Online Appendix for *Anglo-Dutch Premium Auctions in Eighteenth-Century Amsterdam*

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The first part of Theorem 1 states that for any $N \geq 2$, $\rho > 0$, and $\beta > 0$, an equilibrium exists. The proof is constructive: for $N \geq 4$, equilibrium strategies are given in the appendix of the paper. Here, we show that these strategies do indeed make up an equilibrium, and deal separately with the cases $N = 3$ and $N = 2$.

First Round Play

First-round strategies (and beliefs) for player 1 are

- Do not drop out at $P = 0$
- At any history where $P > 0$ and there is one other active bidder, believe that he has signal $s_i = 1$, and
  - If $P < P^*$, remain active
  - If $P \geq P^*$, drop out

and for player $i \neq 1$ are

- Drop out at $P = 0$
- At any history where $P > 0$ and bidder i is still active along with bidder 1,
  - If $P \leq P^*$, believe $s_1 \sim U[0, 1]$; remain active if $s_i = 1$ (and drop out if $s_i < 1$)
  - If $P^* < P < P^{**}$, believe $s_1 = 1$; remain active if $s_i = 1$ (and drop out if $s_i < 1$)
  - If $P \geq P^{**}$, drop out immediately

where

$$P^* = \begin{cases} c + \frac{\beta}{N}\frac{N+1}{2} + \rho & if \quad \rho \geq \frac{\beta}{2N} \\ c + \frac{\beta}{N}\left(1 + \frac{N}{2}\left(\frac{2N\rho}{\beta}\right)^{\frac{1}{N}}\right) & if \quad \rho < \frac{\beta}{2N} \end{cases}$$

and

$$P^{**} = \begin{cases} c + \frac{\beta}{N}\frac{N+2}{2} + \rho & if \quad \rho \geq \frac{\beta}{2N} \\ c + \frac{\beta}{N}\left(2 + \frac{N-1}{2}\left(\frac{2N\rho}{\beta}\right)^{\frac{1}{N-1}}\right) & if \quad \rho < \frac{\beta}{2N} \end{cases}$$

Since all bidders other than bidder 1 are expected to drop out at $P = 0$, any history where $P > 0$ and more than one of bidders 2, 3, …, N are still active could only be reached following simultaneous deviations by more than one player. Strategies at such histories therefore does not affect equilibrium play, and we do not specify what strategies are played following such deviations.

Next, we will describe second-round beliefs and strategies, both on and off the equilibrium path, and show that these are equilibrium strategies. After that, we will show that these first-round strategies are best-responses given that continuation play.

Second Round Play On the Equilibrium Path

Bidder 1 won the first round at a price of 0. Common beliefs are that $s_i \sim U[0,1]$ for all i. Second-round bidding functions are

$$b(s_i) \quad = \quad c + \tfrac{\beta}{N}\left(\tfrac{N-1}{N}s_i + (N-1)\tfrac{s_i}{2}\right)$$

To see that these bid functions are an equilibrium, first note that bids above $b(1)$ are strictly dominated by bidding $b(1)$, since this will still win with probability 1; and bids below $b(0)$ earn expected payoff of 0. For $x \in [0,1]$, a bidder with signal s_i who plans to bid $b(x)$ will win whenever $s_j < x$ for every $j \neq i$, which occurs with probability x^{N-1}; in that event, the security is worth, on average, $c + \frac{\beta}{N}\left(s_i + (N-1)\frac{x}{2}\right)$, so his expected payoff is

$$x^{N-1}\left[c + \tfrac{\beta}{N}\left(s_i + (N-1)\tfrac{x}{2}\right) - \left(c + \tfrac{\beta}{N}\left(\tfrac{N-1}{N}x + (N-1)\tfrac{x}{2}\right)\right)\right] \quad = \quad \tfrac{\beta}{N}x^{N-1}\left[s_i - \tfrac{N-1}{N}x\right]$$

Differentiating with respect to x gives $\frac{\beta}{N}\left[(N-1)s_i x^{N-2} - (N-1)x^{N-1}\right] = \frac{\beta}{N}(N-1)x^{N-2}(s_i - x)$, which is positive when $x < s_i$ and negative when $x > s_i$, so expected payoff is maximized by setting $x = s_i$. Note also (plugging this into the payoff function) that this gives nonnegative expected payoff, so bidding less than $b(0)$ and losing for sure is not a profitable deviation.

Should bidder 1 choose to drop out at $P = 0$ along with the other bidders (so that the first-round winner is chosen by a random tiebreaker), second-round beliefs and strategies are unchanged.

Second Round Play After a First-Round Deviation By Bidder $i \geq 2$

We refer to this continuation game as CG2. As in the text, assume without loss of generality that bidder 2 is the one who deviated. Since bidder 1's equilibrium strategy following such a deviation is to bid up to price P^*, assume that either the deviating bidder has dropped out at a price $P \in (0, P^*]$ or has won the first round at the price P^*. All bidders commonly believe that $s_2 = 1$; beliefs about the other bidders' signals (including bidder 1) match the prior $s_i \sim U[0,1]$. Strategies are:

- Bidder 2 does not bid (regardless of his actual type)
- If $P \geq c + \frac{\beta}{N}\left(1+\frac{N}{2}\right)$, then nobody bids
- If $P \in \left(c + \frac{\beta}{N}, c + \frac{\beta}{N}\left(1+\frac{N}{2}\right)\right)$, then let s^* solve $P = c + \frac{\beta}{N}\left(1 + \frac{N}{2}s^*\right)$. Bidder $i \neq 2$ does not bid when $s_i \leq s^*$; when $s_i > s^*$, he bids

$$b(s_i) \quad = \quad c + \frac{\beta}{N}\left(\frac{N-2}{N-1}s_i + \frac{s_i}{N-1}\left(\frac{s^*}{s_i}\right)^{N-1} + (N-2)\frac{s_i}{2} + 1\right)$$

- If $P \leq c + \frac{\beta}{N}$, then bidders $i \neq 2$ each bid

$$b(s_i) \quad = \quad c + \frac{\beta}{N}\left(\frac{N-2}{N-1}s_i + (N-2)\frac{s_i}{2} + 1\right)$$

In the first case, $P \geq c + \frac{\beta}{N}(1+\frac{N}{2})$, each bidder $i \neq 2$ expects nobody to bid; winning the object would require bidding strictly more than P, therefore strictly more than $c + \frac{\beta}{N}(1+\frac{N}{2})$, for an object that will be worth, on average, $c + \frac{\beta}{N}\left(s_i + 1 + \frac{N-2}{2}\right) \leq c + \frac{\beta}{N}(1+\frac{N}{2})$, so not bidding is a best-response. Bidder 2 expects the object to be worth $c + \frac{\beta}{N}\left(s_2 + \frac{N-1}{2}\right) < c + \frac{\beta}{N}(1+\frac{N}{2})$, so not bidding is a best-response for him as well.

In the second case, $b(s_i)$ is strictly increasing, so bids above $b(1)$ are again strictly dominated. By construction, $b(s^*) = P$. For $x \in [s^*, 1]$, a bidder with signal s_i who bids $b(x)$ expects to win when the other $N-2$ bidders (excluding himself and bidder 2) have signals below x; he therefore earns expected payoff

$$\begin{aligned} & x^{N-2}\left[c + \frac{\beta}{N}\left(s_i + 1 + (N-2)\frac{x}{2}\right) - c - \frac{\beta}{N}\left(\frac{N-2}{N-1}x + \frac{x}{N-1}\left(\frac{s^*}{x}\right)^{N-1} + (N-2)\frac{x}{2} + 1\right)\right] \\ &= \quad x^{N-2}\frac{\beta}{N}\left[s_i - \frac{N-2}{N-1}x - \frac{x}{N-1}\left(\frac{s^*}{x}\right)^{N-1}\right] \quad = \quad \frac{\beta}{N}\left[s_i x^{N-2} - \frac{N-2}{N-1}x^{N-1} - \frac{(s^*)^{N-1}}{N-1}\right] \end{aligned}$$

Differentiating with respect to x gives $(N-2)x^{N-3}(s_i - x)$, which is positive for $x < s_i$ and negative for $x > s_i$, so expected payoff is maximized at $b(s_i)$. Bidding $P = b(s^*)$ (or a tiny bit above that) wins only if nobody else bids, giving expected payoff

$$\begin{aligned} & (s^*)^{N-2}\left[c + \frac{\beta}{N}\left(s_i + 1 + (N-2)\frac{s^*}{2}\right) - c - \frac{\beta}{N}\left(\frac{N-2}{N-1}s^* + \frac{s^*}{N-1}\left(\frac{s^*}{s^*}\right)^{N-1} + (N-2)\frac{s^*}{2} + 1\right)\right] \\ &= \quad (s^*)^{N-2}\frac{\beta}{N}\left[s_i - \frac{N-2}{N-1}s^* - \frac{s^*}{N-1}\right] \quad = \quad (s^*)^{N-2}\frac{\beta}{N}\left[s_i - s^*\right] \end{aligned}$$

making it worse than not bidding when $s_i < s^*$. So for $s_i < s^*$, any bid equal or greater than P would earn negative expected profits; and for $s_i > s^*$, bidding $b(s_i)$ earns greater payoff than $b(s^*)$, which is itself positive, so $b(s_i)$ is a best-response.

(Note that if bidder 2 dropped out at a price $P < P^*$ but above $c + \frac{\beta}{N}$, then bidder 1 is the first-round winner. The fact that he will win the object if nobody bids in the second round means that by not bidding, he pays $b(s^*)$ for the object whenever all other bidders (aside from 2) have signals $s_i < s^*$. This is equivalent to bidder 1 having to bid $b(s^*)$ if he does not bid higher than that; his optimal move is therefore still to bid $b(s_1)$ when $s_1 > s^*$, and to not bid in the second round when $s_1 \leq s^*$.)

As for bidder 2, bidding $b(x)$ $(x \geq x^*)$ would give expected payoff

$$\begin{array}{rl} x^{N-1} \left[c + \frac{\beta}{N} \left(s_2 + (N-1)\frac{x}{2} \right) - c - \frac{\beta}{N} \left(\frac{N-2}{N-1}x + \frac{x}{N-1} \left(\frac{s^*}{x} \right)^{N-1} + (N-2)\frac{x}{2} + 1 \right) \right] & \\ = \quad x^{N-1} \frac{\beta}{N} \left[s_2 - 1 + \frac{x}{2} - \frac{N-2}{N-1}x - \frac{x}{N-1} \left(\frac{s^*}{x} \right)^{N-1} \right] \quad < \quad 0 \end{array}$$

so not bidding is a best-response.

In the third case, bids above $b(1)$ are again dominated, and a bid of $b(x)$ for $x \in [0,1]$ earns expected payoff

$$\begin{array}{rl} x^{N-2} \left[c + \frac{\beta}{N} \left(s_i + 1 + (N-2)\frac{x}{2} \right) - c - \frac{\beta}{N} \left(\frac{N-2}{N-1}x + (N-2)\frac{x}{2} + 1 \right) \right] & \\ = \quad x^{N-2} \frac{\beta}{N} \left[s_i - \frac{N-2}{N-1}x \right] \end{array}$$

The derivative with respect to x is again $(n-2)x^{N-3}(s_i - x)$, so expected payoff is again maximized at $s_i = x$, which gives positive expected payoff (so bidding less than $b(0)$ is not a profitable deviation). For bidder 2, bidding $b(x)$ would give

$$\begin{array}{rl} x^{N-1} \left[c + \frac{\beta}{N} \left(s_2 + (N-1)\frac{x}{2} \right) - c - \frac{\beta}{N} \left(\frac{N-2}{N-1}x + (N-2)\frac{x}{2} + 1 \right) \right] & \\ = \quad x^{N-1} \frac{\beta}{N} \left[s_2 - 1 + \frac{x}{2} - \frac{N-2}{N-1}x \right] \quad < \quad 0 \end{array}$$

so not bidding is again a best-response.

Bidder 2's Payoff From Deviating In the First Round

By deviating in the first round and dropping out before P^*, bidder 2 guarantees entry into CG2 discussed above, in which he does not bid and earns expected payoff 0; so remaining active past price 0, but planning to drop out before P^*, is not a profitable deviation.

On the other hand, if bidder 2 deviates in the first round and does not drop out before P^*, he expects bidder 1 to drop out at price P^*. We next show that this would earn expected profit of 0 or less for bidder 2.

If $\rho \geq \frac{\beta}{2N}$, then $P^* = c + \frac{\beta}{N}\frac{N+1}{2} + \rho \geq c + \frac{\beta}{N}\frac{N+2}{2}$, so if the first round reaches price P^*, nobody bids in the second round. In that case, by winning the first round at price P^* (and not updating beliefs about s_1), bidder 2 earns second-round payoff

$$c + \frac{\beta}{N}\left(s_2 + \frac{N-1}{2}\right) - \left[c + \frac{\beta}{N}\frac{N+1}{2} + \rho\right] \quad = \quad \frac{\beta}{N}(s_2 - 1) - \rho$$

Combined with winning the premium, then, bidder 2's expected payoff would be nonpositive.

If $\rho < \frac{\beta}{2N}$, then if $P = P^* = c + \frac{\beta}{N}\left(1 + \frac{N}{2}\left(\frac{2N\rho}{\beta}\right)^{\frac{1}{N}}\right)$, s^* solves

$$c + \frac{\beta}{N}\left(1 + \frac{N}{2}\left(\frac{2N\rho}{\beta}\right)^{\frac{1}{N}}\right) \quad = \quad c + \frac{\beta}{N}\left(1 + \frac{N}{2}s^*\right)$$

and so $s^* = \left(\frac{2N\rho}{\beta}\right)^{\frac{1}{N}}$. After winning the first round at price P^*, bidder 2 expects to be "stuck with" the security with probability $(s^*)^{N-1}$, for second-round expected payoff

$$\begin{aligned}
&(s^*)^{N-1}\left[c + \frac{\beta}{N}\left(s_2 + (N-1)\frac{s^*}{2}\right) - \left(c + \frac{\beta}{N}\left(1 + \frac{N}{2}s^*\right)\right)\right] \\
&= \quad \frac{\beta}{N}(s^*)^{N-1}\left[s_2 - 1 - \frac{1}{2}s^*\right] \quad = \quad -\frac{\beta}{N}(s^*)^{N-1}(1-s_2) - \frac{\beta}{2N}(s^*)^N \\
&= \quad -\frac{\beta}{N}(s^*)^{N-1}(1-s_2) - \frac{\beta}{2N}\frac{2N\rho}{\beta} \quad = \quad -\frac{\beta}{N}(s^*)^{N-1}(1-s_2) - \rho
\end{aligned}$$

So including winning the premium in the first round, winning the first round at price P^* gives expected payoff 0 to bidder 2 if $s_2 = 1$, and negative expected payoff otherwise.

(Note therefore that if $s_2 = 1$ and bidder 2 has already deviated by failing to drop out immediately, he expects payoff 0 whether he drops out before P^* or waits for bidder 1 drop out at P^*. This means that once he has already deviated, it is a weak best response not to drop out before P^* (which makes bidder 1 willing to "punish" him by staying in until P^*); but also that deviating in the first place cannot be profitable.)

Second Round Play After Deviations By Bidders $i \geq 2$ and 1

Again, assume $i = 2$ is the first deviating bidder. If bidder 2 does not drop out immediately, and bidder 1 either drops out at a price $P < P^*$, or fails to drop out at price P^* (so that either bidder 1 or 2 wins the first round at a price $P > P^*$), we refer to the resulting continuation game as CG3. Beliefs and strategies in the second round are as follows:

- Everyone believes $s_1 = s_2 = 1$
- Bidders 1 and 2 (regardless of their actual types) do not bid

- If $P \geq c + \frac{\beta}{N}\left(1 + \frac{N-3}{2} + 2\right)$, then nobody bids
- If $P \in \left(c + \frac{2\beta}{N}, c + \frac{\beta}{N}\left(1 + \frac{N-3}{2} + 2\right)\right)$, then let s^* solve

$$P = c + \frac{\beta}{N}\left(s^* + (N-3)\frac{s^*}{2} + 2\right)$$

Bidder $i \geq 3$ does not bid when $s_i \leq s^*$; when $s_i > s^*$, he bids

$$b(s_i) = c + \frac{\beta}{N}\left(\frac{N-3}{N-2}s_i + \frac{s_i}{N-2}\left(\frac{s^*}{s_i}\right)^{N-2} + (N-3)\frac{s_i}{2} + 2\right)$$

- If $P \leq c + \frac{2\beta}{N}$, then bidders $i \geq 3$ each bid

$$b(s_i) = c + \frac{\beta}{N}\left(\frac{N-3}{N-2}s_i + (N-3)\frac{s_i}{2} + 2\right)$$

In the first case, if $P \geq c + \frac{\beta}{N}(1 + \frac{N-3}{2} + 2)$, then a bidder $i \geq 3$ with signal s_i expects no one else to bid, and believes that the security is worth in expectation $c + \frac{\beta}{N}(s_i + \frac{N-3}{2} + 1 + 1) \leq P$, and therefore does not bid. Bidder $i \in \{1, 2\}$ believes the security to be worth in expectation $c + \frac{\beta}{N}(s_i + \frac{N-2}{2} + 1) < P$ and therefore does not bid either.

For the second case, a bid above $c + \frac{\beta}{N}\left(1 + \frac{N-3}{2} + 2\right)$ is dominated; a bid by bidder $i \geq 3$ equal to $b(x)$ $(x > s^*)$ wins whenever $s_j < x$ for all $j \in \{3, 4, \ldots, N\} - \{i\}$, which is with probability x^{N-3}, and therefore delivers expected payoff of

$$\begin{aligned} & x^{N-3}\left[c + \frac{\beta}{N}\left(s_i + (N-3)\frac{x}{2} + 2\right) - \left(c + \frac{\beta}{N}\left(\frac{N-3}{N-2}x + \frac{x}{N-2}\left(\frac{s^*}{x}\right)^{N-2} + (N-3)\frac{x}{2} + 2\right)\right)\right] \\ &= x^{N-3}\frac{\beta}{N}\left[s_i - \frac{N-3}{N-2}x - \frac{(s^*)^{N-2}}{N-2}x^{-(N-3)}\right] = \frac{\beta}{N}\left[s_i x^{N-3} - \frac{N-3}{N-2}x^{N-2} - \frac{(s^*)^{N-2}}{N-2}\right] \end{aligned}$$

Differentiating with respect to x gives

$$\frac{\beta}{N}\left[(N-3)s_i x^{N-2} - (N-3)x^{N-3}\right]$$

which is positive for $x < s_i$ and negative for $x > s_i$, so $b(s_i)$ is a best-response among bids above P. Bidding P (or slightly above P) would give expected payoff

$$\frac{\beta}{N}\left[s_i(s^*)^{N-3} - \frac{N-3}{N-2}(s^*)^{N-2} - \frac{(s^*)^{N-2}}{N-2}\right] = \frac{\beta}{N}\left[s_i(s^*)^{N-3} - (s^*)^{N-2}\right]$$

which is positive for $s_i > s^*$ and negative for $s_i < s^*$. So not bidding is a best-response for $s_i \leq s^*$, and $b(s_i)$ a best-response for $s_i > s^*$. Bidder $i \in \{1, 2\}$ who is *not* the first-round winner would, by bidding $b(x)$, earn expected payoff

$$x^{N-2}\left[c+\frac{\beta}{N}\left(s_i+(N-2)\frac{x}{2}+1\right)-\left(c+\frac{\beta}{N}\left(\frac{N-3}{N-2}x+\frac{x}{N-2}\left(\frac{s^*}{x}\right)^{N-2}+(N-3)\frac{x}{2}+2\right)\right)\right]$$
$$= \quad \frac{\beta}{N}x^{N-2}\left[s_i-1+\frac{x}{2}-\frac{N-3}{N-2}x-\frac{x}{N-2}\left(\frac{s^*}{x}\right)^{N-2}\right] \quad \leq \quad 0$$

and so prefers not to bid. Bidder $i \in \{1,2\}$ who *is* the first-round winner is effectively bidding $b(s^*)$ by not making a new bid; but since

$$\frac{d}{dx}\left\{\frac{\beta}{N}\left[(s_i-1)x^{N-2}+x^{N-1}\left(\frac{1}{2}-\frac{N-3}{N-2}\right)-\frac{x}{N-2}(s^*)^{N-2}\right]\right\}$$
$$= \quad \frac{\beta}{N}\left[(N-2)x^{N-3}(s_i-1)+(N-1)x^{N-2}\left(\frac{1}{2}-\frac{N-3}{N-2}\right)-\frac{1}{N-2}(s^*)^{N-2}\right] \quad < \quad 0$$

he prefers $b(s^*)$ to a higher bid, and does not bid again.

For the third case, a bid of $b(x)$ by bidder $i \geq 3$ gives expected payoff

$$x^{N-3}\left[c+\frac{\beta}{N}\left(s_i+(N-3)\frac{x}{2}+2\right)-c-\frac{\beta}{N}\left(\frac{N-3}{N-2}x+(N-3)\frac{x}{2}+2\right)\right] \quad = \quad \frac{\beta}{N}x^{N-3}\left[s_i-\frac{N-3}{N-2}x\right]$$

which has derivative $\frac{\beta}{N}\left((N-3)x^{N-2}s_i-(N-3)x^{N-3}\right)$, which once again is positive for $x < s_i$ and negative for $x > s_i$; setting $x = s_i$ gives strictly positive payoff, so $b(s_i)$ is a best-response. For bidder $i \in \{1,2\}$ (whether or not the first-round winner, since the first-round winner will for sure be outbid if he does not bid again, but counting the premium received as a "sunk benefit"), bidding $b(x)$ gives payoff

$$x^{N-2}\left[c+\frac{\beta}{N}\left(s_i+(N-2)\frac{x}{2}+1\right)-c-\frac{\beta}{N}\left(\frac{N-3}{N-2}x+(N-3)\frac{x}{2}+2\right)\right]$$
$$= \quad \frac{\beta}{N}x^{N-2}\left[s_i-1+\frac{x}{2}-\frac{N-3}{N-2}x\right] \quad \leq \quad 0$$

so not bidding is a best-response.

Bidder 1's Expected Payoff From Failing To Punish

Next, we show that if $s_1 = s_2 = 1$ and either bidder 1 or 2 wins the first round at price P^{**}, his expected payoff (including the first-round premium) is 0. This ensures that (i) once the price passes P^* and bidder 2 believes $s_1 = 1$, then assuming $s_2 = 1$ as well, bidder 2 is willing to stay in until P^{**}; and (ii) bidder 1 cannot gain by "failing to punish" a deviation by bidder 2, since he expects bidder 2 to stay in until P^{**}, and even if $s_1 = s_2 = 1$, winning the first round would therefore give bidder 1 zero expected payoff.

If $\rho \geq \frac{\beta}{2N}$, then $P^{**} \geq c + \frac{\beta}{N}\frac{N+3}{3}$, so if $P = P^{**}$, nobody will bid in the second round; in that case, the first-round winner wins the security for sure (regardless of others' signals), so if $s_1 = s_2 = 1$, his expected payoff is

$$c + \frac{\beta}{N}\left(2 + \frac{N-2}{2}\right) - \left[c + \frac{\beta}{N}\frac{N+2}{2} + \rho\right] \quad = \quad -\rho$$

which, combined with winning the premium ρ in the first round, gives a net expected payoff of 0.

If $\rho < \frac{\beta}{2N}$, then when $P = P^{**}$, s^* solves

$$c + \frac{\beta}{N}\left(2 + \frac{N-1}{2}\left(\frac{2N\rho}{\beta}\right)^{\frac{1}{N-1}}\right) \quad = \quad c + \frac{\beta}{N}\left(s^* + \frac{N-3}{2}s^* + 2\right)$$

and therefore $s^* = \left(\frac{2N\rho}{\beta}\right)^{\frac{1}{N-1}}$. So with probability $(s^*)^{N-2}$, nobody bids in the second round, and the security is worth, in expectation (assuming $s_1 = s_2 = 1$), $c + \frac{\beta}{N}\left((N-2)\frac{s^*}{2} + 2\right)$. So the first-round winner $i \in \{1, 2\}$ at price P^{**}, assuming $s_1 = s_2 = 1$, gets expected payoff (gross of winning the premium)

$$\begin{aligned}
&(s^*)^{N-2}\left[c + \frac{\beta}{N}\left((N-2)\frac{s^*}{2} + 2\right) - c - \frac{\beta}{N}\left((N-1)\frac{s^*}{2} + 2\right)\right] \\
= \quad & \frac{\beta}{N}(s^*)^{N-2}\left[-\frac{s^*}{2}\right] \quad = \quad -\frac{\beta}{2N}(s^*)^{N-1} \quad = \quad -\frac{\beta}{2N}\frac{2N\rho}{\beta} \quad = \quad -\rho
\end{aligned}$$

and so including winning the premium, the first-round winner would get ex-ante expected payoff 0.

First Round Strategies Are Equilibrium Strategies

Finally, we need to show the strategies given earlier for the first round are indeed best-responses. First, consider bidder 1 at $P = 0$. If he drops out immediately (leading to the first round being determined by a random tiebreaker), he wins the premium with probability $\frac{1}{N}$ instead of 1, and does not affect second-round play, so this is not a profitable deviation.

Following a deviation by bidder 2, bidder 1 believes $s_2 = 1$, and expects bidder 2 to stay active until the price reaches P^{**}. Dropping out at price P^* would lead to CG2 and nonnegative payoffs. Dropping out before or after P^* would lead to CG3 and zero payoffs; staying in until P^{**} would lead to zero payoffs (or negative payoffs if $s_1 < 1$). So sticking to equilibrium strategies is a best-response.

As for bidder 2 (representing all bidders $i > 1$), dropping out immediately leads to nonnegative (strictly positive if $s_2 > 0$) expected payoffs in the second round. Not dropping out immediately, bidder 2 would expect bidder 1 to remain active until P^* and then drop out; dropping out before then would lead to CG2 and zero payoffs; winning the first round at price P^* would likewise lead to CG2 and zero payoffs (or negative if $s_2 < 1$). So having already failed to drop out at $P = 0$, it remains a best-response to stay active until $P = P^*$. If bidder 1 fails to drop out at $P = P^*$, bidder 2 then believes $s_1 = 1$; provided $s_2 = 1$ as well, bidder 2 expects zero payoffs if he stays in and wins at price P^{**}, strictly positive payoffs if he stays in and wins at a lower price than that; dropping out prior to P^{**} would lead to CG3, and zero payoffs for bidder 2. So having stayed in past $P = 0$, bidder 2 has no reason to drop out prior to P^{**}. Once the price passes P^{**}, the winner of the first round is assured negative payoffs, so both players best-respond by dropping out immediately.

As mentioned above, any history not explicitly addressed in the strategies above could only be reached via simultaneous deviations by multiple bidders, and therefore cannot affect play on the equilibrium path; we therefore do not specify what happens at these histories.

What if $N = 3$?

If $N = 3$, define

$$P^{**} = \max\left\{P^*,\ c + \frac{\beta}{3}\left(2 + \sqrt{\frac{6\rho}{\beta}}\right)\right\}$$

If $P^{**} = P^*$, modify both bidder 1's strategy following a deviation by bidder 2 to be, "stay in till P^*, drop out at P^*, and drop out immediately at any $P > P^*$", and bidder 2's strategy following a deviation by himself to be, "stay in till P^*, do not drop out at P^*, drop out immediately at any $P > P^*$." Play at all histories other than CG3 are otherwise unchanged.

In CG3 when $N = 3$, bidders effectively play the "drainage tract auction" equilibrium considered in Engelbrecht-Wiggans, Milgrom and Weber (1983), Milgrom and Weber (1982b), and Hendricks, Porter and Wilson (1994), where the one bidder with private information plays a pure strategy and the uninformed bidders play mixed strategies and earn zero expected profit. We will construct such an equilibrium where the round-one winner plays a mixed strategy while the third bidder – whiever of bidders 1 and 2 lost the first round – does not bid.

Let i denote the identity of the first-round winner, and j whichever of 1 and 2 is not i. Recall that this continuation game includes the common beliefs that $s_1 = s_2 = 1$. Let $\underline{s}_3$ solve $P = c + \frac{\beta}{3}(2 + \underline{s}_3)$, or $\underline{s}_3 = 0$ if $P < c + \frac{2\beta}{3}$. Define a function $\Gamma : [\underline{s}_3, 1] \to \Re^+$ by

$$\Gamma(t) = c + \frac{\beta}{3}\left(2 + \frac{1}{2}t + \frac{1}{2}\frac{\underline{s}_3^2}{t}\right)$$

and note that $\Gamma'(t) = \frac{\beta}{3}\left(\frac{1}{2} - \frac{1}{2}\frac{\underline{s}_3^2}{t^2}\right) > 0$ for $t \in (\underline{s}_3, 1]$, so Γ is strictly increasing. Also note that $\Gamma(\underline{s}_3) = P$.

Define a probability distribution G with support $[P, \Gamma(1)]$ by

$$G(x) = \begin{cases} 1 & if \quad x \geq \Gamma(1) \\ \exp\left(-\int_x^{\Gamma(1)} \left(\frac{1}{c+\frac{\beta}{3}(2+\Gamma^{-1}(t))-t}\right) dt\right) & if \quad x \in [P, \Gamma(1)) \\ 0 & if \quad x < P \end{cases}$$

(Note that this distribution has a point mass at P.)

Bidding strategies are as follows:

- Bidder 3 bids $\Gamma(s_3)$ if $s_3 > \underline{s}_3$, and does not bid if $s_3 \leq \underline{s}_3$
- If $s_i = 1$, bidder i mixes such that the CDF of his bids is G (with the point mass at P indicating no new bid); if $s_i < 1$, bidder i does not bid
- Bidder j does not bid

To see this is an equilibrium, we first show that if bidder 3 plays this strategy, bidder i is indifferent between not bidding and bidding any number in the support of b_3. Treat the first-round bonus won by bidder i as a "sunk benefit", and note that (since bidder 3 never bids exactly P) not bidding is identical to bidding $P = \Gamma(\underline{s}_3)$ for bidder i. By bidding $\Gamma(x)$ for $x \in [\underline{s}_3, 1]$, bidder i wins when $s_3 < x$; if bidder i has signal s_i and believes $s_j = 1$, this gives expected payoff

$$\begin{aligned} x\left[c + \frac{\beta}{3}\left(1 + s_i + \frac{x}{2}\right) - \Gamma(x)\right] &= x\left[c + \frac{\beta}{3}\left(1 + s_i + \frac{x}{2}\right) - c - \frac{\beta}{3}\left(2 + \frac{1}{2}x + \frac{1}{2}\frac{\underline{s}_3^2}{x}\right)\right] \\ &= \frac{\beta}{3}x\left[s_i - 1 - \frac{1}{2}\frac{\underline{s}_3^2}{x}\right] \quad = \quad -\frac{\beta}{3}(1 - s_i)x - \frac{\beta}{6}\underline{s}_3^2 \end{aligned}$$

If $s_i = 1$, then this is the same for all x, so bidder i is indifferent among all bids in the range of Γ and not bidding (effectively bidding P), and is therefore best-responding by playing a mixed strategy.

As for bidder j, by those same calculations, if $s_i = 1$, bidding $\Gamma(x)$ would give an expected payoff of $\left(-\frac{\beta}{3}(1 - s_j)x - \frac{\beta}{6}\underline{s}_3^2\right) G(x)$, which is nonpositive for any x and any s_j, so j best-responds by not bidding.

Finally, knowing the value of the security is $c + \frac{\beta}{3}(2 + s_3)$, bidder 3 solves the maximization problem

$$\max_{b \in (P, \Gamma(1)] \cup \{0\}} \ln\left[G(b)\left(c + \frac{\beta}{3}(2 + s_3) - b\right)\right]$$

Taking the derivative with respect to b gives

$$\frac{g(b)}{G(b)} - \frac{1}{c + \frac{\beta}{3}(2 + s_3) - b} \quad = \quad \frac{1}{c + \frac{\beta}{3}(2 + \Gamma^{-1}(b)) - b} - \frac{1}{c + \frac{\beta}{3}(2 + s_3) - b}$$

This is positive when $\Gamma^{-1}(b) < s_3$, or $b < \Gamma(s_3)$ and negative when $b > \Gamma(s_3)$; so bidder 3's log-profit problem is strictly quasiconcave and solved at $b = \Gamma(s_3)$. For $s_3 > \underline{s}_3$,

$$c + \frac{\beta}{3}(2 + s_3) - \Gamma(s_3) \quad = \quad c + \frac{\beta}{3}(2 + s_3) - c - \frac{\beta}{3}\left(2 + \frac{1}{2}s_3 + \frac{1}{2}\frac{\underline{s}_3^2}{s_3}\right) \quad = \quad \frac{\beta}{3}\left(\frac{1}{2}s_3 - \frac{1}{2}\frac{\underline{s}_3^2}{s_3}\right) \quad > \quad 0$$

so bidder 3 prefers bidding to not bidding; and for any $s_3 \leq \underline{s}_3$, $c + \frac{\beta}{3}(2 + s_3) \leq P$, so bidder 3 prefers not bidding to bidding.

What we need from CG3 is for bidder j (the one who is not the first-round winner) to earn 0 payoffs (which is true since he doesn't bid), and if $P \geq P^{**}$, for player i to earn non-postiive ex-ante payoffs. To see the latter, note that

$$c + \frac{\beta}{3}(2 + \underline{s}_3) \quad = \quad P \quad \geq \quad P^{**} \quad = \quad c + \frac{\beta}{3}\left(2 + \sqrt{\frac{6\rho}{\beta}}\right)$$

means $\underline{s}_3 \geq \sqrt{\frac{6\rho}{\beta}}$; bidder i's second-round profit is therefore

$$-\frac{\beta}{3}(1 - s_i)x - \frac{\beta}{6}\underline{s}_3^2 \quad \leq \quad -\rho$$

and therefore even counting the first-round bonus, bidder i's profits are not positive.

What if $N = 2$?

For $N = 2$, the game is modified in the following way:

- First-round strategies are the same, but with P^* and P^{**} modified to be $P^* = c + \frac{\beta}{2}\left(1 + 2\sqrt{\frac{\rho}{\beta}}\right)$ and $P^{**} = c + \beta + \rho$
- If bidder 2 deviates in the first round but drops out at a price $P \leq P^*$, leaving bidder 1 as the first-round winner, CG2 is replaced by a drainage tract auction where $s_2 = 1$ and $s_1 \sim U[0, 1]$. Bidder 1's strategy is to bid $\Gamma_2(s_i) = c + \frac{\beta}{2}(1 + \frac{s_1}{2})$ if this is above P, and not bid otherwise. Bidder 2's strategy is to mix over not bidding and bidding in the range of Γ, such that the

distribution of his bids is

$$G_2(x) = \begin{cases} 1 & if \quad x \geq \Gamma_2(1) \\ \exp\left(-\int_x^{\Gamma(1)} \left(\frac{1}{c+\frac{\beta}{2}(1+\Gamma_2^{-1}(t))-t}\right) dt\right) & if \quad x \in [P, \Gamma_2(1)) \\ 0 & if \quad x < P \end{cases}$$

(with the point mass at P corresponding to not bidding).

- If bidder 2 deviates in the first round and wins when bidder 1 drops out at price P^*, CG2 is replaced by a different drainage tract auction where $s_2 = 1$ and $s_1 \sim U[0,1]$. Bidder 1's strategy is to bid $\Gamma_3(s_1) = c + \frac{\beta}{2}(1 + \frac{s_1}{2}) + \frac{\rho}{s_1}$ if $s_1 > 2\sqrt{\frac{\rho}{\beta}}$, and to not bid if $s_1 \leq 2\sqrt{\frac{\rho}{\beta}}$; bidder 2 mixes such that the distribution of his bids is

$$G_3(x) = \begin{cases} 1 & if \quad x \geq \Gamma_3(1) \\ \exp\left(-\int_x^{\Gamma(1)} \left(\frac{1}{c+\frac{\beta}{2}(1+\Gamma_3^{-1}(t))-t}\right) dt\right) & if \quad x \in [P, \Gamma_3(1)) \\ 0 & if \quad x < P \end{cases}$$

- If both bidders deviate in the first round, leading to either bidder 2 winning at a price $P < P^*$ or either bidder winning a price $P > P^*$, beliefs are $s_1 = s_2 = 1$; if $P < c + \beta$, both bidders bid $c + \beta$ in the second round, and if $P \geq c + \beta$, neither bidder bids in the second round.

In the case where 2 deviated but still dropped out at $P \leq P^*$, losing the premium, bidder 1's strategy implies that a bid of $\Gamma_2(x)$ by bidder 2 gives expected payoff

$$x\left(c + \frac{\beta}{2}(s_2 + \frac{x}{2}) - \Gamma_2(x)\right) \quad = \quad x\frac{\beta}{2}(s_2 - 1)$$

so if $s_2 = 1$, bidder 2 earns zero payoffs from any bid (and is thus content to mix). Given bidder 2's mixed strategy, bidder 1 (having already won the premium and believing that $s_2 = 1$), by bidding b, earns additional payoff $G_2(b)\left(c + \frac{\beta}{2}(1 + s_1) - b\right)$. Taking the log and then the first-order condition gives $\frac{1}{c+\frac{\beta}{2}(1+\Gamma_2^{-1}(b))-b} - \frac{1}{c+\frac{\beta}{2}(1+s_1)-b}$, which is positive when $b < \Gamma_2(s_1)$ and negative when $b > \Gamma_2(s_1)$, making $\Gamma_2(s_1)$ a best-response when it is greater than P (and making not bidding, equivalent to bidding P, a best-response when $\Gamma_2(s_1) < P$).

In the case where bidder 2 won the first round at price P^*, the logic is similar, but now bidding $\Gamma_3(x)$ earns bidder 2 a second-round payoff of

$$x\left(c+\frac{\beta}{2}(s_2+\frac{x}{2})-\Gamma_3(x)\right) \quad = \quad x\left(c+\frac{\beta}{2}(s_2+\frac{x}{2})-c-\frac{\beta}{2}(1+\frac{x}{2})-\frac{\rho}{x}\right) \quad = \quad x\frac{\beta}{2}(s_2-1)-\rho$$

so if $s_2 = 1$, any second-round bid gives the same payoff of $-\rho$, exactly countering the win of the premium in the first round. (Since bidder 2 won the premium, not bidding is the same as bidding P, and gives the same payoff of $-\rho$.) Again given bidder 2's mixing strategy, Γ_3 is a best-response for bidder 1.

If bidder 2 deviates, bidder 1 expects to drop out of the first round at price P^* and earn positive payoff in the second round; by dropping out earlier, he would induce bidder 2 to bid $c+\beta$ in the second round, so he would earn 0. So it is credible for bidder 1 to stay in until P^*. If bidder 2 deviates, then, he expects to either drop out before P^*, in which case he earns nothing; or to win the first round at P^*, in which case he wins the premium ρ but then earns payoff $-\rho$ in the second round. So deviating in the first round is not profitable for bidder 2. Second-round play on the equilibrium path is the same symmetric equilibrium as in the $N > 2$ cases.

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