_n(v))^n \leq v$ .

light of Assumption 4.1(iii), and since  $\mathcal{N}$  is compact,  $\bar{\pi}_n(r|x) - \underline{\pi}_n(r|x)$  will become arbitrarily close to zero over our testing range if and only if  $\mathcal{R}$  includes values arbitrarily close to  $v_0$ .

Under Assumption 4 and if the conditions in Theorem 1 are satisfied, the following confidence interval has uniformly valid asymptotic coverage of  $(1 - \alpha)\%$  for the true value of  $\pi_n(r|x)$ :

$$CI_\alpha^1 \equiv \left[ \hat{\underline{\pi}}_n(r|x) - c_\alpha^1 \cdot \frac{\hat{\sigma}_n(r|x)}{\sqrt{Lh_L^z}}, \hat{\bar{\pi}}_n(r|x) + c_\alpha^1 \cdot \frac{\hat{\sigma}_n(r|x)}{\sqrt{Lh_L^z}} \right] \quad (11)$$

where  $c_\alpha^1$  solves

$$\Phi \left( c_\alpha^1 + \frac{\sqrt{Nh_L^z} \hat{\Delta}_n(r|x)}{\max\{\hat{\sigma}_n(r|x), \hat{\sigma}_n(r|x)\}} \right) - \Phi(-c_\alpha^1) = 1 - \alpha$$

where  $\Phi$  is the Standard Normal cdf. In light of our previous asymptotic results, establishing the validity of (11) would follow the same steps as those in Proposition 1 in Stoye (2009). A particular case of interest (which was relevant in our counterfactual experiments) is when  $r = v_0$ , which yields  $\underline{\pi}_n(r|x) = \bar{\pi}_n(r|x)$  for any  $x$  and  $n$  and point-identification of  $\pi_n(r|x)$ . By construction of our estimators, if  $r = v_0$  we automatically have  $\hat{\underline{\pi}}_n(r|x) = \hat{\bar{\pi}}_n(r|x) \equiv \hat{\pi}_n(r|x)$  and  $\hat{\Delta}_n(r|x) = 0$ . From here, it is easy to verify that the confidence interval described in (11) automatically becomes the one we should use under point-identification. More generally, however, the construction in (11) yields, uniformly, the correct asymptotic coverage probability whether or not  $\mathcal{R}$  includes reservation prices  $r$  arbitrarily close to  $v_0$ .

#### A.4.3 Results under IPV

If IPV is assumed, expected profits are point-identified and the estimator  $\hat{\pi}_n^{IPV}(r|x)$  defined in (7) is valid. Let

$$\begin{aligned} \psi^{IPV}(r, U_i|x, n) &= \frac{(\max\{r, B_i\} - T_{n-1:n}(r|x))}{q_{X,N}(x, n)} \mathbb{1}\{N_i = n\} \\ &- (r - v_0) \cdot n \cdot (\phi_n(F_{n-1:n}(r|x)))^{n-1} \cdot \phi'_n(F_{n-1:n}(r|x)) \cdot \frac{(\mathbb{1}\{B_i \leq r\} - F_{n-1:n}(r|x))}{q_{X,N}(x, n)} \mathbb{1}\{N_i = n\} \end{aligned} \quad (12)$$

As we did above, let  $\mu_K^2 \equiv \int K^2(\xi) d\xi$  and define

$$\Sigma_n^2(r|x) = E [\psi^{IPV}(r, U_i|x, n)^2 | X_i = x] f_X(x) \mu_K^2$$

Under IPV and Assumption 4,

$$\sqrt{Lh_L^z} (\hat{\pi}_n^{IPV}(r|x) - \pi_n(r|x)) \xrightarrow{d} \mathcal{N}(0, \Sigma_n^2(r|x)) \quad (13)$$

As before, a consistent estimator of  $\Sigma_n^2(r|x)$  can be obtained as

$$\hat{\Sigma}_n^2(r|x) \equiv \frac{1}{Lh_L^z} \sum_{i=1}^L \left\{ \hat{\psi}^{IPV}(r, U_i|x, n) \cdot \mathbb{1}\{X_i^d = x^d\} \cdot K\left(\frac{X_i^c - x^c}{h_L}\right) \right\}^2$$

Since profits are point-identified in this scenario, a  $(1 - \alpha)\%$  confidence interval for  $\pi_n(r|x)$  under IPV can be constructed simply as

$$CI_{\alpha}^{IPV} \equiv \left[ \hat{\pi}_n^{IPV}(r|x) - c_{\alpha} \cdot \frac{\widehat{\Sigma}_n^2(r|x)}{\sqrt{Lh_L^z}}, \hat{\pi}_n^{IPV}(r|x) + c_{\alpha} \cdot \frac{\widehat{\Sigma}_n^2(r|x)}{\sqrt{Lh_L^z}} \right] \quad (14)$$

where  $c_{\alpha}$  satisfies  $\Phi(c_{\alpha}) - \Phi(-c_{\alpha}) = 1 - \alpha$ , with  $\Phi$  denoting the Standard Normal cdf.

#### A.4.4 Kernels and Bandwidths Used

The expected profit curves presented in Figures 3, 5, 6 and 7 were calculated using a multiplicative kernel  $K(X) = \prod_{\ell=1}^4 k(x_{\ell})$ , where  $k(\cdot)$  is a quartic kernel of the form

$$k(\psi) = b \cdot (s^2 - \psi^2)^2 \cdot \mathbb{1}\{|\psi| \leq s\}.$$

The support of  $k$  is the compact set  $[-s, s]$ , and the constant  $b$  was chosen so that  $\int_{-s}^s k(\psi) d\psi = 1$ . Given the multiplicative nature of  $K$  and the fact that  $X$  is four-dimensional, we found that the smoothness of the resulting graphs depended on the width of the support used, with larger widths resulting in smoother graphs. The results shown in the paper correspond to  $s = 15$ , but their qualitative features were not found to differ significantly for larger (or moderately smaller) values of  $s$ . The kernel used satisfies Assumption 4. Bandwidths used are detailed below.

The functionals we estimate for the portfolio-level counterfactuals are highly demanding of the data, since they require estimation of nonparametric functionals for each observation in our sample. Furthermore, these functionals are all functions of  $N$ , which means that for the estimation of the relevant functionals evaluated at the  $i^{th}$  observation we only utilize the subsample of observations with auction size  $N_i$ . This, along with the use of four continuously distributed covariates in  $X$ , implies that while our asymptotic results are invariant to the choice of kernel and bandwidth (as long as the conditions described in the Supplementary Appendix are satisfied), our choice of these tuning parameters matters for the estimates obtained in the sample at hand.

For the counterfactuals, a bias-reducing kernel is needed to achieve the desired asymptotic properties (described in the Supplementary Appendix). To this end, we employed a bias-reducing version of the kernel used for the graphs. Namely, we used a multiplicative kernel of the form  $K(X) = \prod_{\ell=1}^4 k(x_{\ell})$ , where  $k$  is a bias-reducing kernel of the form

$$k(\psi) = \sum_{j=1}^6 b_j \cdot (s^2 - \psi^2)^{2j} \cdot \mathbb{1}\{|\psi| \leq s\}$$

By construction,  $k(\psi)$  is symmetric around zero and therefore  $\int_{-s}^s \psi^{\ell} k(\psi) d\psi = 0$  for all odd  $\ell$ . Given  $s$ , the coefficients  $(b_j)_{j=1}^5$  are chosen such that

$$\int_{-s}^s k(\psi) d\psi = 1, \quad \text{and} \quad \int_{-s}^s \psi^{\ell} k(\psi) d\psi = 0 \quad \text{for } \ell = 2, 4, 6, 8, 10.$$

This gives rise to a bias-reducing kernel of order 12 which satisfies the conditions described in the Supplementary Appendix. To extract as much information as possible from our data (see above), we used a kernel with compact but relatively large support: specifically, we used  $s = 30$  for the results shown in the paper, which were robust to moderately large changes in  $s$ .

In all cases (both graphs and counterfactual analysis), the bandwidth  $h_L$  was chosen so that  $h_L = 0.3 \cdot \hat{\sigma}(X)$  for the sample size  $L = 2,036$ . Our findings were robust to moderately large changes in these tuning parameters. As a trimming set  $\mathcal{S}$ , we used

$$\mathcal{S} = \left\{ x : \hat{q}_{X,N}(x, n) \geq 2.8 \times 10^{-7} \text{ for all } n \in \{2, 3, \dots, 11\} \right\} \times \{2, 3, \dots, 11\}$$

for all counterfactuals, where  $\hat{q}_{X,N}(x, n)$  is the estimated density of  $(X, N)$  at  $(x, n)$ ,

$$\hat{q}_{X,N}(x, n) = \frac{1}{Lh_L^z} \sum_{i=1}^L K\left(\frac{X_i^c - x^c}{h_L}\right) \cdot \mathbf{1}\{X_i^d = x^d\} \cdot \mathbf{1}\{N_i = n\}$$

The set  $\mathcal{S}$  contains 95.6% of the auctions in the data.

## A.5 Expected Profit Estimates at Other $X$ and $N$

Figure 5: Profit estimates and confidence intervals for  $n = 3$  and  $5$  and  $x = X_{(0.10)}$  and  $X_{(0.25)}$  (the element-wise 10<sup>th</sup> and 25<sup>th</sup> percentile of the sample)

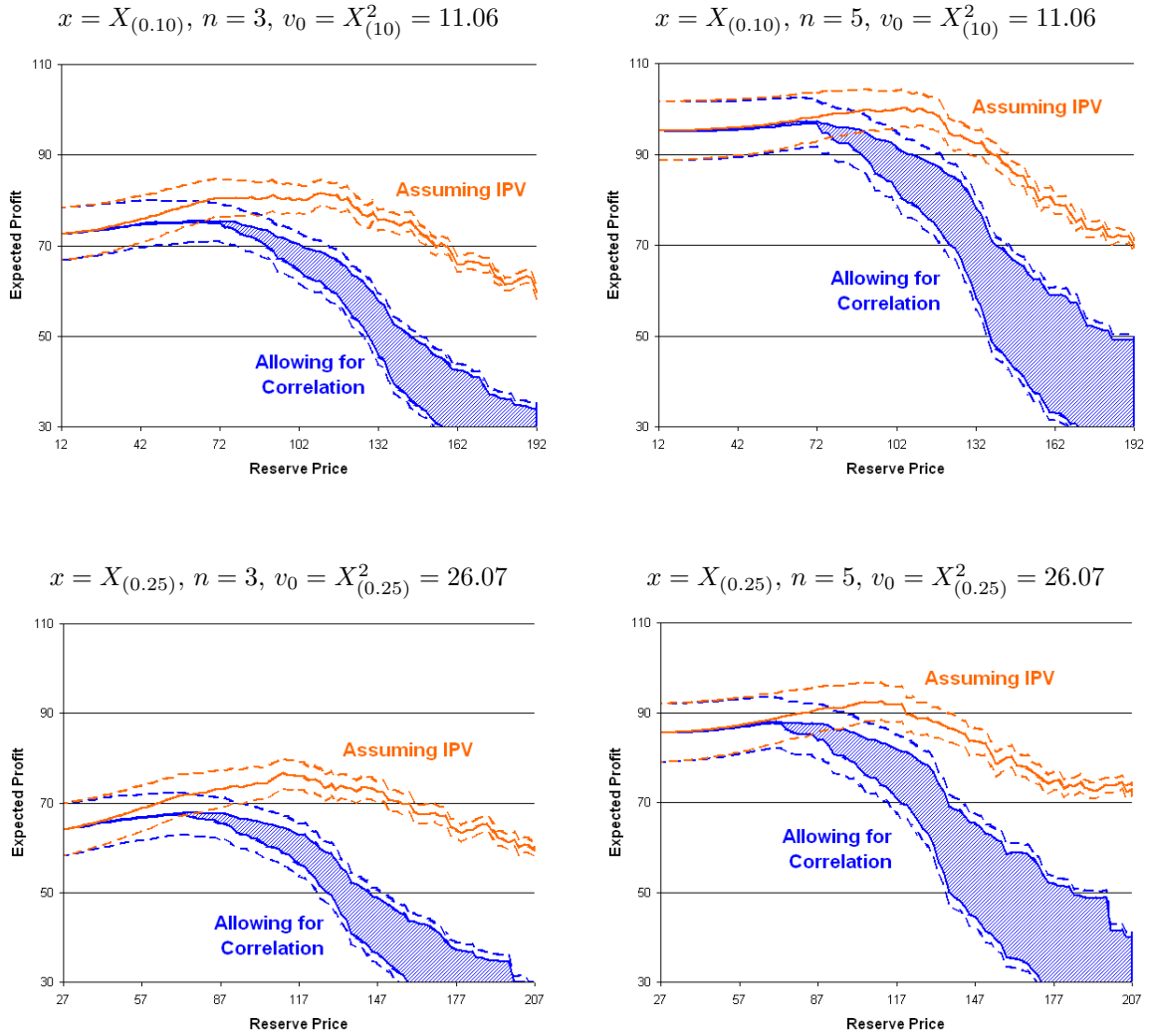


Figure 6: Profit estimates and confidence intervals for  $n = 3$  and  $5$  and  $x = X_{(0.75)}$  and  $X_{(0.90)}$  (the element-wise 75<sup>th</sup> and 90<sup>th</sup> percentile of the sample)

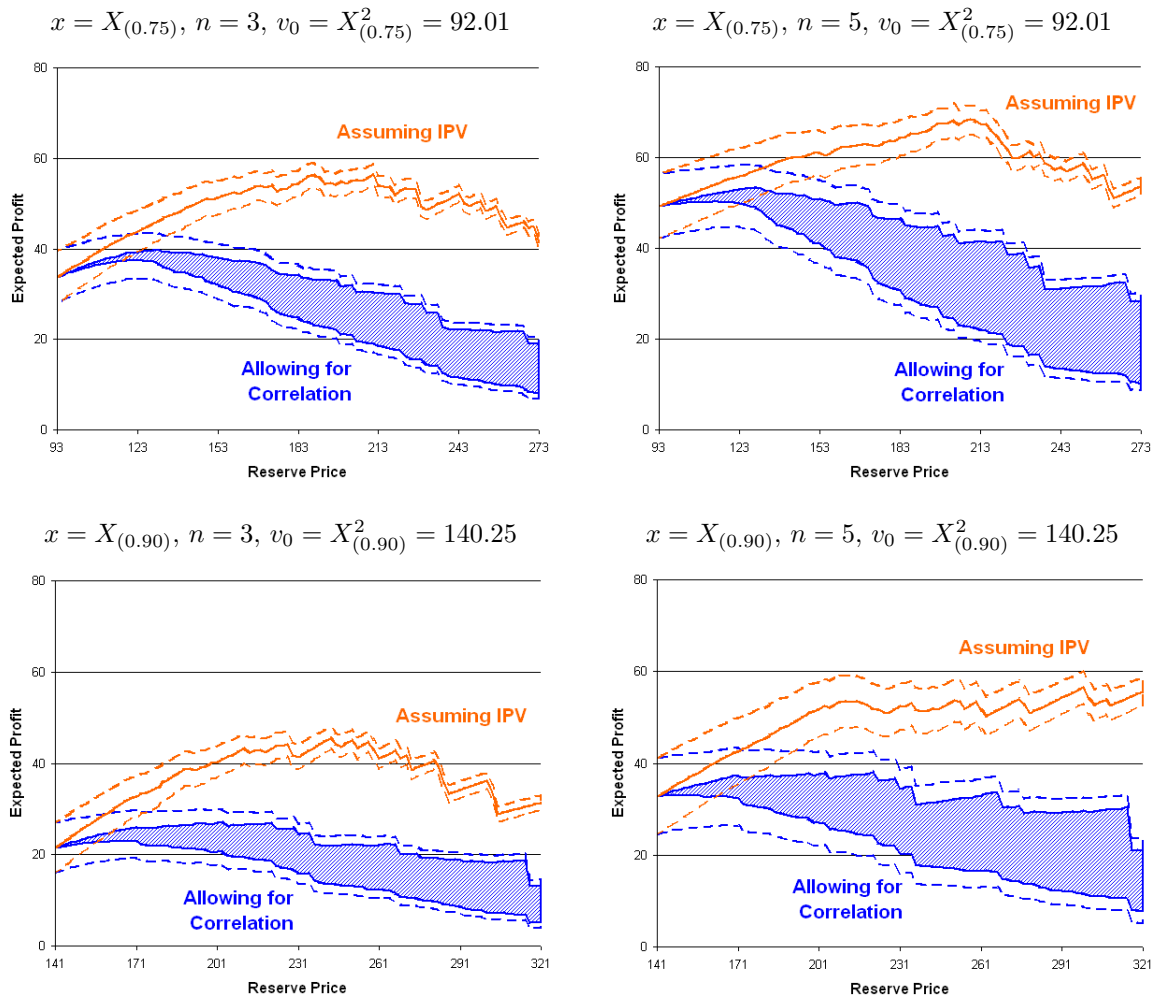
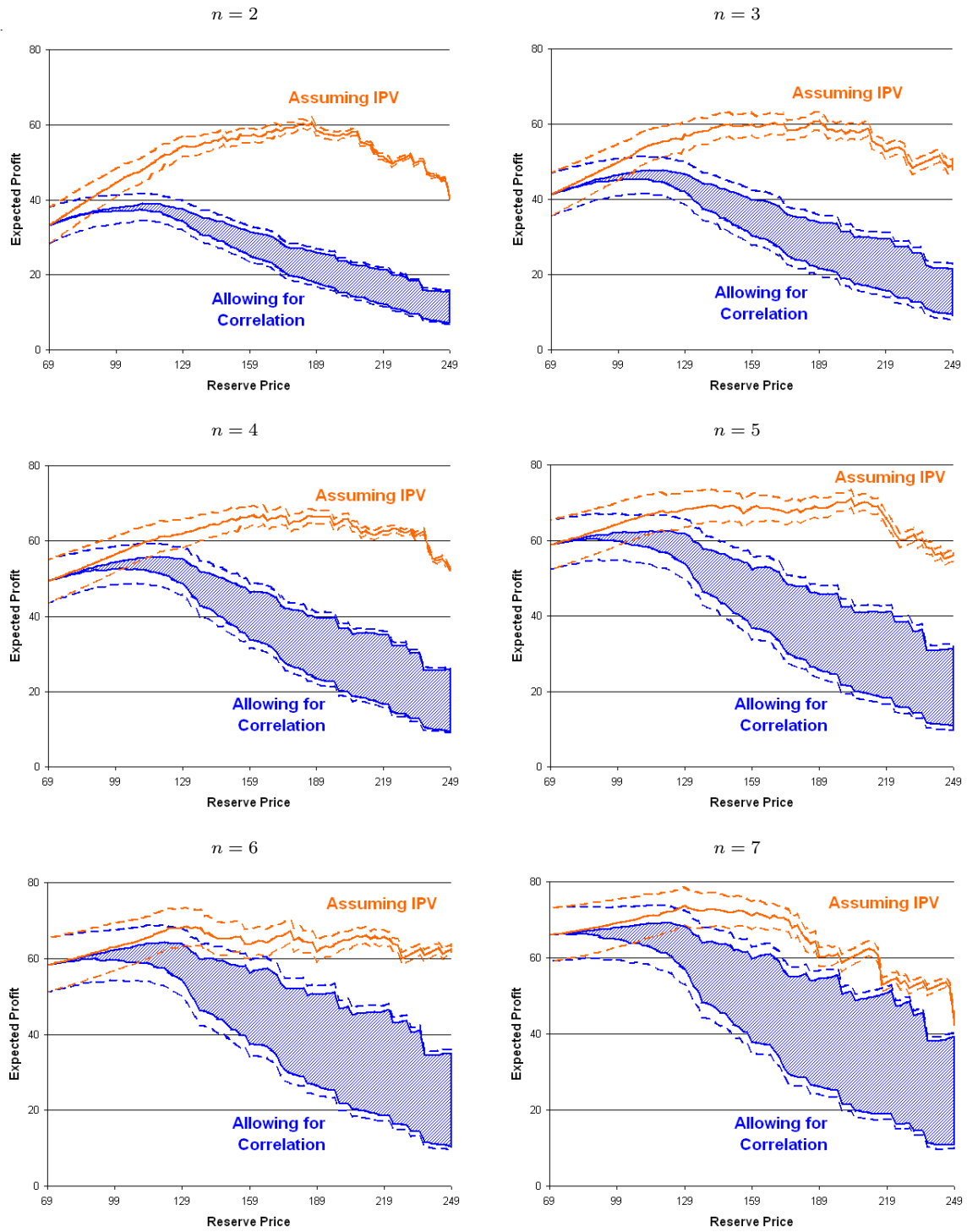


Figure 7: Profit estimates and confidence intervals for  $n \in \{2, \dots, 7\}$  and  $x = \bar{X}$  (the element-wise sample average)



## B Proofs

### B.1 Proof of Lemma 1

Let  $V_{-i} = (V_1, \dots, V_{i-1}, V_{i+1}, \dots, V_n)$ . If values are affiliated, then  $\Pr(V_i < v | V_{-i})$  is decreasing in  $V_{-i}$ , since it is the expected value of  $\mathbf{1}\{V_i < v\}$ , a decreasing function of  $V_i$ ; Assumption 3 follows.

As noted in the text, conditional independence and independence with unobserved heterogeneity are, for our purposes only, equivalent models. Suppose values are *i.i.d.*  $\sim F(\cdot | \theta)$ ; let  $P$  be the prior distribution of  $\theta$  and  $p$  its density, and let all integrals below be over the support of  $p$ . Fix  $v$ ; conditional on exactly  $m$  out of  $m + k$  bidders having valuations weakly less than  $v$ , Bayes' Law puts posterior density

$$p(\theta | m, k) = \frac{p(\theta) \binom{m+k}{m} (F(v|\theta))^m (1 - F(v|\theta))^k}{\int p(\theta') \binom{m+k}{m} (F(v|\theta'))^m (1 - F(v|\theta'))^k d\theta'}$$

on the value of  $\theta$ ; so conditional on  $m$  bidders having valuations below  $v$  and  $k$  having values above, the probability that the next bidder has value below  $v$  is

$$\Pr(V_i < v | m, k) = \int p(\theta | m, k) F(v|\theta) d\theta = \frac{\int p(\theta) (F(v|\theta))^{m+1} (1 - F(v|\theta))^k d\theta}{\int p(\theta') (F(v|\theta'))^m (1 - F(v|\theta'))^k d\theta'}$$

Next, let  $q(\theta) = p(\theta) (F(v|\theta))^{m-1} (1 - F(v|\theta))^k d\theta$ ,  $\bar{q} = \int q(\theta) d\theta$ , and  $\tilde{q}(\theta) = \frac{q(\theta)}{\bar{q}}$ . Since  $q(\theta)$  is everywhere nonnegative, so is  $\tilde{q}(\theta)$ ; since  $\tilde{q}(\cdot)$  (by construction) integrates to 1, it is a density function. Letting  $\tilde{\theta}$  be a random variable with density  $\tilde{q}$ ,

$$\begin{aligned} 0 &\leq \text{Var}(F(v|\tilde{\theta})) \\ &= \int (F(v|\theta))^2 \tilde{q}(\theta) d\theta - \left( \int F(v|\theta) \tilde{q}(\theta) d\theta \right)^2 \\ &= \frac{1}{\bar{q}^2} \left( \int (F(v|\theta))^2 q(\theta) d\theta \int q(\theta') d\theta' - \left( \int F(v|\theta) q(\theta) d\theta \right)^2 \right) \\ &= \frac{\int q(\theta') d\theta' \int F(v|\theta) q(\theta) d\theta}{\bar{q}^2} \left( \frac{\int (F(v|\theta))^2 q(\theta) d\theta}{\int F(v|\theta) q(\theta) d\theta} - \frac{\int F(v|\theta) q(\theta) d\theta}{\int q(\theta) d\theta} \right) \\ &= \frac{\int F(v|\theta) q(\theta) d\theta}{\bar{q}} \left( \frac{\int p(\theta) (F(v|\theta))^{m+1} (1 - F(v|\theta))^k d\theta}{\int p(\theta') (F(v|\theta'))^m (1 - F(v|\theta'))^k d\theta'} - \frac{\int p(\theta) (F(v|\theta))^m (1 - F(v|\theta))^k d\theta}{\int p(\theta') (F(v|\theta'))^{m-1} (1 - F(v|\theta'))^k d\theta'} \right) \end{aligned}$$

This means that  $\frac{\int p(\theta) (F(v|\theta))^{m+1} (1 - F(v|\theta))^k d\theta}{\int p(\theta') (F(v|\theta'))^m (1 - F(v|\theta'))^k d\theta'} \geq \frac{\int p(\theta) (F(v|\theta))^m (1 - F(v|\theta))^k d\theta}{\int p(\theta') (F(v|\theta'))^{m-1} (1 - F(v|\theta'))^k d\theta'}$ , meaning that  $\Pr(V_i < v | m, k)$  (the probability  $V_i < v$  given  $m$  bidders have values below  $v$  and  $k$  have values above  $v$ ) is increasing in  $m$ . By exactly analogous steps, we can show it decreasing in  $k$ , and so  $\Pr(V_i < v | m, n - 1 - m) \geq \Pr(V_i < v | m - 1, n - m)$ , which is exactly Assumption 3.

### B.2 Proof of Lemma 2

Under Assumption 2, profits are  $V_{n-1:n} - v_0$  if  $V_{n:n} > V_{n-1:n} > r$  and  $r - v_0$  if  $V_{n:n} > r > V_{n-1:n}$ , so

$$\begin{aligned} \pi_n(r) &= \int_r^\infty (v - v_0) dF_{n-1:n}(v) + (F_{n-1:n}(r) - F_{n:n}(r)) (r - v_0) \\ &= \int_0^\infty \max\{r, v\} dF_{n-1:n}(v) - v_0 - F_{n:n}(r) (r - v_0) \end{aligned}$$

The bidder with the highest value wins whenever  $V_{n:n} > r$  and pays  $\max\{V_{n-1:n}, r\}$ , so

$$u_n(r) = \frac{1}{n} \int_r^\infty [v - E(\max\{V_{n-1:n}, r\} | V_{n:n} = v)] dF_{n:n}(v)$$

Separating the two terms and using iterated expectations, this is

$$u_n(r) = \frac{1}{n} \int_r^\infty v dF_{n:n}(v) - \frac{1}{n} (1 - F_{n:n}(r)) E(\max\{V_{n-1:n}, r\} | V_{n:n} > r)$$

For any  $a > r$ ,  $V_{n-1:n} = a$  implies  $V_{n:n} > a > r$ , so the conditional density  $f_{n-1:n}(a | V_{n:n} > r) = \frac{f_{n-1:n}(a)}{1 - F_{n:n}(r)}$ ; and  $F_{n-1:n}(r | V_{n:n} > r) = \frac{\Pr(V_{n:n} > r > V_{n-1:n})}{\Pr(V_{n:n} > r)} = \frac{F_{n-1:n}(r) - F_{n:n}(r)}{1 - F_{n:n}(r)}$ . So

$$\begin{aligned} u_n(r) &= \frac{1}{n} \int_r^\infty v dF_{n:n}(v) - \frac{1}{n} (1 - F_{n:n}(r)) \left[ \frac{F_{n-1:n}(r) - F_{n:n}(r)}{1 - F_{n:n}(r)} r + \int_r^\infty v \frac{f_{n-1:n}(v)}{1 - F_{n:n}(r)} dv \right] \\ &= \frac{1}{n} \left[ \int_r^\infty v dF_{n:n}(v) - (F_{n-1:n}(r) - F_{n:n}(r)) r - \int_r^\infty v f_{n-1:n}(v) dv \right] \\ &= \frac{1}{n} \left[ \int_r^\infty v dF_{n:n}(v) + r F_{n:n}(r) - r F_{n-1:n}(r) - \int_r^\infty v f_{n-1:n}(v) dv \right] \\ &= \frac{1}{n} \left[ \int_0^\infty \max\{r, v\} dF_{n:n}(v) - \int_0^\infty \max\{r, v\} dF_{n-1:n}(v) \right] \end{aligned}$$

### B.3 Proof of Lemma 3

As in Quint (2008). Fix  $n$  and  $v$ , and let  $P_k^n = F_{n-k:n}(v) - F_{n-k+1:n}(v)$  be the probability that exactly  $k$  of  $n$  bidders have valuations above  $v$  ( $P_0^n = F_{n:n}(v)$ ). Given symmetry,

$$\begin{aligned} \frac{\binom{n}{k+1}^{-1} P_{k+1}^n}{\binom{n}{k}^{-1} P_k^n} &= \frac{\Pr(V_1, \dots, V_k > v; V_{k+1}, \dots, V_{n-1} \leq v) \Pr(V_n > v | V_1, \dots, V_k > v; V_{k+1}, \dots, V_{n-1} \leq v)}{\Pr(V_1, \dots, V_k > v; V_{k+1}, \dots, V_{n-1} \leq v) \Pr(V_n \leq v | V_1, \dots, V_k > v; V_{k+1}, \dots, V_{n-1} \leq v)} \\ &= \frac{\Pr(V_n > v | V_1, \dots, V_k > v; V_{k+1}, \dots, V_{n-1} \leq v)}{\Pr(V_n \leq v | V_1, \dots, V_k > v; V_{k+1}, \dots, V_{n-1} \leq v)} \end{aligned}$$

which by Assumption 3 is weakly increasing in  $k$ .

Next, let  $p = \phi_n(F_{n-1:n}(v))$ , so that  $F_{n-1:n}(v) = \phi_n^{-1}(p) = np^{n-1} - (n-1)p^n$ . We will show that  $P_0^n \geq p^n$ . Suppose not.

Note that  $F_{n-1:n}(v) = P_0^n + P_1^n$ . So if  $P_0^n < p^n$ , then

$$P_1^n = (np^{n-1} - (n-1)p^n) - P_0^n > (np^{n-1} - (n-1)p^n) - p^n = np^{n-1}(1-p)$$

and so  $\frac{1}{n} \frac{P_1^n}{P_0^n} > \frac{p^{n-1}(1-p)}{p^n} = \frac{1-p}{p}$ .

Since  $\frac{\binom{n}{2}^{-1} P_2^n}{\binom{n}{1}^{-1} P_1^n} \geq \frac{\binom{n}{1}^{-1} P_1^n}{\binom{n}{0}^{-1} P_0^n} > \frac{1-p}{p}$  and  $P_1^n > np^{n-1}(1-p)$ , we get  $\binom{n}{2}^{-1} P_2^n > p^{n-2}(1-p)^2$ .

Likewise  $\frac{\binom{n}{3}^{-1} P_3^n}{\binom{n}{2}^{-1} P_2^n} \geq \frac{\binom{n}{2}^{-1} P_2^n}{\binom{n}{1}^{-1} P_1^n} > \frac{1-p}{p}$ , so  $\binom{n}{3}^{-1} P_3^n > p^{n-3}(1-p)^3$ , and so on.

This gives us  $P_1^n + P_0^n = np^{n-1} - (n-1)p^n = \binom{n}{1} p^{n-1}(1-p) + \binom{n}{0} p^n$ , and  $P_k^n > \binom{n}{k} p^{n-k}(1-p)^k$  for  $k > 1$ ; together, these imply  $\sum_{k=0}^n P_k^n > \sum_{k=0}^n \binom{n}{k} p^{n-k}(1-p)^k = 1$ , a contradiction, proving  $P_0^n \geq p^n$ .

## B.4 Proof of Lemmas 4 and 4'

The result is basically successive application of the relation  $F_{n:n}(v) = \frac{1}{n+1}F_{n:n+1}(v) + \frac{n}{n+1}F_{n+1:n+1}(v)$  (in the case of independence) or  $F_{n:n}(v) \geq \frac{1}{n+1}F_{n:n+1}(v) + \frac{n}{n+1}F_{n+1:n+1}(v)$  (in the case of stochastic monotonicity). To be formal, we prove Lemma 4' by induction on  $\bar{n}$ . The base case ( $\bar{n} = n + 1$ ) amounts to

$$F_{n:n}(v) \geq \frac{1}{n+1}F_{n:n+1}(v) + \frac{n}{n+1}F_{n+1:n+1}(v)$$

which is implied by Equation 4 and the assumption that  $F_{n:n}^{n+1} \succ_{FOSD} F_{n:n}$ . For the inductive step, if  $F_{n:n}(v) \geq \sum_{m=n+1}^K \frac{n}{(m-1)m} F_{m-1:m}(v) + \frac{n}{K} F_{K:K}(v)$ , then

$$\begin{aligned} & \sum_{m=n+1}^{K+1} \frac{n}{(m-1)m} F_{m-1:m}(v) + \frac{n}{K+1} F_{K+1:K+1}(v) \\ &= \sum_{m=n+1}^K \frac{n}{(m-1)m} F_{m-1:m}(v) + \frac{n}{K(K+1)} F_{K:K+1}(v) + \frac{n}{K+1} F_{K+1:K+1}(v) \\ &= \sum_{m=n+1}^K \frac{n}{(m-1)m} F_{m-1:m}(v) + \frac{n}{K} F_{K:K}(v) - \frac{n}{K} F_{K:K}(v) + \frac{n}{K(K+1)} F_{K:K+1}(v) + \frac{n}{K+1} F_{K+1:K+1}(v) \\ &\leq F_{n:n}(v) - \frac{n}{K} F_{K:K}(v) + \frac{n}{K(K+1)} F_{K:K+1}(v) + \frac{n}{K+1} F_{K+1:K+1}(v) \\ &= F_{n:n}(v) + \frac{n}{K} \left( -F_{K:K}(v) + \frac{1}{K+1} F_{K:K+1}(v) + \frac{K}{K+1} F_{K+1:K+1}(v) \right) \\ &\leq F_{n:n}(v) \end{aligned}$$

with the last inequality coming again from  $F_{K:K}(v) \geq F_{K:K}^{K+1}(v) = \frac{1}{K+1} F_{K:K+1}(v) + \frac{K}{K+1} F_{K+1:K+1}(v)$ .

Lemma 4 is the same proof, with all inequalities replaced by equalities.

## B.5 Proof of Theorems 1 and 1'

From Lemmas 4 and 4', if valuations are independent of or stochastically increasing in  $N$ ,

$$\begin{aligned} F_{n:n}(v) &\geq \sum_{m=n+1}^{\bar{n}} \frac{n}{(m-1)m} F_{m-1:m}(v) + \frac{n}{\bar{n}} F_{\bar{n}:\bar{n}}(v) \\ &\geq \sum_{m=n+1}^{\bar{n}} \frac{n}{(m-1)m} F_{m-1:m}(v) + \frac{n}{\bar{n}} (\phi_{\bar{n}}(F_{\bar{n}-1:\bar{n}}(v)))^{\bar{n}} \end{aligned}$$

and if valuations are independent of  $N$ , then in addition

$$\begin{aligned} F_{n:n}(v) &= \sum_{m=n+1}^{\bar{n}} \frac{n}{(m-1)m} F_{m-1:m}(v) + \frac{n}{\bar{n}} F_{\bar{n}:\bar{n}}(v) \\ &\leq \sum_{m=n+1}^{\bar{n}} \frac{n}{(m-1)m} F_{m-1:m}(v) + \frac{n}{\bar{n}} F_{\bar{n}-1:\bar{n}}(v) \end{aligned}$$

Plugging these bounds into (1) gives the first result. For the second, let  $r_n^* = \arg \max \pi_n(r)$  and  $r_* = \arg \max \underline{\pi}_n(r)$ ; if  $\underline{\pi}_n(\cdot) \leq \pi_n(\cdot) \leq \bar{\pi}_n(\cdot)$ , then

$$\bar{\pi}_n(r_n^*) \geq \pi_n(r_n^*) \geq \pi_n(r_*) \geq \underline{\pi}_n(r_*) = \max_{r'} \underline{\pi}_n(r')$$

so  $r_n^* \in \{r \geq v_0 : \bar{\pi}_n(r) \geq \max_{r'} \underline{\pi}_n(r')\}$ .

## B.6 Proof of Proposition A1

The proof of Lemma 3 above showed that under Assumption 3,  $\frac{\binom{n}{k+1}^{-1}(F_{n-(k+1):n}(v)-F_{n-k:n}(v))}{\binom{n}{k}^{-1}(F_{n-k:n}(v)-F_{n-(k-1):n}(v))}$  is increasing in  $k$ . This means

$$\frac{\binom{n}{2}^{-1}(F_{n-2:n}(v)-F_{n-1:n}(v))}{\frac{1}{n}(F_{n-1:n}(v)-F_{n:n}(v))} \geq \frac{\frac{1}{n}(F_{n-1:n}(v)-F_{n:n}(v))}{F_{n:n}(v)}$$

Under the assumptions of Haile and Tamer (2003),  $F_{n-1:n}(v) \geq G_{n:n}^\Delta(v)$  and  $F_{n-2:n}(v) \leq G_{n-2:n}(v)$ , so

$$\frac{F_{n:n}(v)}{\frac{1}{n^2}(G_{n:n}^\Delta(v)-F_{n:n}(v))^2} \geq \frac{F_{n:n}(v)}{\frac{1}{n^2}(F_{n-1:n}(v)-F_{n:n}(v))^2} \geq \frac{\binom{n}{2}}{F_{n-2:n}(v)-F_{n-1:n}(v)} \geq \frac{\binom{n}{2}}{G_{n-2:n}(v)-G_{n:n}^\Delta(v)}$$

Since  $\frac{x}{\frac{1}{n^2}(G_{n:n}^\Delta(v)-x)^2}$  is strictly increasing in  $x$ , the outer inequality gives a lower bound on  $F_{n:n}(v)$ .

For the second result,  $\frac{\binom{n}{3}^{-1}(F_{n-3:n}(v)-F_{n-2:n}(v))}{\binom{n}{2}^{-1}(F_{n-2:n}(v)-F_{n-1:n}(v))} \geq \frac{\binom{n}{2}^{-1}(F_{n-2:n}(v)-F_{n-1:n}(v))}{\binom{n}{1}^{-1}(F_{n-1:n}(v)-F_{n:n}(v))} \geq \frac{\binom{n}{1}^{-1}(F_{n-1:n}(v)-F_{n:n}(v))}{\binom{n}{0}^{-1}F_{n:n}(v)}$  implies  $\frac{\binom{n}{3}^{-1}(F_{n-3:n}(v)-F_{n-2:n}(v))}{\binom{n}{1}^{-1}(F_{n-1:n}(v)-F_{n:n}(v))} \geq \left( \frac{\binom{n}{1}^{-1}(F_{n-1:n}(v)-F_{n:n}(v))}{\binom{n}{0}^{-1}F_{n:n}(v)} \right)^2$ . Letting  $x = F_{n:n}(v)$ , this is

$$\frac{x^2}{\left(\frac{1}{n}(F_{n-1:n}(v)-x)\right)^3} \geq \frac{\binom{n}{3}}{F_{n-3:n}(v)-F_{n-2:n}(v)}$$

We get a lower bound for  $F_{n-2:n}(v)$  from  $\frac{\binom{n}{2}^{-1}(F_{n-2:n}(v)-F_{n-1:n}(v))}{\frac{1}{n}(F_{n-1:n}(v)-x)} \geq \frac{\frac{1}{n}(F_{n-1:n}(v)-x)}{x}$ , which gives

$$F_{n-2:n}(v) \geq F_{n-1:n}(v) + \binom{n}{2} \frac{\left(\frac{1}{n}(F_{n-1:n}(v)-x)\right)^2}{x} \geq G_{n:n}^\Delta(v) + \binom{n}{2} \frac{\left(\frac{1}{n}(G_{n:n}^\Delta(v)-x)\right)^2}{x}$$

Combining  $\frac{x^2}{\left(\frac{1}{n}(F_{n-1:n}(v)-x)\right)^3} \geq \frac{\binom{n}{3}}{F_{n-3:n}(v)-F_{n-2:n}(v)}$  with  $F_{n-1:n}(v) \geq G_{n:n}^\Delta(v)$ ,  $F_{n-3:n}(v) \leq G_{n-3:n}(v)$ , and  $F_{n-2:n}(v) \geq G_{n:n}^\Delta(v) + \binom{n}{2} \frac{\left(\frac{1}{n}(G_{n:n}^\Delta(v)-x)\right)^2}{x}$  gives

$$\frac{x^2}{\left(\frac{1}{n}(G_{n:n}^\Delta(v)-x)\right)^3} \geq \frac{\binom{n}{3}}{G_{n-3:n}(v) - \left(G_{n:n}^\Delta(v) + \binom{n}{2} \frac{\left(\frac{1}{n}(G_{n:n}^\Delta(v)-x)\right)^2}{x}\right)}$$

The left-hand side is again strictly increasing in  $x$ , and the right-hand side is strictly decreasing in  $x$ , giving the result.

## B.7 Proof of Propositions A2–A4

For the Levin-Smith result, suppose  $\theta$  is continuous; let  $P(\cdot)$  denote its prior distribution and  $p$  its density, and  $P(\cdot|N=n)$  its posterior distribution conditional on exactly  $n$  of the  $\bar{n}$  potential bidders choosing to participate. We will show that  $q_\theta$  increasing in  $\theta$  implies  $P(\cdot|N=n)$  decreasing in  $n$ . Bayes' Law gives

$$P(\tilde{\theta}|n) = \frac{\int_{-\infty}^{\tilde{\theta}} p(\theta) \binom{\bar{n}}{n} q_\theta^n (1-q_\theta)^{\bar{n}-n} d\theta}{\int_{-\infty}^{+\infty} p(\theta) \binom{\bar{n}}{n} q_\theta^n (1-q_\theta)^{\bar{n}-n} d\theta} = R \left( \frac{\int_{\tilde{\theta}}^{+\infty} p(\theta) \binom{\bar{n}}{n} q_\theta^n (1-q_\theta)^{\bar{n}-n} d\theta}{\int_{-\infty}^{\tilde{\theta}} p(\theta) \binom{\bar{n}}{n} q_\theta^n (1-q_\theta)^{\bar{n}-n} d\theta} \right)$$

where  $R(x) \equiv \frac{1}{1+x}$ . Consider the change to the argument of  $R$  as  $n$  increases: the integrand in the numerator gets multiplied by  $\frac{q_\theta}{1-q_\theta} \geq \frac{q_{\tilde{\theta}}}{1-q_{\tilde{\theta}}}$ , while the integrand in the denominator gets multiplied by  $\frac{q_\theta}{1-q_\theta} \leq \frac{q_{\tilde{\theta}}}{1-q_{\tilde{\theta}}}$ , so the argument of  $R$  increases; since  $R$  is decreasing, this means  $P(\tilde{\theta}|N = n)$  is decreasing in  $n$ , so  $n > n'$  implies  $P(\cdot|N = n) \succsim_{FOSD} P(\cdot|N = n')$ . (With discrete  $\theta$ , the same proof applies with sums replacing integrals.) Since  $F_{m:m}^n(v) = E_{\theta|N=n}\{F^m(v|\theta)\}$  and (by assumption)  $F^m(v|\theta)$  is decreasing in  $\theta$ , this then implies  $F_{m:m}^n(v) \leq F_{m:m}^{n'}(v)$ , so valuations are stochastically increasing in  $N$ .

For the sufficient condition, let  $u_n(\theta)$  denote each bidder's expected surplus in an  $n$ -bidder auction given a realization  $\theta$ , and  $q_\theta$  the equilibrium entry probability.  $q_\theta$  is the value of  $q$  that solves  $c = \sum_{n=0}^{\bar{n}-1} \left[ \binom{\bar{n}-1}{n} q^n (1-q)^{\bar{n}-1-n} \right] u_{n+1}(\theta)$ . Since  $u_n(\theta)$  is decreasing in  $n$ , if it's increasing in  $\theta$  for each  $n$ , then the right-hand side is increasing in  $\theta$  and decreasing in  $q$ , which would make the solution  $q_\theta$  increasing in  $\theta$ .

For the Samuelson result, the same logic from above – higher  $\theta$  means more entry, therefore higher  $N$  means higher  $\theta$ , which gives the result – still holds, but with two changes. Let  $v^*(\theta)$  denote the equilibrium entry threshold at a realized value of  $\theta$ ; then  $1 - F(v^*(\theta)|\theta)$  replaces  $q_\theta$  as each potential bidder's entry probability, and  $F(v|\theta, v > v^*(\theta)) = \frac{F(v|\theta) - F(v^*(\theta)|\theta)}{1 - F(v^*(\theta)|\theta)}$  replaces  $F(v|\theta)$  as the distribution of valuations among those bidders who enter.  $v^*(\theta)$  is the solution to  $c = (v - r)(F(v|\theta))^{\bar{n}-1}$ ; since  $F(v|\theta)$  is decreasing in  $\theta$ ,  $v^*(\theta)$  must be increasing in  $\theta$  and  $F(v^*(\theta)|\theta)$  decreasing in  $\theta$ ; so the entry probability is increasing in  $\theta$ , and so (as above)  $P(\theta|N = n)$  is decreasing in  $n$ . What remains is to show that for  $\theta > \theta'$ ,  $\frac{F(\cdot|\theta) - F(v^*(\theta)|\theta)}{1 - F(v^*(\theta)|\theta)} \succsim_{FOSD} \frac{F(\cdot|\theta') - F(v^*(\theta')|\theta')}{1 - F(v^*(\theta')|\theta')}$ .

The MLRP implies that  $\frac{f(\cdot|\theta)}{1 - F(\cdot|\theta)}$  is decreasing in  $\theta$ , and therefore that  $\log(1 - F(v|\theta))$  has increasing differences in  $v$  and  $\theta$ . This means that for  $\theta > \theta'$ ,  $\log(1 - F(v|\theta)) - \log(1 - F(v|\theta'))$  is increasing in  $v$ , or  $\log\left(\frac{1 - F(v|\theta)}{1 - F(v|\theta')}\right)$  is increasing in  $v$ , or  $\frac{1 - F(v|\theta)}{1 - F(v|\theta')}$  is increasing in  $v$ . Since  $v^*(\theta)$  is increasing in  $\theta$ , then, for any  $v > v^*(\theta)$ ,

$$\frac{1 - F(v|\theta)}{1 - F(v|\theta')} \geq \frac{1 - F(v^*(\theta)|\theta)}{1 - F(v^*(\theta)|\theta')} \geq \frac{1 - F(v^*(\theta)|\theta)}{1 - F(v^*(\theta')|\theta')}$$

from which we get  $\frac{F(v|\theta) - F(v^*(\theta)|\theta)}{1 - F(v^*(\theta)|\theta)} = 1 - \frac{1 - F(v|\theta)}{1 - F(v^*(\theta)|\theta)} \leq 1 - \frac{1 - F(v|\theta')}{1 - F(v^*(\theta')|\theta')} = \frac{F(v|\theta') - F(v^*(\theta')|\theta')}{1 - F(v^*(\theta')|\theta')}$ , and therefore  $F(v|\theta, v > v^*(\theta)) \succsim_{FOSD} F(v|\theta', v > v^*(\theta'))$ , giving the result.

Finally, for the model of Marmer et. al., let  $e_i$  be an indicator function taking value 1 if bidder  $i$  participates in the auction and 0 otherwise. If a symmetric cutoff-strategy equilibrium exists with cutoff  $\bar{s}$ , then  $e_i = \mathbf{1}\{S_i \geq \bar{s}\}$  is a nondecreasing transformation of  $S_i$ , so  $(V_1, \dots, V_{\bar{n}}, e_1, \dots, e_{\bar{n}})$  are affiliated. Given symmetry, we can write  $F_{m:m}^n(v)$  as

$$\Pr(V_1, V_2, \dots, V_m \leq v \mid e_1, e_2, \dots, e_n = 1, e_{n+1}, \dots, e_{\bar{n}} = 0)$$

Since  $\Pr(V_1, V_2, \dots, V_m \leq v)$  is the expected value of a function which is decreasing in every  $V_i$ , under affiliation, it is decreasing in each  $e_i$ , and so  $F_{m:m}^n(v) \leq F_{m:m}^{n'}(v)$  for  $n > n'$ .

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