
- Tuesday, we introduced mechanisms, direct-revelation mechanisms, and the revelation principle
- We showed that in the independent private values world, feasibility of a direct revelation mechanism was equivalent to four conditions:
  4. Monotonicity: \( Q_i(t_i) \) nondecreasing in \( t_i \)
  5. Envelope theorem: \( U_i(t_i) = U_i(a_i) + \int_{a_i}^{t_i} Q_i(s_i) ds_i \)
  6. Individual rationality for the lowest types: \( U_i(a_i) \geq 0 \)
  7. Non-stupidity of the allocation rule: \( \sum_i p_i(t) \leq 1, p_i(t) \geq 0 \)
- We then formulated the mechanism designer’s problem – maximize expected revenue, subject to these four conditions – and imposed the second one to recast the problem as maximizing
  \[
  t_0 + E_t \left\{ \sum_{i \in N} p_i(t) \left( t_i - \frac{1 - F_i(t_i)}{f_i(t_i)} - t_0 \right) \right\} - \sum_{i \in N} U_i(p, x, a_i)
  \]
  subject to conditions 4, 6, and 7

  This brings us to Lemma 3 from Myerson:

  **Lemma 1.** If \( p \) maximizes
  \[
  t_0 + E_t \left\{ \sum_{i \in N} p_i(t) \left( t_i - \frac{1 - F_i(t_i)}{f_i(t_i)} - t_0 \right) \right\}
  \]
  subject to \( Q_i \) increasing in \( t_i \) and \( p \) possible, and
  \[
  x_i(t) = t_i p_i(t) - \int_{a_i}^{t_i} p_i(t_{-i}, s_i) ds_i
  \]
  then \((p, x)\) is an optimal auction.

  The transfers are chosen to set \( U_i(a_i) = 0 \) and to satisfy the expected payoffs required by the envelope theorem. To see this, fix \( t_i \) and take the expectation over \( t_{-i} \), and we find
  \[
  E_{t_{-i}} x_i(t) = t_i Q(t_i) - \int_{a_i}^{t_i} Q_i(s_i) ds_i
  \]
or
\[ \int_{t_i}^{t_i^*} Q_i(s_i) ds_i = t_i Q_i(t_i) - E_{t-i} x_i(t_i, t-i) = U_i(t_i) \]
which is exactly the envelope theorem combined with \( U_i(a_i) = 0 \).

(The exact transfers \( x_i(t) \) are not uniquely determined by incentive compatibility and the allocation rule \( p \); what is uniquely pinned down is \( E_{t-i} x_i(t_i, t-i) \), because this is what’s payoff-relevant to bidder \( i \). The transfers above are just one rule that works.)

**Regularity**

With one additional assumption, things fall into place very nicely.

Suppose that each bidder’s bid distribution is such that
\[ t_i - 1 - \frac{F_i(t_i)}{F_i(t_i)} \]
is strictly increasing in \( t_i \) for each \( i \).

This is not that crazy an assumption. Most familiar distributions have increasing hazard rates – that is, \( \frac{f_i}{1 - F_i} \) is increasing, which would imply \( \frac{1 - E_i}{F_i} \) decreasing – this is a weaker condition, since \( \frac{f_i}{1 - F_i} \) is allowed to decrease, just not too quickly.

When the bid distributions are all regular, the optimal auction becomes this:

- Calculate \( c_i(t_i) = t_i - 1 - \frac{F_i(t_i)}{F_i(t_i)} \) for each player
- Give the good to the guy with the highest \( c_i(t_i) \), provided it’s higher than \( t_0 \); if not, keep the object
- Charge the transfers determined by incentive compatibility

Since \( c_i \) is increasing in \( t_i \), under this rule, \( Q_i(t_i) = 0 \) when \( t_i < c_i^{-1}(t_0) \), and the rest of the time,
\[ Q_i(t_i) = \prod_{j \neq i} F_j \left( c_j^{-1}(c_i(t_i)) \right) \]
which is increasing in \( t_i \), so the rule satisfies our two constraints, and it’s obvious that it maximizes the seller’s objective function. All that’s left is calculating \( x_i(t) \). But fixing everyone else’s type, \( p_i \) is 0 when \( c_i(t_i) < \max\{t_0, \max\{c_j(t_j)\}\} \) and 1 when \( c_i(t_i) > \max\{t_0, \max\{c_j(t_j)\}\} \), so this is just
\[ x_i(t) = t_i - \int_{t_i}^{t_i^*} ds_i = t_i - (t_i - t_i^*) = t_i^* \]

where \( t_i^* \) is the lowest type that \( i \) could have reported (given everyone else’s reports) and still won the object.
Symmetric IPV

In the case of symmetric IPV, each bidder’s $c$ function is the same as a function of his type, that is,

$$c_i(t_i) = c(t_i) = t_i - \frac{1 - F(t_i)}{f(t_i)}$$

which is strictly increasing in $t_i$. This means the bidder with the highest type has the highest $c_i$, and therefore gets the object; and so his payment is the reported type of the next-highest bidder, since this is the lowest type at which he would have won the object. Which brings us to our first claim:

**Theorem 1.** With symmetric independent private values, the optimal auction is a second-price auction with a reserve price of $c^{-1}(t_0)$.

Note, though, that even when $t_0 = 0$, this reserve price will be positive. The optimal auction is not efficient – since $c(t_i) < t_i$, the seller will sometimes keep the object even though the highest bidder values it more than him – but he never allocates it to the “wrong” bidder.

Also interesting is that the optimal reserve price under symmetric IPV does not depend on the number of bidders – it’s just $c^{-1}(t_0)$, regardless of $N$.

Asymmetric IPV

When the bidders are not symmetric, things are different. With different $F_i$, it will not always be true that the bidder with the highest $c_i$ also has the highest type; so sometimes the winning bidder will not be the bidder with the highest value. (As we’ll see later, efficiency is not standard in auctions with asymmetric bidders: even a standard first-price auction is sometimes not be won by the bidder with the highest value.)

One special case that’s easy to analyze: suppose every bidder’s bid is drawn from a uniform distribution, but uniform over different intervals. That is, suppose each $F_i$ is the uniform distribution over a (potentially) different interval $[a_i, b_i]$. Then

$$c_i(t_i) = t_i - \frac{1 - F_i(t_i)}{f_i(t_i)} = t_i - \frac{(b_i - t_i)/(b_i - a_i)}{1/(b_i - a_i)} = t_i - (b_i - t_i) = 2t_i - b_i$$

So the optimal auction, in a sense, penalizes bidders who have high maximum valuations. This is to force them to bid more aggressively when they have high values, in order to extract more revenue; but the price of this is that sometimes the object goes to the wrong bidder.

The Not-Regular Case

Myerson does solve for the optimal auction in the case where $c_i$ is not increasing in $t_i$, that is, where the auction above would not be feasible. Read the paper if you’re interested.
Finally, What About with Correlated Private Values

Myerson gives an example with correlated private values where the seller is able to hold the efficient auction (allocate the object to the bidder with the highest value), and extract the entire surplus from that bidder, so every bidder has expected payoff of 0 at every type. Obviously, this is the best that a seller could possibly hope to do, given individual rationality.

The example is discrete, but the intuition extends to continuous cases as well.
Two bidders, joint distribution of their types \((t_1, t_2)\) is \(Pr(100, 100) = Pr(10, 10) = \frac{1}{3}, \ Pr(10, 100) = \ Pr(100, 10) = \frac{1}{6}\). Suppose \(t_0 = 0\).

Consider the following mechanism:

- If both bidders report high types, flip a coin, and give one of them the object for 100
- If one reports high and one reports low, sell the high guy the object for 100, charge the low guy 30 and give him nothing
- If both report low, flip a coin, give one 15 and give the other 5 plus the object

First, we’ll verify that this is feasible, that is, that truthful revelation is an equilibrium; then we’ll consider the logic behind it.

If you have a low value, you know, conditional on that, the other guy is low with probability \(\frac{2}{3}\). So if you declare low, you get an expected payoff of

\[
\frac{1}{3}(-30) + \frac{2}{3}(15) = 0
\]

If you declared high, then more than two-thirds of the time, you have to pay 100 for the object which you value at 10, so there’s no temptation to deviate.

If you have a high value, you know that the other guy is also high with probability \(\frac{2}{3}\). By declaring high, you either get the object for 100 or get nothing, your payoff is 0. If you instead declared low, then with probability \(\frac{2}{3}\), you get charged 30 for nothing. With probability \(\frac{1}{3}\), you both declare low, so you either get 15 or you get the object plus 5, worth 105. So your expected payoff if you’re high and declare low is

\[
\frac{2}{3}(-30) + \frac{1}{3} \left( \frac{1}{2}(15) + \frac{1}{2}(105) \right) = -20 + \frac{1}{6}(120) = 0
\]

So this mechanism is individually rational and incentive-compatible, so it’s feasible. Also note that when the bidders reveal their true types, the auction is efficient (the higher-type bidder gets the object, and someone always gets it), and the bidders have 0 expected surplus, so the seller is sucking all the value possible out of the situation.

Now, where did this auction come from? In a “normal” setting, bidders with high types get positive expected payoffs, because you have to “leave them” some surplus in order to prevent them from lying and saying they have a low value. However, in this case, a bidder with a high type has information about the other type’s likely value. So what you do here is this: when a bidder claims
to have a low type, you also force him to accept a side-bet with you about the other bidder’s type; this bet has 0 expected-value when his type is actually low, but negative expected-value when his type is actually high. Since bidders are risk-neutral, this doesn’t hurt low types, but it does take back some surplus from high types.

Reimagine the auction above as a first-price auction, with bids constrained to be either 10 or 100 and ties broken by coin-flip, and where in addition, a bidder who bids 10 accepts the following bet: “If the other bidder also bids 10, I get an extra 15. If the other bidder bids 100, I lose 30.” For a bidder with a low type, this bet has 0 expected value; but for a bidder with a high type who is considering pretending to be low, who expects his opponent to bid truthfully, it has an expected value of \( \frac{1}{3}(15) + \frac{2}{3}(-30) = -15. \) Since a high bidder who bids low only wins with probability \( \frac{1}{6} \), this wipes out the surplus of 90 he would get the times he did win.

Cremer and McLean (1988, “Full Extraction of the Surplus in Bayesian and Dominant Strategy Auctions”, Econometrica 56.6) generalize this, showing when full surplus extraction is possible with correlated types, and their conclusion is that in terms of Bayesian implementation, it generally is. (Basically, you just need nondegenerate correlation “everywhere”.) However, this sort of mechanism does not seem very robust – more than most auctions, it is extremely sensitive to the seller being right about the true distribution of types. It also clearly requires risk-neutrality, since you need the low types to be willing to accept a large zero-expected-value bet. Finally, it’s very sensitive to collusion – both bidders bidding low is profitable for them (it’s even an equilibrium!). I don’t think Myerson is suggesting you would actually run this auction in this way – just making the point that when bidder values are correlated, the optimal auction may be complicated but may outperform anything you would think of as a “regular-looking” auction.
Bulow and Klemperer, “Auctions versus Negotiations”

I had initially put it later on the syllabus, but it seems pretty topical now. We just learned that with symmetric IPV and risk-neutral bidders, the best you can possibly do is to choose the perfect reserve price and run a second-price auction. This might suggest that choosing the perfect reserve price is important. There’s a paper by Bulow and Klemperer, “Auctions versus Negotiations,” that basically says: nah, it’s not that important. Actually, what they say is, it’s better to attract one more bidder than to run the perfect auction.

Suppose we’re in a symmetric IPV world where bidders’ values are drawn from some distribution $F$ on $[a, b]$, and the seller values the object at $t_0$. Bulow and Klemperer show the following: as long as $a \geq t_0$ (all bidders are “serious”), the optimal auction with $N$ bidders gives lower revenue than a second-price auction with no reserve price and $N+1$ bidders.

To see this, recall that we wrote the auctioneer’s expected revenue as

$$t_0 + E_t \left\{ \sum_{i \in N} p_i(t) \left( t_i - \frac{1 - F_i(t_i)}{f_i(t_i)} - t_0 \right) \right\} - \sum_{i \in N} U_i(p, x, a_i)$$

Consider mechanisms where $U_i(p, x, a_i) = 0$, and define the marginal revenue of bidder $i$ as

$$MR_i = t_i - \frac{1 - F_i(t_i)}{f_i(t_i)}$$

so expected revenue is

$$E_t \left\{ \sum_{i \in N} p_i(t)MR_i(t) + \left( 1 - \sum_{i \in N} p_i(t) \right) t_0 \right\}$$

So if we think of the seller as being another possible buyer, with marginal revenue of $t_0$, then the expected revenue is simply the expected value of the marginal revenue of the winner.

Jump back to the symmetric case, so $F_i = F$. Continue to assume regularity. In an ordinary second-price or ascending auction, with no reserve price, the object sells to the bidder with the highest type, which is also the bidder with the highest marginal revenue; so the expected revenue in this type of auction (what Bulow and Klemperer call an “absolute English auction”) is

$$\text{Expected Revenue} = E_t \max\{MR(t_1), MR(t_2), \ldots, MR(t_N)\}$$

(This is Bulow and Klemperer Lemma 1.)

The fact that expected revenue = expected marginal revenue of winner also makes it clear why the optimal reserve price is $MR^{-1}(t_0)$ – this replaces bidders with marginal revenue less than $t_0$ with $t_0$. So (counting the seller’s value from keeping the unsold object) an English auction with an optimal reserve price has expected revenue

$$\text{Expected Revenue} = E_t \max\{MR(t_1), MR(t_2), \ldots, MR(t_N), t_0\}$$

So here’s the gist of Bulow and Klemperer, “Auctions Versus Negotiations.” They compare the simple ascending auction with $N+1$ bidders, to the optimal auction with $N$ bidders. (We
discovered last week that with symmetric independent private values, the optimal auction is an ascending auction with a reserve price of $MR^{-1}(t_0)$. The gist of Bulow and Klemperer is that the former is higher, that is, that

$$E \max \{MR(t_1), MR(t_2), \ldots, MR(t_N), MR(t_{N+1})\} \geq E \max \{MR(t_1), MR(t_2), \ldots, MR(t_N), t_0\}$$

so the seller gains more by attracting one more bidder than by holding the “perfect” auction. (They normalize $t_0$ to 0, but this doesn’t change anything.)

Let’s prove this. The proof has a few steps.

First of all, note that the expected value of $MR(t_i)$ is $a$, the lower bound of the support. This is because

$$E_t MR(t_i) = E_t \left( t_i - \frac{1-F(t_i)}{f(t_i)} \right) = \int_a^b \left( t_i - \frac{1-F(t_i)}{f(t_i)} \right) f(t_i) dt_i = \int_a^b (t_i f(t_i) - 1 + F(t_i)) dt_i$$

Now, $tf(t) + F(t)$ has integral $tF(t)$, so this integrates to

$$t_i F(t_i) \big|_{t_i=a}^{b} - (b-a) = b - 0 - (b-a) = a$$

which by assumption is at least $t_0$. So $E(MR(t_i)) \geq t_0$.

Next, note that for fixed $x$, the function $g(y) = \max\{x, y\}$ is convex, so by Jensen’s inequality,

$$E_y \max\{x, y\} \geq \max\{x, E(y)\}$$

If we take an expectation over $x$, this gives us

$$E_x \{E_y \max\{x, y\}\} \geq E_x \max\{x, E(y)\}$$

or

$$E \max\{x, y\} \geq E \max\{x, E(y)\}$$

Now let $x = \max\{MR(t_1), MR(t_2), \ldots, MR(t_N)\}$ and $y = MR(t_{N+1})$;

$$E \max\{MR(t_1), MR(t_2), \ldots, MR(t_N), MR(t_{N+1})\} \geq$$

$$E \max\{MR(t_1), MR(t_2), \ldots, MR(t_N), E(MR(t_{N+1}))\} \geq$$

$$E \max\{MR(t_1), MR(t_2), \ldots, MR(t_N), t_0\}$$

and that’s the proof.

Bulow and Klemperer require independence and symmetry, but they do relax private values. They allow each bidder’s value to depend in pretty much any symmetric way on his own and his opponents’ signals, provided the game stays symmetric and the signals stay independent. In that case, the optimal auction is not an ascending auction with reserve price, but an ascending auction followed by a take-it-or-leave-it offer to the last man standing after everyone else has dropped out. They show that once again, with independent signals and risk-neutral bidders, adding one
more bidder and running a straight ascending auction is better in expectation than the optimal mechanism.

We just saw an example with correlated types where the optimal auction extracted all bidder surplus, and you can’t outperform that by finding one more bidder and going back to an ascending auction; but Bulow and Klemperer cite another result that when types are affiliated (a particular type of positive correlation that we’ll cover next week), the ascending-plus-offer auction is optimal among all mechanisms where losers don’t pay anything, the winner (when someone wins) has the highest type, and his payment is weakly increasing in his own type; so in this setting (affiliated types), adding one more bidder is still better than running the best among all “standard-looking” mechanisms.

So the results of Bulow and Klemperer are basically:

- Assume symmetry, risk-neutrality, and “serious bidders”, that is, $v \geq t_0$. If either (i) bidders have private values, or (ii) bidders’ signals are independent or affiliated, expected revenue is higher in an “absolute” $N+1$-bidder English auction than an $N$-bidder English auction followed by a take-it-or-leave-it offer

- With independent signals, the latter auction is optimal, so adding an extra bidder and running a simple ascending auction outperforms any feasible sales mechanism

- With affiliated signals, the latter auction is not optimal, and the optimal auction outperforms the former; but the latter is optimal among auctions where losers don’t pay, the winner has the highest signal, and his payment is weakly increasing in his own signal for any realization of his opponents’ signals; so adding a bidder outperforms any sales mechanism that meets these criteria

Finally (and leading to the title of the paper), Bulow and Klemperer point out that “negotiations” – really, any process for allocating the object and determining the price – cannot outperform the optimal mechanism, and therefore leads to lower expected revenue than a simple ascending auction with one more bidder. They therefore argue that a seller should never agree to an early “take-it-or-leave-it” offer from one buyer when the alternative is an ascending auction with at least one more buyer, etc.
Reserve Prices

We haven’t really done much with reserve prices yet. In our symmetric IPV world, we’ve shown that the optimal reserve price in a second-price auction is $MR^{-1}(t_0)$, which does not depend on the number of bidders. A couple of points worth making.

- In a second-price auction with a reserve price $r$, bidders with values $t > r$ still have a dominant strategy of bidding their type. Bidders below $r$ won’t submit serious bids. (Exactly what they do is undetermined, but doesn’t affect the outcome.)

- In a first-price auction with a reserve price $r$, bidders with values $t < r$ won’t submit serious bids. It’s also clear that $V(r) = 0$, that is, a bidder with value $t$ equal to the reserve price has expected payoff 0. Expected payoffs, and therefore equilibrium bids, can be calculated for types above $r$ using the envelope theorem, since the equilibrium will be symmetric and bids strictly increasing in types above $r$:

$$V(t) = V(r) + \int_r^t F^{-1}(s)ds = \int_r^t F^{-1}(s)ds = F^{-1}(t)(t - b(t))$$

- By the usual envelope-theorem logic, first- and second-price auctions with the same reserve price will be revenue-equivalent (and so the same reserve price is optimal) (The proofs are all basically identical to the proofs without reserve prices.)

The other thing we can do is calculate the ex-ante expected profit for the seller, and maximize it with respect to $r$, to show why the optimal reserve price is $c^{-1}(t_0)$ and does not depend on $N$. If we let $F_1$ be the CDF of the highest of all the bidders’ types, and $F_2$ the CDF of the second-highest, then we can write expected profit as

$$\pi(r) = (F_2(r) - F_1(r))(r - t_0) + \int_r^b (v - t_0)dF_2(v)$$

where $b$ is the upper limit of the support of the distribution of types. Differentiating with respect to $r$ gives

$$\pi'(r) = (f_2(r) - f_1(r))(r - t_0) + F_2(r) - F_1(r) - (r - t_0)f_2(r)$$

$$= F_2(r) - F_1(r) - (r - t_0)f_1(r)$$

Now, with symmetric IPV, $F_1(r) = F^N(r)$, and $F_2(r) = NF^{N-1}(r) - (N - 1)F^N(r)$; so the first-order condition becomes

$$0 = NF^{N-1}(r)(1 - F(r)) - (r - t_0)NF^{N-1}(r)f(r)$$

$$0 = 1 - F(r) - (r - t_0)f(r)$$

That is, the $N$ terms cancel in the first-order condition, and solving what’s left gives $r - t_0 = \frac{1 - F(r)}{f(r)}$, or $t_0 = c(r)$ (or $r = c^{-1}(t_0)$), as we already knew.