Appendix for: Lifecycle Wage Growth and Heterogeneous Human Capital

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January 16, 2012

In this appendix we expand upon the model presented in the text of the paper. First we generalize the model to allow for bargaining on the job rather than taking wages as exogenous. We then consider some special cases, first considering general human capital with search. We then show that firm and occupation/industry specific human capital can be considered special cases of our model. Finally, we generalize the model to allow for exogenous job separations.

Nash Bargaining Model with Home Production as Threat Point

There are many ways to model the wage determination process in an on-the-job search model such as Burdett and Mortensen (1998) and Cahuc et al. (2006).[1] To explore how the results are altered, we assume that wages and human capital investment, s_t, are determined by a generalized Nash bargaining model. We consider two different cases for the threat point. In this subsection we take it to be non-employment. One could justify our assumption either by assuming that other firms will not bargain when they know they will lose because there is some cost to making an offer or that outside offers are not verifiable to the current firm. The second case which we consider in the next subsection is based on Cahuc et al. (2006) in which the threat point is potentially determined by a the outside threat of another firm. Let δ denote the bargaining power of the worker in both cases. We assume that productivity at a firm can be written as $\pi' H_t$. This will imply that the model in the text is a special version of this model in which $\delta = 1$.

There are many extensions one could add to this model including more complicated commitment devices. For example, one could allow for back-loading of wages which would solve

[1] Burdett and Coles (2010)and Fu (2012) provide equilibrium versions of human capital models with search frictions.

one problem, but cause another: inefficient turnover. This new problem could be solved by allowing firms to pay other more productive firms to take on these contracts. We have not allowed for any of this here. In this sense this appendix represents a starting point for this type of model.

As in the text, we work backwards starting with the period 2 wage. Home production is linear with productivity $\pi'_h H_2$. Under generalized Nash bargaining, the wage at time 2 for a worker at a π firm is

$$w_2(H_2,\pi) = \delta\pi' H_2 + (1-\delta)\,\pi'_h H_2. \tag{1}$$

Now consider the first period. We focus on workers in the market sector whose outside option is the home sector. We define $V_1^h(H_1)$ to be the value function of the home sector. Presumably it involves the gains to human capital investment and search that we see for workers in the labor market. However, all that will be relevant to our analysis is $V_1^h(H_1)$ so we do not explicitly model where it comes from.[2]

As in the text, we let w_1 be the first period wage which is endogenous. Being somewhat loose about notation, the value function at time 1 for a worker employed during period 1 at a productivity π_1 firm is

$$V_1(H_1,\pi_1,w_1,s_1) = w_1 + \frac{1}{R}E_\pi \max\{\delta\pi'_1\mathscr{H}(s_1) + (1-\delta)\pi'_h\mathscr{H}(s_1)\,, \delta\pi'\mathscr{H}(s_1) + (1-\delta)\pi'_h\mathscr{H}(s_1)\}. \tag{2}$$

Everything except the productivity of the offering firm in the second period π is known in period 1, so the worker takes the expectation over draws of the outside offer's productivity π. We have implicitly assumed that the worker prefers the current firm to the home production sector during period 2.[3]

[2] A natural case would be that $V_1^h(H_1) = \pi'_h H_1 + \frac{1}{R}E_\pi \max\{\delta\pi' H_2 + (1-\delta)\pi'_h H_2, \pi'_h H_2\}$ and one would have to be explicit about the human capital production function while at home and the offer distribution for the unemployed.

[3] This is not guaranteed even though the worker is in the market during the first period. For example, if the worker could invest in human capital during the first period at the firm but not at home, it is possible that the worker would prefer the firm during the first period and then choose to be at home in the second. We rule this case out.

We also need to calculate the value of the match to the firm as of period 1 which is

$$\Pi_1(H_1,\pi_1,w_1,s_1) = \left(1 - \sum_{m=1}^{M} s_1^{(m)}\right)\pi_1' H_1 - w_1 + \frac{1}{R} Pr\left(\pi_1'\mathcal{H}(s_1) > \pi'\mathcal{H}(s_1)\right)(1-\delta)\left[\pi_1' - \pi_h'\right]\mathcal{H}(s_1). \tag{3}$$

The first terms on the right hand side of equation (3) represent the rents accrued during the first period. The last part represents the discounted expected rents from the second period where $Pr\left(\pi_1'\mathcal{H}(s_1) > \pi'\mathcal{H}(s_1)\right)$ is the probability that the worker will remain with the firm and $(1-\delta)\left[\pi_1' - \pi_h'\right]\mathcal{H}(s_1)$ are the second period rents if the worker remains.

The generalized Nash bargaining problem in this case is to choose w_1 and s_1 in the first period to solve

$$\max_{s_1,w_1}\left[V_1(H_1,\pi_1,w_1,s_1) - V_1^h(H_1)\right]^{\delta}\left[\Pi_1(H_1,\pi_1,w_1,s_1)\right]^{1-\delta} \tag{4}$$

subject to the human capital production function and $0 \le \sum_{m=1}^{M} s_1^{(m)} \le 1$.

The wage in this model is

$$\begin{aligned} w_1 =& \delta\left[\left(1 - \sum_{m=1}^{M} s_1^{(m)}\right)\pi_1' H_1 + \frac{1}{R} Pr\left(\pi_1'\mathcal{H}(s_1) > \pi'\mathcal{H}(s_1)\right)(1-\delta)\left[\pi_1' - \pi_h'\right]\mathcal{H}(s_1)\right] \\ &+ (1-\delta)\left[V_1^h(H_1) - \frac{1}{R} E_\pi\left(\max\{\delta\pi_1'\mathcal{H}(s_1) + (1-\delta)\pi_h'\mathcal{H}(s_1), \delta\pi'\mathcal{H}(s_1) + (1-\delta)\pi_h'\mathcal{H}(s_1)\}\right)\right] \end{aligned} \tag{5}$$

To get some intuition for expression (5) notice that the first term in brackets is the discounted present value of rent (net of the first period wage) that the firm receives. If $\delta = 1$, the worker would capture all of these rents. The second term in brackets is the reservation wage of the worker. It is the first period wage that makes $V_1^h(H_1) = V_1(H_1,\pi_1,w_1,s_1)$. If $\delta = 0$, the firm would pay the worker his reservation wage.

For any particular skill m, assuming we are at an interior, the first order condition for human capital is

$$\begin{aligned} \pi_1' H_1 =& \frac{1}{R}\Big[\Pr\left(\pi_1'\mathcal{H}(s_1) \le \pi'\mathcal{H}(s_1)\right) E_\pi\left(\pi^{(m)} - (1-\delta)\left(\pi^{(m)} - \pi_h^{(m)}\right) \mid \pi_1'\mathcal{H}(s_1) \le \pi'\mathcal{H}(s_1)\right) \\ &+ \Pr\left(\pi_1'\mathcal{H}(s_1) > \pi'\mathcal{H}(s_1)\right)\pi_1^{(m)} + \\ &+ \frac{\partial \Pr\left(\pi_1'\mathcal{H}(s_1) > \pi'\mathcal{H}(s_1)\right)}{\partial \mathcal{H}^{(m)}\left(s_1^{(m)}\right)}(1-\delta)\left(\pi_1' - \pi_h'\right)\mathcal{H}(s_1)\Big]\frac{\partial \mathcal{H}^{(m)}\left(s_1^{(m)}\right)}{\partial s_1^{(m)}}. \end{aligned} \tag{6}$$

The first terms in the brackets in equation (6) represent the worker's expected private return to human capital in cases in which the worker switches to a different firm in the second period. The second term represents the expected joint worker/firm return in the case in which the worker stays at the current firm. The final term arises from the fact that the first period firm loses all its revenue in the second period if the worker leaves. Thus the firm has some incentive to encourage the worker to invest in skills that are likely to make him stay. That is if

$$\frac{\partial \Pr\left(\pi_1' \mathscr{H}(s_1) > \pi' \mathscr{H}(s_1)\right)}{\partial \mathscr{H}^{(m)}\left(s_1^{(m)}\right)} > 0,$$

then skill m is more valuable at firm 1 than in the outside market and this leads to more investment in it. By contrast if this term is negative then this skill is more valuable to the outside market than the current firm and would lead to less investment.

In general, human capital investment will not be efficient. This is because there are three interested parties: the first period firm, the worker, and the second period firm. Since one can not contract with the second period firm during the first period and when $\delta < 1$ the second period firm receives some of the returns from human capital, there will in general be inefficient investment in human capital. To see the departure from the planner's problem note that in this framework if $\delta = 1$ the worker would capture all of the surplus both from the first period firm and from subsequent firms. Thus the case in which $\delta = 1$ in expression (6) represents the socially efficient amount of human capital investment. Now consider various scenarios.

First consider the case in which $\Pr\left(\pi_1' \mathscr{H}(s_1) > \pi' \mathscr{H}(s_1)\right) = 1$ so that the worker will surely remain with the first period firm. In this case δ would not enter expression (6) and human capital investment would be efficient.

Next consider the opposite extreme in which $\Pr\left(\pi_1' \mathscr{H}(s_1) > \pi' \mathscr{H}(s_1)\right) = 0$ so that the worker definitely leaves. The term in brackets in (6) would simplify to

$$E_\pi(\pi^{(m)} - (1-\delta)\left[\pi^{(m)} - \pi_h^{(m)}\right] \mid \pi_1' \mathscr{H}(s_1) \leq \pi' \mathscr{H}(s_1)).$$

We expect skills to typically be more valuable at the outside firm than at home so that $\pi^{(m)} > \pi_h^{(m)}$. In that case this expression is increasing in δ which means that if $\delta < 1$, workers underinvest in human capital. This is the classic holdup problem. Period 2 firms would secure some of the returns from the human capital investment but have no way to influence time 1 human capital accumulation. As we show below, this result comes completely from Nash bargaining as opposed to the heterogeneous human capital. That is, even with homogeneous human capital one would still see underinvestment in human capital when $\delta < 1$.

The other inefficiency appears in the third term in brackets in (6) above. The firm encourages the worker to over-invest in the types of skills that are more likely to lead him to remain with the first period firm and to under-invest in those that would encourage him to leave. If the worker has all the bargaining power ($\delta = 1$) this term would be zero. However, if $\delta < 1$ it leads to inefficient investment.[4] Note that in this case alternative assumptions about bargaining may eliminate this inefficiency. To see why, consider a case in which $\delta < 1$ and the worker receives an outside offer to which she is essentially indifferent. The first period firm is not indifferent as they receive positive rents in the second period. Since the worker can not commit to stay, ex-ante the firm encourages the worker to over-invest in a type of skill that makes this scenario less likely to happen. By contrast, we will show that the bargaining framework of Cahuc et al. (2006) yields a different result.

Nash Bargaining Model with Outside Firm as Threat Point

This subsection is a modified version of the Cahuc et al. (2006) model. We assume that in the first period the worker gets one offer so when they bargain with the firm over first period wages the threat point is non-employment. They begin the second period working for the period one firm and then get a new offer from a type π firm. At this point one of three things can happen:

1. If $\pi' H_2 > \pi_1' H_2$ they accept an offer from the new firm and for reasons justified in Cahuc et al. (2006), their threat point is the period one firm so their wage is

$$w_2(H_2, \pi_1, \pi) = \delta \pi' H_2 + (1-\delta)\pi_1' H_2.$$

2. The second possibility is that if the highest wage the new firm would pay is lower than home production (i.e. $\pi' H_2 < \pi_h' H_2$), the threat point is home production. In this case

$$w_2(H_2, \pi_1, \pi) = \delta \pi_2' H_2 + (1-\delta)\pi_h' H_2.$$

3. The third possibility is the intermediate case in which they stay at the current firm, but

[4] It also only appears when $\frac{\partial Pr(\pi_1' H_2 > \pi' H_2)}{\partial H_2^{(m)}} \neq 0$ so it will only enter if the density of $\pi' H_2$ evaluated at $\pi' H_1$ is not zero.

use their new wage offer as their threat point (i.e. $\pi_h' H_2 \le \pi' H_2 \le \pi_1' H_2$), then

$$w_2(H_2, \pi_1, \pi) = \delta \pi_1' H_2 + (1-\delta)\pi' H_2.$$

We make a couple of comments. First, we are assuming that productivity at the time one firm is higher than home production. As in the previous case this doesn't have to be the case, but considering the alternative would change things a bit without making it fundamentally different. The more important point is that the nature of our contracts is not identical to Cahuc et al. (2006). In their model, the firm and the worker agree to a contract and pay a certain wage until worker gets an outside offer. A problem here is that since both workers and firms are risk neutral there are many ways to write this contract that gives the same expected wage payment. We solve this problem by essentially getting rid of this commitment and assuming that in the absence of an acceptable outside offer, the firm uses home production as the threat point. This can just as easily be interpreted as a contract that pays $\delta\pi_1' H_2 + (1-\delta)\pi' H_2$ in the second period, which, in our view, is not obviously better or worse than assuming the first and second period wages are the same.

The value function at time 1 for a worker employed during period 1 at a productivity π_1 firm is

$$V_1(H_1, \pi_1, w_1, s_1) = w_1 + \frac{1}{R} E_\pi \left[w_2 \left(\mathscr{H}(s_1), \pi_1, \pi \right) \right].$$

We also need to calculate the value of the match to the firm as of period 1 which is

$$\begin{aligned}
&\Pi_1(H_1, \pi_1, w_1, s_1) \\
&= \left(1 - \sum_{m=1}^{M} s_1^{(m)}\right) \pi_1' H_1 - w_1 \\
&\quad + \frac{1}{R}(1-\delta)\left[\Pr\left(\pi' \mathscr{H}(s_1) < \pi_h' \mathscr{H}(s_1)\right)\left(\pi_1' \mathscr{H}(s_1) - \pi_h' \mathscr{H}(s_1)\right)\right. \\
&\quad\quad \left. + \Pr\left(\pi_h' \mathscr{H}(s_1) \le \pi' \mathscr{H}(s_1) \le \pi_1' \mathscr{H}(s_1)\right) E\left(\pi_1' \mathscr{H}(s_1) - \pi' \mathscr{H}(s_1) \mid \pi_h' \mathscr{H}(s_1) \le \pi' \mathscr{H}(s_1) \le \pi_1' \mathscr{H}(s_1)\right)\right]
\end{aligned}$$

The first terms on the right hand side of this equation represent the rents accrued during the first period. The last two terms represent the discounted expected rents from the second period for the cases in which the worker does not or does get a raise in response to an outside offer.

We solve the Generalized Nash bargaining problem as above which yields the wage

$$
\begin{aligned}
w_1 =& \delta \left[\left(1 - \sum_{m=1}^{M} s_1^{(m)} \right) \pi_1' H_1 \right. \\
&+ \frac{1}{R} (1-\delta) \left[\Pr\left(\pi' \mathscr{H}(s_1) < \pi_h' \mathscr{H}(s_1) \right) \left(\pi_1' \mathscr{H}(s_1) - \pi_h' \mathscr{H}(s_1) \right) \right. \\
&\left.\left. + \Pr\left(\pi_h' \mathscr{H}(s_1) \le \pi' \mathscr{H}(s_1) \le \pi_1' \mathscr{H}(s_1) \right) E\left(\pi_1' \mathscr{H}(s_1) - \pi' \mathscr{H}(s_1) \mid \pi_h' \mathscr{H}(s_1) \le \pi' \mathscr{H}(s_1) \le \pi_1' \mathscr{H}(s_1) \right) \right] \right] \\
&+ (1-\delta) \left[V_1^0(H_1) - \frac{1}{R} E_\pi \left[w_2 \left(\mathscr{H}(s_1), \pi_1, \pi \right) \right] \right]
\end{aligned}
$$

This is analogous to the case above.

For any particular skill m we can write the first order condition for human capital as

$$
\begin{aligned}
\pi_1' H_1 =& \frac{1}{R} \left[Pr\left(\pi_1' \mathscr{H}(s_1) \le \pi' \mathscr{H}(s_1) \right) E_\pi \left(\pi^{(m)} - (1-\delta) \left(\pi^{(m)} - \pi_h^{(m)} \right) \mid \pi_1' \mathscr{H}(s_1) \le \pi' \mathscr{H}(s_1) \right) \right. \\
&\left. + \Pr\left(\pi_1' \mathscr{H}(s_1) > \pi' \mathscr{H}(s_1) \right) \pi_1^{(m)} \right] \frac{\partial \mathscr{H}^{(m)}\left(s_1^{(m)} \right)}{\partial s_1^{(m)}}.
\end{aligned}
$$

This is very similar to the expression in the previous case except that we do not have the last term that corresponded to the inefficiency from the firm having a desire to keep the worker to stay. In this case the only inefficiency arrives from the holdup problem.

Special Case: General Human Capital Only

We consider the first generalized Nash bargaining model in which the threat point is home production but treat H_t as one dimensional. To justify people switching jobs we continue to allow for heterogeneity across firms in the one dimensional π. Investment is also one dimensional so we do not need to index it by m. In this case wages are

$$
\begin{aligned}
w_1 =& \delta \left[(1-s_1) H_1 + \frac{1}{R} \Pr(\pi_1 > \pi) (1-\delta) [\pi_1 - \pi_h] \mathscr{H}(s_1) \right] \\
&+ (1-\delta) \left[V_1^h(H_1) - \frac{1}{R} E_\pi \left(\max\{\delta\pi_1 + (1-\delta)\pi_h, \delta\pi + (1-\delta)\pi_h\} \right) \mathscr{H}(s_1) \right] \qquad (7)
\end{aligned}
$$

The first order condition for human capital is

$$\pi_1' H_1 = \frac{1}{R}\left[\Pr\left(\pi_1 \le \pi\right) E_\pi \left(\pi - (1-\delta)\left(\pi - \pi_h\right) \mid \pi_1 \le \pi\right)\right.$$
$$\left. + \left[\Pr\left(\pi_1 > \pi\right)\pi_1\right]\right] \frac{\partial \mathcal{H}^{(m)}\left(s_1^{(m)}, H_1\right)}{\partial s_1^{(m)}}.$$

One can see in this case that the holdup problem still arises. The second inefficiency from workers investing in specialized skill no longer holds because turnover does not depend on human capital investment.

Special Case: Firm Specific Human Capital and General Human Capital

In this case we take H_t to be two dimensional. The first dimension is general human capital and has value at both firms. The second dimension is only valuable at firm 1. Thus we can use the notation π to denote the one dimensional payoff to this general skill. Home production works much the same way as the first dimension is valuable at home, but the second is not. The wage in the first period is

$$w_1 = \delta\left[\left(1 - \sum_{m=1}^{2} s_1^{(m)}\right)\pi_1' H_1 + \frac{1}{R}\Pr\left(\pi_1' H_2 > \pi H_2^{(1)}\right)(1-\delta)\left[\pi_1' H_2 - \pi_h H_2^{(1)}\right]\right]$$
$$+ (1-\delta)\left[V_1^h(H_1) - \frac{1}{R}E_\pi\left(\max\{\delta\pi_1' H_2 + (1-\delta)\pi_h H_2^{(1)}, \delta\pi' H_2 + (1-\delta)\pi_h H_2^{(1)}\}\right)\right] \qquad (8)$$

What is particularly nice about this case is the contrast between the two first order conditions. First for general human capital:

$$\pi_1' H_1 = \frac{1}{R}\left[\Pr\left(\pi_1' \mathcal{H}(s_1) \le \pi \mathcal{H}^{(1)}\left(s_1^{(1)}\right)\right) E_\pi\left(\pi - (1-\delta)\left(\pi - \pi_h\right) \mid \pi_1' \mathcal{H}(s_1) \le \pi \mathcal{H}^{(1)}\left(s_1^{(1)}\right)\right)\right.$$
$$+ \Pr\left(\pi_1' \mathcal{H}(s_1) > \pi \mathcal{H}^{(1)}\left(s_1^{(1)}\right)\right)\pi_1^{(1)} +$$
$$\left. + \frac{\partial \Pr\left(\pi_1^{(1)}\mathcal{H}^{(1)}\left(s_1^{(1)}\right) + \pi_1^{(2)}\mathcal{H}^{(2)}\left(s_1^{(2)}\right) > \pi \mathcal{H}^{(1)}\left(s_1^{(1)}\right)\right)}{\partial \mathcal{H}^{(1)}\left(s_1^{(1)}\right)}(1-\delta)\left(\pi_1' \mathcal{H}(s_1) - \pi_h \mathcal{H}^{(1)}\left(s_1^{(1)}\right)\right)\right] \frac{\partial \mathcal{H}^{(1)}\left(s_1^{(1)}\right)}{\partial s_1^{(1)}}. \qquad (9)$$

Then for specific:

$$\pi_1' H_1 = \frac{1}{R}\left[\Pr\left(\pi_1'\mathcal{H}(s_1) > \pi\mathcal{H}^{(1)}\left(s_1^{(1)}\right)\right)\pi_1^{(1)} + \right.$$
$$\left. + \frac{\partial \Pr\left(\pi_1^{(1)}\mathcal{H}^{(1)}\left(s_1^{(1)}\right) + \pi_1^{(2)}\mathcal{H}^{(2)}\left(s_1^{(2)}\right) > \pi\mathcal{H}^{(1)}\left(s_1^{(1)}\right)\right)}{\partial\mathcal{H}^{(2)}\left(s_1^{(2)}\right)}(1-\delta)\left(\pi_1' - \pi_h'\right)\mathcal{H}(s_1)\right]\frac{\partial\mathcal{H}^{(2)}\left(s_1^{(2)}\right)}{\partial s_1^{(2)}}. \tag{10}$$

The holdup problem arises for the general model but not for the specific one since the outside firm does not benefit from specific human capital invest.

We can also sign the other inefficiency. The specific human capital model is straight forward as

$$\frac{\partial \Pr\left(\pi_1^{(1)}\mathcal{H}^{(1)}\left(s_1^{(1)}\right) + \pi_1^{(2)}\mathcal{H}^{(2)}\left(s_1^{(2)}\right) > \pi\mathcal{H}^{(1)}\left(s_1^{(1)}\right)\right)}{\partial\mathcal{H}^{(2)}\left(s_1^{(2)}\right)}$$

is clearly positive. This means that firms will over-invest in the specific skill. Somewhat more subtly, the inequality $\pi_1^{(1)}\mathcal{H}^{(1)}\left(s_1^{(1)}\right) + \pi_1^{(2)}\mathcal{H}^{(2)}\left(s_1^{(2)}\right) > \pi\mathcal{H}^{(1)}\left(s_1^{(1)}\right)$ is only possible if $\pi > \pi_1^{(1)}$ because $\pi_1^{(2)}\pi_1^{(2)}\mathcal{H}^{(2)}\left(s_1^{(2)}\right) > 0$. This means that

$$\frac{\partial Pr\left(\pi_1^{(1)}\mathcal{H}^{(1)}\left(s_1^{(1)}\right) + \pi_1^{(2)}\mathcal{H}^{(2)}\left(s_1^{(2)}\right) > \pi\mathcal{H}^{(1)}\left(s_1^{(1)}\right)\right)}{\partial\mathcal{H}^{(1)}\left(s_1^{(1)}\right)}$$

must be negative and thus there is underinvestment in general human capital.

Special Case: Industry or Occupational Specific Human Capital

In this subsection we consider a case in which human capital is either completely industry or occupation specific. We do not explicitly distinguish between occupation and industry specific human capital, but allow two types of jobs (1 and 2) with job specific human capital. That is, human capital is two-dimensional and only the first dimension is value at type 1 jobs and only the second is available at type 2 jobs. Without loss of generality, we assume that the worker is at a type 1 job during the first period. Let μ_1 be the probability that the job offer in period 2 is of type 1. Since neither type of human capital should be useful at home, we write home production as the one dimensional object H_h.

In this case the value function as

$$
\begin{aligned}
&V_1(H_1,\pi_1,w_1,s_1)\\
&= w_1+\frac{1}{R}\Big[\mu_1 E_\pi \max\left\{\delta\pi_1^{(1)}\mathscr{H}^{(1)}\left(s_1^{(1)}\right),\delta\pi^{(1)}\mathscr{H}^{(1)}\left(s_1^{(1)}\right)\right\}\\
&\quad +(1-\mu_1)E_\pi \max\left\{\delta\pi_1^{(1)}\mathscr{H}^{(1)}\left(s_1^{(1)}\right),\delta\pi^{(2)}\mathscr{H}^{(2)}\left(s_1^{(2)}\right)\right\}+(1-\delta)H_h\Big].
\end{aligned}
$$

This is just a special case of our model above. It gives the wage

$$
\begin{aligned}
w_1 =&\delta\Bigg[\left(1-\sum_{m=1}^{2}s_1^{(m)}\right)\pi_1^{(1)}H_1^{(1)}\\
&+\frac{1}{R}\Big[\mu_1 \Pr\left(\pi_1^{(1)}\mathscr{H}_2^{(1)}\left(s_2^{(1)}\right)>\pi^{(1)}\mathscr{H}_2^{(1)}\left(s_2^{(1)}\right)\right)\\
&\quad +(1-\mu_1)\Pr\left(\pi_1^{(1)}\mathscr{H}_2^{(1)}\left(s_2^{(1)}\right)>\pi^{(2)}\mathscr{H}_2^{(2)}\left(s_2^{(2)}\right)\right)\Big](1-\delta)\left[\pi_1^{(1)}H_2^{(1)}-H_h\right]\Bigg]\\
&+(1-\delta)\Big[V_1^h(H_1)-\frac{\mu_1}{R}E_\pi \max\left\{\delta\pi_1^{(1)}H_2^{(1)},\delta\pi^{(1)}\mathscr{H}_2^{(1)}\left(s_2^{(1)}\right)\right\}\\
&-\frac{(1-\mu_1)}{R}E_\pi \max\left\{\delta\pi_1^{(1)}\mathscr{H}_2^{(1)}\left(s_2^{(1)}\right),\delta\pi^{(2)}\mathscr{H}_2^{(2)}\left(s_2^{(2)}\right)\right\}-\frac{(1-\delta)H_h}{R}\Big].
\end{aligned}
$$

This is analogous to what we have shown with wages before. The first order conditions are more interesting,

$$
\begin{aligned}
\pi_1' H_1 =&\frac{1}{R}\Big[\mu_1 \Pr\left(\pi_1^{(1)}\le\pi^{(1)}\right)\delta E_\pi\left(\pi^{(1)}\mid\pi_1^{(1)}\le\pi^{(1)}\right)\\
&+\left[\mu_1 \Pr\left(\pi_1^{(1)}>\pi^{(1)}\right)+(1-\mu_1)\Pr\left(\pi_1^{(1)}\mathscr{H}^{(1)}\left(s_1^{(1)}\right)>\pi^{(2)}\mathscr{H}^{(1)}\left(s_1^{(1)}\right)\right)\right]\pi_1^{(1)}+\\
&+(1-\mu_1)\frac{\partial \Pr\left(\pi_1^{(1)}\mathscr{H}^{(1)}\left(s_1^{(1)}\right)>\pi^{(2)}\mathscr{H}^{(2)}\left(s_1^{(2)}\right)\right)}{\partial\mathscr{H}^{(1)}\left(s_1^{(1)}\right)}(1-\delta)\left(\pi_1^{(1)}\mathscr{H}^{(1)}\left(s_1^{(1)}\right)-H_h\right)\Big]\frac{\partial\mathscr{H}^{(1)}\left(s_1^{(1)}\right)}{\partial s_1^{(1)}}.
\end{aligned}
$$

Note that in this case $\frac{\partial \Pr\left(\pi_1^{(1)}\mathscr{H}^{(1)}\left(s_1^{(1)}\right)>\pi^{(2)}\mathscr{H}^{(2)}\left(s_1^{(2)}\right)\right)}{\partial\mathscr{H}^{(1)}\left(s_1^{(1)}\right)}>0$. Thus the third term leads to over-investment in skill 1 relative to the social optimum. Since the first term represents underinvestment due to the holdup problem we can not sign the overall effect.

Next consider the first order condition for skill 2

$$
\begin{aligned}
\pi_1' H_1 =&\frac{1}{R}\Big[(1-\mu_1)\Pr\left(\pi_1^{(1)}\mathscr{H}^{(1)}\left(s_1^{(1)}\right)\le\pi^{(2)}\mathscr{H}^{(2)}\left(s_1^{(2)}\right)\right)\delta E_\pi\left(\pi^{(2)}\mid\pi_1^{(1)}\mathscr{H}^{(1)}\left(s_1^{(1)}\right)\le\pi^{(2)}\mathscr{H}^{(2)}\left(s_1^{(2)}\right)\right)\\
&+\frac{\partial \Pr\left(\pi_1^{(1)}\mathscr{H}^{(1)}\left(s_1^{(1)}\right)>\pi^{(2)}\mathscr{H}^{(2)}\left(s_1^{(2)}\right)\right)}{\partial\mathscr{H}^{(2)}\left(s_1^{(2)}\right)}(1-\delta)\left(\pi_1^{(1)}\mathscr{H}^{(1)}\left(s_1^{(1)}\right)-H_h\right)\Big]\frac{\partial\mathscr{H}^{(2)}\left(s_1^{(2)}\right)}{\partial s_1^{(2)}}.
\end{aligned}
$$

This second expression might not make sense since it may be reasonable to believe that one could not invest in type 2 human capital if one is at a type 1 job. In that case we would typically assume that $\frac{\partial \mathcal{H}^{(2)}\left(s_1^{(2)}\right)}{\partial s_1^{(2)}} = 0$ in which this expression would be degenerate. However, if we do allow for such investment we would get underinvestment in it. As usual the holdup problem leads to underinvestment. Furthermore, $\frac{\partial \Pr\left(\pi_1^{(1)}\mathcal{H}^{(1)}\left(s_1^{(1)}\right)>\pi^{(2)}\mathcal{H}^{(2)}\left(s_1^{(2)}\right)\right)}{\partial \mathcal{H}^{(2)}\left(s_1^{(2)}\right)} < 0$ so this term also leads to underinvestment in human capital.

Allowing Exogenous Destruction of Jobs

In this subsection we extend the model by allowing jobs to be exogenously destroyed at the end of the period at rate λ. After a job has been destroyed, the worker makes a decision about home production or an offer from a market job.

In this case the value function is now

$$\begin{aligned} V_1(H_1,\pi_1,w_1,s_1) =& w_1 + \frac{1}{R}(1-\lambda)\left(\delta E_\pi \max\{\pi_1'\mathcal{H}(s_1),\pi'\mathcal{H}(s_1)\} + (1-\delta)\pi_h'\mathcal{H}(s_1)\right) \\ &+ \frac{1}{R}\lambda\left(\delta E_\pi \max\{\pi_h'\mathcal{H}(s_1),\pi'\mathcal{H}(s_1)\} + (1-\delta)\pi_h'\mathcal{H}(s_1)\right) \end{aligned}$$

Solving a similar problem as before we see the wage:

$$\begin{aligned} w_1 =& \delta\left[\left(1-\sum_{m=1}^{M} s_1^{(m)}\right)\pi_1' H_1 + \frac{1}{R}\Pr\left(\pi_1'\mathcal{H}(s_1) > \pi'\mathcal{H}(s_1)\right)(1-\delta)\left[\pi_1'-\pi_h'\right]\mathcal{H}(s_1)\right] \\ &+ (1-\delta)\left[V_1^0(H_1) - \frac{1}{R}\left((1-\lambda)\,\delta E_\pi \max\{\pi_1'\mathcal{H}(s_1),\pi'\mathcal{H}(s_1)\}\right.\right. \\ &\left.\left. + \lambda\left(\delta E_\pi \max\{\pi_h'\mathcal{H}(s_1),\pi'\mathcal{H}(s_1)\} + (1-\delta)\pi_h'\mathcal{H}(s_1)\right)\right)\right]. \end{aligned}$$

This is analogous to the standard case. The first order condition can now be written as

$$\begin{aligned} \pi_1' H_1 =& \frac{1}{R}\left[(1-\lambda)\frac{1}{R}\left[\frac{\partial \Pr\left(\pi_1'\mathcal{H}(s_1) > \pi'\mathcal{H}(s_1)\right)}{\partial H_2^{(m)}}(1-\delta)\left[\pi_1'-\pi_h'\right]\mathcal{H}(s_1)\right]\right. \\ &+ (1-\lambda)\frac{1}{R}\left[\Pr\left(\pi_1'\mathcal{H}(s_1) > \pi'\mathcal{H}(s_1)\right)\pi_1^{(m)}\right] \\ &+ (1-\lambda)\left[\Pr(\pi_1'\mathcal{H}(s_1) < \pi'\mathcal{H}(s_1))E_\pi(\delta\pi^{(m)} + (1-\delta)\pi_h^{(m)} \mid \pi_1'\mathcal{H}(s_1) < \pi'\mathcal{H}(s_1))\right] \\ &+ \delta\lambda\left[\Pr(\pi_h'\mathcal{H}(s_1) > \pi'\mathcal{H}(s_1))\pi_h^{(m)} + \Pr(\pi_h'\mathcal{H}(s_1) < \pi'\mathcal{H}(s_1))E_\pi(\pi^{(m)} \mid \pi_h'\mathcal{H}(s_1) < \pi'\mathcal{H}(s_1))\right]. \end{aligned}$$

The only major difference is the last term in the expression which picks up the worker's return in the case in which they lose their job. This is another channel through which the holdup problem will discourage human capital investment relative to the efficient level. Other than that the same basic intuition applies.

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