1. Discrete Auctions with Continuous Types

(a) Revenue equivalence does not hold: since types are continuous but bids are discrete, the bidder with the highest valuation does not always win.

(b) We do several iterations of eliminating strictly dominated strategies.
First, note that when \( v < 75 \), bidding 75 is strictly dominated: regardless of the other bidder’s strategy, you will win the auction at least half the time, giving strictly negative payoff, while bidding 0 ensures a payoff of 0.
Second, assume your opponent never bids 75 unless his value is at least 75. Then a bid of 50 will win with at least probability three-eighths. So when \( v < 50 \), a bid of 50 is strictly dominated by a bid of 0.
Third, assume your opponent bids below 50 when his value is below 50. Then a bid of 25 will win with at least probability one-fourth. So when \( v < 25 \), a bid of 25 is strictly dominated by a bid of 0.

(c) A bid of 75 gives an expected payoff of at most \( v - 75 \).
Equilibrium strategies must survive IESDS. Knowing that your opponent bids below 50 half the time, and below 75 three-fourths of the time, a bid of 50 must win with probability at least \( \frac{5}{8} \), and \( \frac{5}{8}(v - 50) > v - 75 \) for all \( v \leq 100 \).

(d) A bid of 0 gets a payoff of 0; a bid of 25 gets a payoff of at least \( \frac{1}{2}(v - 25) \), which is strictly higher for any \( v > 25 \).

(e) Argue by strict single-crossing differences. \( f(x, t) = \Pr(win|bid = x)(t - x) \). Let \( x' > x \).
If \( \Pr(win|bid = x') > \Pr(win|bid = x) \), then \( f(x', t) - f(x, t) \) is strictly increasing in \( t \).
If \( \Pr(win|bid = x') = \Pr(win|bid = x) > 0 \), then \( x' \) gives strictly lower expected payoff than \( x \) and need not be considered. If \( \Pr(win|bid = x') = \Pr(win|bid = x) = 0 \), then \( x' = x = 0 \) and the assumption that \( x' > x \) is violated.

(f) We have reduced potential equilibrium strategies to
\[
b(v) = \begin{cases} 
0 & \text{if } v \in [0, 25) \\
25 & \text{if } v \in (25, v^*) \\
50 & \text{if } v \in (v^*, 100] 
\end{cases}
\]
where \( v^* \) is some threshold value which must be at least 50. \( b(25) \) can be 0, 25, or any mix of the two; \( b(v^*) \) can be 25, 50, or any mix of the two.) To solve for \( v^* \), note that when my opponent plays the strategy above with cutoff \( v^* \),
\[
f(25, v) = \frac{25}{100}(v - 25) + \frac{v^* - 25}{100} \frac{1}{2}(v - 25)
\]
and
\[
f(50, v) = \frac{v^*}{100}(v - 50) + \frac{100 - v^*}{100} \frac{1}{2}(v - 50)
\]
Since we are looking for a symmetric equilibrium, I should be bidding with the same threshold $v^*$, which requires that $f(25,v^*) = f(50,v^*)$, or

$$\frac{25}{100}(v^* - 25) + \frac{v^* - 25}{100} \cdot \frac{1}{2}(v^* - 25) = \frac{v^*}{100}(v^* - 50) + \frac{100 - v^*}{100} \cdot \frac{1}{2}(v^* - 50)$$

Multiplying by 200 gives

$$50(v^* - 25) + (v^* - 25)^2 = 2v^*(v^* - 50) + (100 - v^*)(v^* - 50)$$
$$50v^* - 1250 + (v^*)^2 - 50v^* + 625 = 2(v^*)^2 - 100v^* + 100v^* - 5000 - (v^*)^2 + 50v^*$$
$$-1250 + 625 = -5000 + 50v^*$$
$$v^* = \frac{4375}{50} = 87.5$$

so the effectively unique symmetric equilibrium is to bid 0 when $v < 25$, 25 when $v \in (25, 87.5)$, and 50 when $v > 87.5$.

(g) The only strategies surviving deletion of dominated strategies is to bid 0 when $v < 50$, 50 when $v > 50$, either when $v = 50$.

(h) With probability $\frac{1}{4}$, both bidders have values below 50 and revenue is 0; with probability $\frac{3}{4}$, at least one bidder bids 50. Expected revenue is $\frac{1}{4}0 + \frac{3}{4}50 = 37.5$.

Revenue in an unconstrained second-price auction is $v^2$, which has expectation $\frac{100}{3}$ in this case. By revenue equivalence, expected revenue in an unconstrained first-price auction is also $\frac{100}{3}$. So expected revenue when bids are constrained to be either 50 or 0 is higher.

(i) Expected payoff is 0 for bidders with types $v < 50$, and $\frac{3}{4}(v - 50)$ for types $v > 50$.

In an ordinary first-price auction, $V(v) = \int_{0}^{v} F(s) ds = \int_{0}^{v} \frac{s}{100} ds = \frac{1}{100} \cdot \frac{v^2}{2}$. This is clearly higher when $v < 50$. For $v > 50$, the unconstrained auction is better whenever

$$\frac{1}{200}v^2 \geq \frac{3}{4}(v - 50)$$
$$v^2 \geq 150(v - 50)$$
$$v^2 - 150v + 7500 \geq 0$$
$$(v - 75)^2 + 1875 \geq 0$$

which is always. So every type of bidder prefers the unconstrained auction.

(This need not be universally true. If bids were instead constrained to be either 0 or 10, there would be a range of “intermediate” types who preferred the constrained to the unconstrained auction, while those with very low or very high types preferred the unconstrained.)
2. **K-Unit Auctions with Unit Demand**

(a) In any auction format where the bidders with the highest $K$ valuations win the objects,

$$f(x, t) = t \Pr(\text{win}|\text{bid} = x) - P(x)$$

where $P(x)$ is the unconditional expected payment given a bid of $x$. Then

$$V(t) = V(0) + \int_0^t f_t(b(s), s)ds = V(0) + \int_0^t \Pr(\text{win}|\text{type} = s)ds$$

Given $M$ independent draws from a distribution $F$, the probability that exactly $m$ of them are less than $k$ is

$$\binom{M}{m} F^m(k)(1 - F(k))^{M-m}$$

where (recalling introductory combinatorics)

$$\binom{M}{m} = \frac{M!}{m!(M-m)!}$$

Now, if I have type $t$, the probability of winning an object in an efficient equilibrium is the probability that I have one of the $K$ highest valuations, which is the probability that at least $N-K$ of my $N-1$ opponents have values below $t$. This occurs with probability

$$\sum_{i=N-K}^{N-1} \binom{N-1}{i} F^i(t)(1 - F(t))^{N-1-i}$$

Let $r(t)$ denote this sum.

From its definition, $f_t(b(s), s) = \Pr(\text{win}|\text{bid} = b(s)) = r(s)$, and so by the envelope theorem, $V(t) = V(0) + \int_0^t r(s)ds$, so $V(t) = \int_0^t r(s)ds$ in any auction where $V(0) = 0$.

By the same argument as before, expected revenue can be written as the expected value of the objects sold to their buyers, minus the expected payoffs realized by all the buyers:

$$\text{Expected Revenue} = E(v^1 + \ldots + v^K) - NE_t \left\{ \int_0^t r(s)ds \right\}$$

in any auction which is efficient and has $V(0) = 0$.

(b) The proof is exactly identical to the one we did for a single-object second-price auction. Let $B$ denote the $K^{th}$ highest of your opponents’ bids and $t$ your own private value. When $B > t$, a bid above $B$ gives negative expected payoff, all bids below $B$ (including $b = t$) give 0. When $B < t$, a bid below $B$ gives 0, all bids above $B$ (including $b = t$) give $t - B > 0$. When $B = t$, all bids (including $b = t$) give 0. So there is no $B$ (and therefore no opponent bid profile) at which any other bid gives higher payoff than bidding $b = t$.

(c) By revenue equivalence, the “pay-as-bid” auction has the same expected revenue as the $K + 1^{st}$-price auction described in the last part. When all bidders bid their types, the revenue in the $K+1^{st}$-price auction is $v^{K+1}$, so the expected revenue is simply $E\{v^{K+1}\}$. 


One of the nice properties of the uniform distribution is that if \( M \) random variables are independent draws from the uniform distribution on the interval \([a, b]\), the expected value of the \( m^{th} \) highest of them is \( a + \frac{M+1-m}{M+1}(b-a) \). (A proof of this, for the case where \( a = 0 \), is given at the end of these solutions.) So the expected revenue of this auction is \( 100\frac{N+1-(K+1)}{N+1} = 100\frac{N-K}{N+1} \).

3. Auctions and Price-Discriminating Monopolists

(a) As a function of \( Q \), price is \( P = F^{-1}(1 - Q) \) and revenue is \( \pi = QF^{-1}(1 - Q) \), so differentiating,

\[
\frac{d\pi}{dQ} = F^{-1}(1 - Q) - Q (F^{-1})'(1 - Q) = F^{-1}(1 - Q) - \frac{Q}{F'(F^{-1}(1 - Q))} = P - \frac{1 - F(P)}{f(P)}
\]

(The second equality comes from the fact that for any invertible function \( h \), \( (h^{-1})'(y) = \frac{1}{h'(h^{-1}(y))} \), which you can prove by differentiating both sides of \( h(h^{-1}(y)) = y \); the last is simply substituting \( P \) for \( F^{-1}(1 - Q) \) and, equivalently, \( 1 - F(P) \) for \( Q \)).

(b) If \( f \) is regular, then revenue is strictly increasing in \( Q \) up until \( P - \frac{1 - F(P)}{f(P)} = 0 \), and strictly decreasing from then on. (To put it another way, revenue is the integral of marginal revenue over all the customers served, so maximizing revenue means selling to those with positive marginal revenue; when \( f \) is normal, this is possible.)

(c) If \( v_i > P_i \), the customer’s surplus is

\[
v_i - P_i = \int_{P_i}^{v_i} dx = \int_{\xi_i}^{v_i} p_i(x) dx
\]

since \( p_i(x) \) is 1 above \( P_i \) and 0 below it. If \( v_i \leq P_i \), surplus and the integral are both 0.

(d) The gross value of all the goods sold in market \( i \) are

\[
\int\xi_i v p_i(v) f_i(v) dv
\]

and, by part (c) above, the share of this surplus going to consumers is

\[
\int\xi_i \left( \int_{\xi_i}^{v} p_i(x) dx \right) f_i(v) dv
\]

Subtracting these and summing over markets \( i \) give the seller’s revenue.

(e) Changing the order of integration in the second integral, and then evaluating the inner integral, gives

\[
\int\xi_i \left( \int_{\xi_i}^{v} p_i(x) dx \right) f_i(v) dv = \int\xi_i \int_{x}^{\xi_i} f_i(v) p_i(x) dv dx
\]

\[
= \int\xi_i (1 - F_i(x)) p_i(x) dx = \int\xi_i \frac{1 - F_i(x)}{f_i(x)} p_i(x) f_i(x) dx
\]
so we can rewrite revenue as

\[
\sum_{i=1}^{N} \left[ \int_{v_i}^{v_{i+1}} v p_i(v) f_i(v) dv - \int_{v_i}^{v_{i+1}} \left( \int_{v_i}^{v} p_i(x) dx \right) f_i(v) dv \right]
\]

\[
= \sum_{i=1}^{N} \int_{v_i}^{v_{i+1}} p_i(v) \left( v - \frac{1 - F_i(v)}{f_i(v)} \right) f_i(v) dv
\]

(f) We just showed that expected revenue is the sum of marginal revenue, summed (integrated) over every customer the monopolist sells to. If each \( F_i \) is regular and the monopolist is unconstrained, he should just sell to every customer with positive marginal revenue; he does this by setting \( P_i \) in each market such that

\[
0 = P_i - \frac{1 - F_i(P_i)}{f_i(P_i)}
\]

On the other hand, if the capacity constraint \( \sum_{i=1}^{N} \int_{v_i}^{v_{i+1}} p_i(v) f_i(v) dv \leq Q \) is binding, the monopolist maximizes revenue by selling all his available quantity to those customers with the highest marginal revenue, by setting

\[
P_i - \frac{1 - F_i(P_i)}{f_i(P_i)} = k
\]

where \( k \) solves

\[
\sum_{i=1}^{N} \int_{v_i}^{v_{i+1}} 1_{v - \frac{1 - F_i(v)}{f_i(v)} \geq k} f_i(v) dv = Q
\]

(g) i. The monopolist seeks to maximize \( q_1(1 - q_1) + q_2(2 - q_2) \), subject to \( q_1 + q_2 \leq 1 \). First, note that the marginal revenue in the second market is \( 2 - 2q_2 \), which is strictly positive whenever \( q_2 < 1 \), so there is no reason to set \( q_1 + q_2 < 1 \). The optimum can be found either by replacing \( q_1 \) with \( 1 - q_2 \), or by equating the marginal revenues in the two markets, \( 1 - 2q_1 = 2 - 2q_2 \), and solving this simultaneously with \( q_1 + q_2 = 1 \). In the latter approach, note that \( q_2 - q_1 = \frac{1}{2} \), and so \( q_2 = \frac{3}{4} \) and \( q_1 = \frac{1}{4} \), leading to prices \( p_1 = \frac{3}{4} \) and \( p_2 = \frac{5}{4} \).

ii. Recall that a bidder with type \( t \) drawn from the uniform distribution on \([a, b]\) has “marginal revenue” \( t - \frac{1 - F(t)}{f(t)} = t - \frac{(b-t)/(b-a)}{1/(b-a)} = t - (b - t) = 2t - b \). The optimal auction sells to the buyer with the higher marginal revenue, provided it is nonnegative. The second bidder’s marginal revenue is \( 2t - 2 \), which is always positive, so the seller always sells the item. The second bidder has higher marginal revenue whenever \( 2t_2 - 2 \geq 2t_1 - 1 \), or \( t_2 \geq t_1 + \frac{1}{2} \).

iii. In part (i), the monopolist earns revenue

\[
p_1q_1 + p_2q_2 = \frac{3}{4} \cdot \frac{1}{4} + \frac{5}{4} \cdot \frac{3}{4} = \frac{3}{16} + \frac{15}{16} = \frac{18}{16} = \frac{9}{8}
\]

To calculate part (ii), recall that expected revenue in an auction is the expected value of the marginal revenue of the winner, or

\[
E \max\{2t_1 - 1, 2t_2 - 2\}
\]
A bunch of algebra (or some clever trickery) establishes that this is $\frac{25}{24}$.

To understand the difference, note that the monopolist and the auctioneer face optimization problems with the same objective function, but the auctioneer’s problem has a tighter constraint. The monopolist, who faces deterministic measures of potential customers, faces only a single capacity constraint,

$$\sum_{i=1}^{2} \int_{v_i}^{p_i(v)} f_i(v) dv \leq 1$$

On the other hand, the monopolist, who faces stochastic types, faces a capacity constraint at each type profile:

$$\sum_{i=1}^{2} p_i(t_1, t_2) \leq 1$$

at every realized $(t_1, t_2)$. To put it another way, the monopolist must only satisfy his capacity constraint in expectation over types, while the auctioneer must satisfy the capacity constraint at each type profile. Thus, the monopolist’s problem is less constrained, and therefore has a higher value.

(The “problem” the auctioneer faces that the monopolist does not is that with some probability, he will have two bidders with high marginal revenues, and he can only sell to one of them; the monopolist can simultaneously sell to the high-marginal-revenue customers in each market. The two problems would match up perfectly if the auctioneer had a flexible capacity, and could sell up to one unit in expectation.)
Appendix – Derivation of $E(v^m)$ for the Uniform Distribution

In order to see this, consider the probability that the $m^{th}$ highest of $M$ independent draws from a distribution $F$ lies in the interval $[t, t + \epsilon]$. As $\epsilon$ goes to 0, the probability that more than one of the variables is in $[t, t + \epsilon]$ decreases as $\epsilon^2$, so up to order $\epsilon$, this probability is $M$ times the probability that a given draw from $F$ is in the interval, times the probability that of the remaining $M - 1$ draws, exactly $m - 1$ are above the interval and $M - m$ are below it, or

$$M(F(t + \epsilon) - F(t)) \left( \frac{M - 1}{M - m} \right) F^{M-m}(t)(1 - F(t + \epsilon))^{m-1}$$

Dividing by $\epsilon$ and taking the limit gives the marginal density of the $m^{th}$ highest as

$$f^m(t) = M \left( \frac{M - 1}{M - m} \right) f(t) F^{M-m}(t)(1 - F(t))^{m-1}$$

In the case of the uniform distribution on $[0, T]$, $F(t) = t/T$ and $f(t) = 1/T$, so this is

$$f^m(t) = \frac{1}{T^m} M \left( \frac{M - 1}{M - m} \right) t^{M-m}(T - t)^{m-1}$$

The expected value of the $m^{th}$ highest, therefore, is

$$\int_0^T t f^m(t) dt = \frac{1}{T^m} M \left( \frac{M - 1}{M - m} \right) \int_0^T t \times t^{M-m} \times (T - t)^{m-1} dt$$

Now, integrating $\int_0^T t^a(T - t)^b dt$ by parts, with $u = (T - t)^b$ and $dv = t^a dt$, gives

$$uv - \int v du = (T - t)^b \frac{t^{a+1}}{a+1} \bigg|_{t=0}^{t=T} + \int_0^T \frac{t^{a+1}}{a+1} b(T - t)^{b-1} dt$$

$$= \frac{b}{a+1} \int_0^T t^{a+1}(T - t)^{b-1} dt$$

since the first term vanishes at both $t = T$ and $t = 0$ for $b > 0$. Applying this repeatedly gives

$$\int_0^T t^a(T - t)^b dt = \frac{b \times (b - 1) \times \cdots \times 2 \times 1}{(a + 1)(a + 2) \cdots (a + b)} \int_0^T t^{a+b} dt = \frac{b!a!}{(a + b)!} \frac{1}{a + b + 1} T^{a+b+1}$$

Applying this with $a = M - m + 1$ and $b = m - 1$ gives

$$\int_0^T t f^m(t) dt = \frac{1}{T^m} M \left( \frac{M - 1}{M - m} \right) \frac{(M - 1)!}{(M - m)!(m - 1)!} \frac{(m - 1)!(M - m + 1)!}{M!} \frac{1}{M + 1} T^{M+1}$$

$$= \frac{M - m + 1}{M + 1} T$$