Appendix

Model

Consider the following unobserved-component model of real exchange rate (q_t) which consists of a stationary component (x_t) and a random walk component (y_t).

$$q_t = y_t + x_t \tag{1}$$

$$y_t = y_{t-1} + v_t, \quad v_t \sim N(0, \sigma^2_{1,t}) \tag{2}$$

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + e_t, \quad e_t \sim N(0, \sigma^2_{2,t}), \tag{3}$$

where the variances of the two independent shocks, $\sigma^2_{1,t}$ and $\sigma^2_{2,t}$ are assumed to be heteroskedastic. In particular, we assume that they depend on two independent discrete-valued first-order Markov switching variables, S_{1t} and S_{2t} ($S_{1t} = 1, 2,\ or\ 3$; and $S_{2t} = 1, 2,\ or\ 3$) which evolve independently of v_t and e_t, according to the following transition probabilities:

$$Pr[S_{1t} = j | S_{1,t-1} = i] = p_{1,ij}; \quad i, j = 1, 2, 3; \ \sum_{j=1}^{3} p_{1,ij} = 1 \tag{4}$$

$$Pr[S_{2t} = j | S_{2,t-1} = i] = p_{2,ij}; \quad i, j = 1, 2, 3; \ \sum_{j=1}^{3} p_{2,ij} = 1 \tag{5}$$

Thus, we can write the variances of the shocks in the following ways:

$$\sigma^2_{1,t} = \sigma^2_{v1} S_{1,1t} + \sigma^2_{v2} S_{1,2t} + \sigma^2_{v3} S_{1,3t}, \qquad \sigma^2_{v1} < \sigma^2_{v2} < \sigma^2_{v3}, \tag{6}$$

$$\sigma^2_{2,t} = \sigma^2_{e1} S_{2,1t} + \sigma^2_{e2} S_{2,2t} + \sigma^2_{e3} S_{2,3t}, \qquad \sigma^2_{e1} < \sigma^2_{e2} < \sigma^2_{e3}, \tag{7}$$

where $S_{n,kt} = 1$ if $S_{nt} = k$ and $S_{n,kt} = 0$, otherwise ($k = 1, 2, 3$; $n = 1, 2$).

Writing the above model in a state-space form, we have:

$$\textbf{Measurement Equation}: \; q_t = Hc_t \tag{A1}$$

$$\textbf{Transition Equation}: \; c_t = Fc_{t-1} + \epsilon_t, \tag{A2}$$

$$E(\epsilon_t \epsilon_t') = R_t \tag{A3}$$

$$\text{where } H' = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},\; z_t = \begin{bmatrix} y_t \\ x_t \\ x_{t-1} \end{bmatrix},\; F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \phi_1 & \phi_2 \\ 0 & 1 & 0 \end{bmatrix},\; \epsilon_t = \begin{bmatrix} v_t \\ e_t \\ 0 \end{bmatrix},\; \text{and } R_t = \begin{bmatrix} \sigma_v^2 & 0 & 0 \\ 0 & \sigma_{e,t}^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Inferences of the Model

Denote $\tilde{Q}^t = [q_1\ q_2\ ...\ q_t]'$ as the vector of real exchange rates up to time t, $\tilde{C}^t = [c_1'\ c_2'\ ...\ c_t']'$ as the set of state vectors up to time t (here, the first and second rows of $\tilde{C}^t$ can be defined as $\tilde{Z}^t = [z_1'\ z_2'\ ...\ z_t']'$, where $z_t = [\,y_t \quad x_t\,]'$), $\tilde{S}_n^t = [S_{n1}\ S_{n2}\ ...\ S_{nt}]'$, $n = 1, 2$, as the vectors of Markov-switching variables up to time t, and finally, $\tilde{\theta} = [\tilde{\sigma}_v^{2\prime}\ \tilde{\sigma}_e^{2\prime}\ \tilde{p}_1'\ \tilde{p}_2'\ \tilde{\phi}']'$ as the vector of all parameters of the model, where $\tilde{\sigma}_v^2 = [\,\sigma_{v1}^2 \quad \sigma_{v2}^2 \quad \sigma_{v3}^2\,]'$, $\tilde{\sigma}_e^2 = [\,\sigma_{e1}^2 \quad \sigma_{e2}^2 \quad \sigma_{e3}^2\,]'$, $\tilde{p}_n' = [\,p_{n,11}, p_{n,12}, p_{n,21}, p_{n,22}, p_{n,31}, p_{n,32}\,]'$, $(n = 1, 2)$, and $\tilde{\phi} = [\,\phi_1 \quad \phi_2\,]'$. Our purpose is to make inferences on $\tilde{C}^T$ (more specifically, $\tilde{Z}_t$, the first two rows $\tilde{C}_T$), $\tilde{S}_n^T$, $n = 1, 2$, and on $\tilde{\theta}$ conditional on data on real exchange rates, $\tilde{Q}^T$. Maximum likelihood estimation of similar models and ways to make inferences on $\tilde{C}^T$ conditional on parameter estimates have been proposed by Kim (1994). [For applications of the maximum likelihood estimation of the unobserved-component models with Markov-switching heteroskedasticity, see Kim (1993) and Kim and Kim (1996).] However, Kim's methods are based on approximations. In this paper, we adopt the Bayesian Gibbs sampling approach proposed by Carter and Kohn (1994) in making inferences of the model based on the joint posterior density, $p(\tilde{C}^T, \tilde{S}_1^T, \tilde{S}_2^T, \tilde{\theta}|\tilde{Q}^T)$, of $\tilde{C}^T$, $\tilde{S}_1^T$,$\tilde{S}_1^T$, and $\tilde{\theta}$.

As docomented in Gelfand and Smith (1990) and Carter and Kohn (1994), Gibbs sampling in the present context generates $\tilde{C}^T$, $\tilde{S}_1^T$, $\tilde{S}_2^T$, and $\tilde{\theta}$ from conditional densities $p(\tilde{C}^T|\tilde{S}_1^T, \tilde{S}_2^T, \tilde{\theta}, \tilde{Q}^T)$, $p(\tilde{S}_1^T|\tilde{C}^T, \tilde{S}_2^T, \tilde{\theta}, \tilde{Q}^T)$, $p(\tilde{S}_2^T|\tilde{C}^T, \tilde{S}_1^T, \tilde{\theta}, \tilde{Q}^T)$, $p(\tilde{\theta}|\tilde{S}_1^T, \tilde{S}_2^T, \tilde{C}^T, \tilde{Q}^T)$ until eventually $\{\tilde{C}^T, \tilde{S}_1^T, \tilde{S}_2^T, \tilde{\theta}\}$ is generated from the joint posterior density $p(\tilde{C}^T, \tilde{S}_1^T, \tilde{S}_2^T, \tilde{\theta}|\tilde{Q}^T)$. [Carter and Kohn, 1994, p.542-543]. The following explains how each of $\tilde{C}^T$, $\tilde{S}_1^T$, $\tilde{S}_2^T$, and $\tilde{\theta}$ can be generated from an appropriate conditional distribution, combining ideas in Carter and Kohn (1994), Albert and Chib (1993), and Kim, Nelson, and Startz (1997).

A) Generating $\tilde{Z}^T \equiv \{\tilde{Y}^T\ \tilde{X}^T\}$ Conditional on $\tilde{\theta}$ and $\tilde{S}_1^T$, $\tilde{S}_2^T$, and on Data, $\tilde{Q}^T$

Conditional on $\tilde{S}_1^T$, $\tilde{S}_2^T$, and and on $\tilde{\theta}$, the state space-model in (A1)-(A3) is linear and we can adopt Carter and Kohn's (1994) multi-move Gibbs sampling to generate the first two rows of the state vectors $\tilde{C}^T$. What follows is a detailed description of their algorithm in our context.

Assuming that F and R_t in (A1)-(A3) are known, the joint distribution of $\tilde{C}^T = [c_1' \; c_2' \; .... \; c_T']'$, given the data set $\tilde{Q}^T$ and the prior distribution of c_0, is written as:

$$p(\tilde{C}^T|\tilde{Q}^T) = p(c_T|\tilde{Q}^T) \prod_{t=1}^{T-1} p(c_t|\tilde{Q}^t, z_{t+1}) \qquad (A4)$$

where $\tilde{Q}^t$ is information up to time t. Note that the second element of c_t is linked to the third element of c_{t-1} as an identity in the transition equation (A2). Therefore, the above equation suggests we can generate z_T, and then successively generate z_t conditional on z_{t+1} and Q^t for $t = T-1, T-2, ..., 1$. We can take advantage of the Gaussian Kalman filter to obtain $p(c_T|\tilde{Q}^T)$ and $p(c_t|\tilde{Q}^t, c_{t+1})$, as the state-space model is linear given F and R_t. Summarizing Carter and Kohn's (1994) algorithm in our context, we have the following two steps:

Step 1:

Run the Kalman filter algorithm to calculate $c_{t|t} = E(c_t|\tilde{Q}^t)$ and $V_{t|t} = cov(c_t|\tilde{Q}^t)$ for $t = 1, 2, .., T$ and save them. The last iteration of the Kalman filter provides us with $c_{T|T}$ and $V_{T|T}$. The first two elements of $c_{T|T}$ (denoted by $z_{T|T}$) and the first 2×2 block of $V_{T|T}$ (denoted by $V^*_{T|T}$) can be used to generate the first two elements of c_T ($z_T = [y_T \;\; x_T]'$) from a joint Normal distribution.

Step 2:

For $t = T-1, T-2, ...1$, given $c_{t|t}$ and $V_{t|t}$, if we treat $z_{t+1} = [y_{t+1} \;\; x_{t+1}]'$ generatd from the previous iteration as an additional vector of observations to the system, the distribution $p(c_t|\tilde{Q}^t, z_{t+1})$ is easily derived by applying the updating equations of the Kalman filter. From equation (A2), since z_{t+1} is given by

$$z_{t+1} = F^* c_t + \epsilon^*_{t+1}, \qquad (A5)$$

where F^* is the first two rows of F and ϵ^*_{t+1} is the first two elements of ϵ_{t+1}, updating equations are derived as:

$$c_{t|t,z^*_{t+1}} = c_{t|t} + V_{t|t}F^{*}\text{-}_t^{-1}\eta_t, \tag{A6}$$

$$V_{t|t,z^*_{t+1}} = V_{t|t} - V_{t|t}F^{*\prime}\text{-}_t^{-1}F^{*}V_{t|t}. \tag{A7}$$

Here, $\eta_t = z_{t+1} - F^{*}c_{t|t}$ is treated as a vector of forecast errors and is the difference between z_{t+1} and the forecast of z_{t+1} conditional on information up to time t; and $\text{-}_t = F^{*}V_{t|t}F^{*\prime} + Cov(\epsilon^*_{t+1})$ is the covariance of η_t. Then the first two elements of $c_{t|t,z_{t+1}}$ and the first 2×2 block of $V_{t|t,z_{t+1}}$ can be used to generate z_t from a joint Normal distribution, for $t = T-1, T-2, ..., 1$.

B) Generating $\tilde{S}_1^T$ and Parameters Associated with the Permanaent Component, Conditional on $\tilde{Y}^T$

Conditional on $\tilde{Y}^T = [y_1\ y_2\ ...\ y_T]'$, the Markov switching variable S_{1t} and other parameters associated with the permanent component, y_t, are independent of the data set, $\tilde{Q}^T$ and of the stationary component, x_t, by assumption. This allows us to focus only on equation (2), by treating generated $\tilde{Y}^T$ as a data set. The Bayesian Gibbs sampling approach to a two-state Markov-switching model has been suggested by Albert and Chib (1993). More recently, the Gibbs sampling has successfully been implemented for a three-state Markov-switching variance model of stock returns by Kim, Nelson, and Startz (1996). The following is based on Kim, Nelson, and Startz's (1996) extention of Albert and Chib's (1993) algorithm.

B.1. Generating $\tilde{S}_1^T$, Conditional on $\tilde{Y}^T$ and Parameters of the Permanent Component:

Defining $\tilde{\theta}_1^*$ as a vector of parameters of the permanent component, $\tilde{S}_1^T$ can be generated based on the following distribution, obtained from equation (2):

$$p(\tilde{S}_1^T|\tilde{Y}^T,\tilde{\theta}_1^*) = p(S_{1T}|\tilde{Y}^T,\tilde{\theta}_1^*)\prod_{t=1}^{T-1} p(S_{1t}|\tilde{Y}^t,\tilde{\theta}_1^*,S_{1,t+1}) \qquad (A8)$$

In order to simulate $\tilde{S}_1^T$ from the above distribution, we first run Hamilton's (1989) basic filter for the model given by (2) to get $p(S_{1t}|\tilde{Y}^t,\tilde{\theta}_1^*)$ and $p(S_{1t}|\tilde{Y}^{t-1},\tilde{\theta}_1^*)$, for $t = 1,2,...,T$ and save them. The last iteration of the filter provides us with $p(S_{1T}|\tilde{Y}^T,\tilde{\theta}_1^*)$, from which S_{1T} is generated. Then, we can successively generate S_{1t} from $p(S_{1t}|\tilde{Y}^t,\tilde{\theta}_1^*,S_{1,t+1})$, for $t = T-1, T-2, .., 1$, using:

$$p(S_{1t}|\tilde{Y}^t,\tilde{\theta}_1^*,S_{1,t+1}) = \frac{p(S_{1,t+1}|S_{1,t})p(S_{1,t}|\tilde{Y}^t,\tilde{\theta}_1^*)}{p(S_{1,t+1}|\tilde{Y}^t,\tilde{\theta}_1^*)} \qquad (A9)$$

The uniform distribution can be used to generate the three-state Markov-switching variable S_{1t}, $t = 1,2,...,T$. Conditional on $S_{1,t+1} = k$, $k = 1,2,3$, define $p_j = p_{1,jk} \times p(S_{1t} = j|\tilde{Y}^t,\theta_1^*)$, $j = 1,2,3$, where $p_{1,jk}$ is the transition probability in (4). We first generate a random number from the uniform distribution. If the generated number is less than or equal to $\frac{p_1}{p_1+p_2+p_3}$, we set $S_{1t} = 1$; if it is greater than $\frac{p_1}{p_1+p_2+p_3}$, we generate another random number from the uniform distribution. And then, if the generated number is less than or equal to $\frac{p_2}{p_2+p_3}$, we set $S_{1t} = 2$; if it is greater than $\frac{p_2}{p_2+p_3}$, we set $S_{1t} = 3$.

B.2. Generating σ_{vj}^2, $j = 1,2,3$, Conditional on $\tilde{Y}^T$, $\tilde{S}_1^T$, and on other Parameters of the Permanent Component:

In order to give a constraint that $\sigma_{v1}^2 < \sigma_{v2}^2 < \sigma_{v3}^2$, we may re-define σ_{v2}^2 and σ_{v3}^2 in the following way:

$$\sigma_{v2}^2 = \sigma_{v1}^2(1+g_2) \;\; and \;\; \sigma_{v3}^2 = \sigma_{v1}^2(1+g_2)(1+g_3), \qquad (A10)$$

where $g_2 > 0$ and $g_3 > 0$. We first generate σ_{v1}^2, then generate $1+g_2$ and $1+g_3$.

First, to generate σ_{v1}^2, we transform equation (2) as follows:

$$y_{1t} = \frac{(y_t - y_{t-1})}{\sqrt{(1+S_{1,2t}g_2)(1+S_{1,3t}g_2)(1+S_{1,3t}g_3)}} \qquad (A11)$$

By choosing the inverse gamma distribution as the prior $(IG(\frac{\nu_{11}}{2},\frac{\delta_{11}}{2}))$, one can show that the conditional distribution from which σ_{v1}^2 is generated is given by:

$$[\sigma_{v1}^2|\tilde{Y}^T,\tilde{S}_1^T,\tilde{\theta}^*_{j\neq\sigma_{v1}^2}] \sim IG(\frac{\nu_{11}+(T-1)}{2},\frac{\delta_{11}+\sum_{t=2}^{T}y_{1t}^2}{2}), \qquad (A12)$$

where $\tilde{\theta}^*_{j\neq\sigma_{v1}^2}$ represents a vector of parameters of the permanent component that excludes σ_{v1}^2.

Second, to generate $\bar{g}_2 = 1+g_2$, and thus, σ_{v2}^2, we transform equation (2) to get:

$$y_{2t} = \frac{(y_t - y_{t-1})}{\sqrt{\sigma_{v1}^2(1+S_{1,3t}g_3)}} \qquad (A13)$$

Here, we note that the likelihood function of g_2 depends on the values of y_{2t} for which $S_{1t} = 2\ or 3$. By defining $T_{12} = \{t : S_{1t} = 2\ or 3\}$ and choosing the inverse gamma distributions for the prior of $\bar{g}_2$ ($IG(\frac{\nu_{12}}{2},\frac{\delta_{12}}{2})I_{[\bar{g}_2>1]}$), one can show that the complete conditional is given by:

$$[\bar{g}_2|\tilde{Y}^T,\tilde{S}_1^T,\tilde{\theta}_{j\neq\bar{g}_2}] \sim IG(\frac{\nu_{12}+N_{12}}{2},\frac{\delta_{12}+\sum_{t=1}^{T_{12}}y_{2t}^2}{2})I_{[\bar{g}_2>1]}, \qquad (A14)$$

where $\tilde{\theta}_{j\neq\bar{g}_2}$ represents a vector of parameters of the model that excludes $\bar{g}_2$; I is the indicator function on $[\bar{g}_2 > 1]$; N_{12} are cardinalities of T_{12} and the sum is over the elements T_{12}.

Finally, to generate $\bar{g}_3 = 1+g_3$, and thus, σ_{v3}^2, we transform equation (2) to get:

$$y_{3t} = \frac{(y_t - y_{t-1})}{\sqrt{\sigma_{v1}^2(1+S_{1,3t}g_2)}} \qquad (A15)$$

Here, we note that the likelihood function of g_3 depends only on the values of y_{3t} for which $S_{1t} = 3$. By defining $T_{13} = \{t : S_{1t} = 3\}$ and choosing the inverse gamma distributions for the priors of $\bar{g}_3$ ($IG(\frac{\nu_{13}}{2},\frac{\delta_{13}}{2})I_{[\bar{g}_3>1]}$), one can show that the complete conditional is given by:

$$[\bar{g}_3|\tilde{Y}^T,\tilde{S}_1^T,\tilde{\theta}_{j\neq\bar{g}_3}] \sim IG(\frac{\nu_{13}+N_{13}}{2},\frac{\delta_{13}+\sum_{t=1}^{T_{13}}y_{3t}^2}{2})I_{[\bar{g}_3>1]}, \qquad (A16)$$

where $\tilde{\theta}_{j\neq\bar{g}_3}$ represents a vector of parameters of the model that excludes $\bar{g}_3$; I is the indicator function on $[\bar{g}_3 > 1]$; N_{13} are cardinalities of T_{13} and the sum is over the elements T_{13}.

The quantities ν_{1i}, $i = 1,2,3$, represent the strength of the priors of σ_{v1}^2, $\bar{g}_2$, and of $\bar{g}_3$. For our application, we employ $\nu_{1i} = 0$ and $\delta_{1i} = 0$ for $i = 1,2,3$.

B.3. Generating Transition Probabilities ($p_{1,11}, p_{1,12}, p_{1,21}, p_{1,22}, p_{1,31}, p_{1,32}$) Conditional on $\tilde{Y}^T$, $\tilde{S}_1^T$, and Other Parameters of the Permanent Component:

Conditional on $\tilde{S}_1^T$, the transition probabilities are independent of $\tilde{Y}^T$ and other parameters, as in Albert and Chib (1993). For a two-state Markov-switching model, Albert and Chib (1993) derive the full conditional distributions of the transition probabilities as a product of independent beta distributions. For a three-state Markov-switching model auch as in this paper, Kim, Nelson, and Startz (1996) adopt a slight modification of their approach.

Given $\tilde{S}_1^T$ and the initial state, let $n_{1,ij}$, $i, j = 1, 2, 3$, be the total number of transitions from state $S_{1,t-1} = i$ to $S_{1t} = j$, $t = 2, 3, .., T$. Define $\bar{p}_{1,ii} = Pr(S_{1t} \neq i | S_{1,t-1} = i)$ and $\bar{p}_{1,ij} = Pr(S_{1t} = j | S_{1,t-1} = i, S_{1t} \neq i)$, $i = 1, 2, 3$, $j = 1, 2$. Correspondingly, we have $p_{1,ij} = \bar{p}_{1,ij} \times (1 - p_{1,ii})$ for $i \neq j$. Similarly, define $\bar{n}_{1,ii}$ to be the number of transitions from state $S_{1,t-1} = i$ to $S_{1t} \neq i$ and $\bar{n}_{1,ij}$ to be the number of transitions from state $S_{1,t-1} = i$ to state $S_{1t} = j$, conditional on $S_{1t} \neq i$.

Then, as in Albert and Chib (1993), by taking the beta family of distributions as conjugate priors, it can be shown that the posterior distributions of $p_{1,ii}$ are given by

$$[p_{1,ii} | \tilde{S}_{1T}] \sim beta(u_{1,ii} + n_{1,ii}, \bar{u}_{1,ii} + \bar{n}_{1,ii}),\ i = 1, 2, 3, \qquad (A17)$$

where $u_{1,ii}$ and $\bar{u}_{1,ii}$ are the hyperparameters of the prior. Once $p_{1,ii}$, $i = 1, 2, 3$, are generated from the above distribution, generation of the other parameters is straightforward. For example, given that $p_{1,ii}$ is generated, $p_{1,ij}$ can be calculated by $p_{1,ij} = \bar{p}_{1,ij}\bar{p}_{1,ii}$, where $\bar{p}_{1,ij}$ can be generated from the following beta distribution:

$$[\bar{p}_{1,ij} | \tilde{S}_{1T}] \sim beta(u_{1,ij} + n_{1,ij}, u_{1,ik} + n_{1,ik}),\ i \neq j \neq k, \qquad (A18)$$

where $u_{1,ij}$ and $u_{1,ik}$ are the hyperparameters of the prior. The values of the hyperparameters that we employ are : $u_{1,ii} = 1$ and $\bar{u}_{1,ii} = 1$; and $u_{1,ij} = 1$ and $u_{1,ik} = 1$ for $i \neq j \neq k$.

C) Generating $\tilde{S}_2^T$ and Parameters Associated with the Stationary Component, Conditional on $\tilde{X}^T$

Conditional on $\tilde{X}^T = [x_1 \ x_2 \ ... \ x_T]'$, the Markov switching variable S_{2t} and other parameters associated with the stationary component, x_t, are independent of the data set, $\tilde{Q}^T$ and of the random walk component, y_t, by assumption. This allows us to focus only on equation (3), by treating generated $\tilde{X}^T$ as a data set. The procedures are almost identical to those for generating $\tilde{S}_1^T$ and parameters associated with the permanent component, conditional on $\tilde{Y}^T$.

C.1. Generating $\tilde{S}_2^T$, Conditional on $\tilde{X}^T$ and Parameters of Stationary Component:

Defining $\tilde{\theta}_2^*$ as a vector of parameters of the stationary component, $\tilde{S}_2^T$ can be generated based on the following distribution, obtained from equation (3):

$$p(\tilde{S}_2^T|\tilde{X}^T, \tilde{\theta}_2^*) = p(S_{2T}|\tilde{X}^T, \tilde{\theta}_2^*) \prod_{t=1}^{T-1} p(S_{2t}|\tilde{X}^t, \tilde{\theta}_2^*, S_{2,t+1}). \qquad (A19)$$

The remaining details are the same as in B.1) for generating $\tilde{S}_1^T$ of the permanent component.

C.2. Generating σ_{ej}^2, $j = 1, 2, 3$, Conditional on $\tilde{X}^T$, $\tilde{S}_2^T$, and on other Parameters of the Stationary Component:

As in B.2, in order to give a constraint that $\sigma_{e1}^2 < \sigma_{e2}^2 < \sigma_{e3}^2$, we re-define σ_{e2}^2 and σ_{e3}^2 in the following way:

$$\sigma_{e2}^2 = \sigma_{e1}^2(1 + h_2) \ \ and \ \ \sigma_{e3}^2 = \sigma_{e1}^2(1 + h_2)(1 + h_3), \qquad (A20)$$

where $h_2 > 0$ and $h_3 > 0$. We first generate σ_{e1}^2, then generate $1 + h_2$ and $1 + h_3$.

First, to generate σ_{e1}^2, we transform equation (3) as follows:

$$x_{1t} = \frac{(x_t - \phi_1 x_{t-1} - \phi_2 x_{t-2})}{\sqrt{(1 + S_{2,2t}h_2)(1 + S_{2,3t}h_2)(1 + S_{2,3t}h_3)}} \qquad (A21)$$

By choosing the inverse gamma distribution as the prior ($IG(\frac{\nu_{21}}{2}, \frac{\delta_{21}}{2})$), one can show that the conditional distribution from which σ_{e1}^2 is generated is given by:

$$[\sigma_1^2|\tilde{X}^T, \tilde{S}_2^T, \tilde{\theta}^*_{j\neq\sigma_{e1}^2}] \sim IG(\frac{\nu_{21} + (T-2)}{2}, \frac{\delta_{21} + \sum_{t=3}^{T} x_{1t}^2}{2}), \qquad (A22)$$

where $\tilde{\theta}^*_{j\neq\sigma_{e1}^2}$ represents a vector of parameters of the stationary component that excludes σ_{e1}^2.

Second, to generate $\bar{h}_2 = 1 + h_2$, and thus, σ_{e2}^2, we transform equation (3) to get:

$$x_{2t} = \frac{(x_t - \phi_1 x_{t-1} - \phi_2 x_{t-2})}{\sqrt{\sigma_{e1}^2(1 + S_{2,3t}h_3)}} \qquad (A23)$$

Here, we note that the likelihood function of h_2 depends on the values of x_{2t} for which $S_{2t} = 2\ or 3$. By defining $T_{22} = \{t : S_{2t} = 2\ or 3\}$ and choosing the inverse gamma distributions for the prior of $\bar{h}_2$ ($IG(\frac{\nu_{22}}{2}, \frac{\delta_{22}}{2})I_{[\bar{h}_2>1]}$), one can show that the complete conditional is given by:

$$[\bar{h}_2|\tilde{X}^T, \tilde{S}_2^T, \tilde{\theta}_{j\neq\bar{h}_2}] \sim IG(\frac{\nu_{22} + N_{22}}{2}, \frac{\delta_{22} + \sum_{t=1}^{T_{22}} x_{2t}^2}{2})I_{[\bar{h}_2>1]}, \qquad (A24)$$

where $\tilde{\theta}_{j\neq\bar{h}_2}$ represents a vector of parameters of the model that excludes $\bar{h}_2$; I is the indicator function on $[\bar{h}_2 > 1]$; N_{22} are cardinalities of T_{22} and the sum is over the elements T_{22}.

Finally, to generate $\bar{h}_3 = 1 + h_3$, and thus, σ_{e3}^2, we transform equation (3) to get:

$$x_{3t} = \frac{(x_t - \phi_1 x_{t-1} - \phi_2 x_{t-2})}{\sqrt{\sigma_{e1}^2(1 + S_{2,3t}h_2)}} \qquad (A25)$$

Here, we note that the likelihood function of h_3 depends only on the values of x_{3t} for which $S_{2t} = 3$. By defining $T_{23} = \{t : S_{2t} = 3\}$ and choosing the inverse gamma distributions for the priors of $\bar{h}_3$ ($IG(\frac{\nu_{23}}{2}, \frac{\delta_{23}}{2})I_{[\bar{h}_3>1]}$), one can show that the complete conditional is given by:

$$[\bar{h}_3|\tilde{X}^T, \tilde{S}_2^T, \tilde{\theta}_{j\neq\bar{h}_3}] \sim IG(\frac{\nu_{23} + N_{23}}{2}, \frac{\delta_{23} + \sum_{t=1}^{T_{23}} x_{3t}^2}{2})I_{[\bar{h}_3>1]}, \qquad (A26)$$

where $\tilde{\theta}_{j\neq\bar{h}_3}$ represents a vector of parameters of the model that excludes $\bar{h}_3$; I is the indicator function on $[\bar{h}_3 > 1]$; N_{23} are cardinalities of T_{23} and the sum is over the elements T_{23}.

The quantities ν_{2i}, $i = 1, 2, 3$, represent the strength of the priors of σ_{e1}^2, $\bar{h}_2$, and of $\bar{h}_3$. For our application, we employ $\nu_{2i} = 0$ and $\delta_{2i} = 0$ for $i = 1, 2, 3$.

C.3. Generating Transition Probabilities ($p_{2,11}, p_{2,12}, p_{2,21}, p_{2,22}, p_{2,31}, p_{2,32}$) Conditional on $\tilde{X}^T$, $\tilde{S}_2^T$, and Other Parameters of the Stationary Component:

As in B.3, conditional on $\tilde{S}_2^T$, the associated transition probabilities are independent of $\tilde{X}^T$ and other parameters. The procedure for generating the transition probabilities for the transitory component is exactly the same as in the case of the permanent component in Section B.3.

C.4. Generating $\tilde{\phi} = [\phi_1\ \phi_2]'$, Conditional on $\tilde{X}^T$, $\tilde{S}_2^T$, and on other Parameters of the Stationary Component:

Rewriting equation (3) to have homoskedastic errors, we have:

$$\frac{x_t}{\sigma_{e,t}} = \phi_1 \frac{x_{t-1}}{\sigma_{e,t}} + \phi_2 \frac{x_{t-2}}{\sigma_{2,t}} + e_t^*,\ t = 3, 4, ..., T \tag{A27}$$

where $e_t^* \sim iid\ N(0, 1)$. We adopt the multivariate Normal prior distribution for $\tilde{\phi}$, $\tilde{\phi} \sim N(\alpha, A^{-1})$, and denote $\tilde{X}$ to be the vector of left-hand-side variables and $\tilde{W}$ to be the matrix of right-hand-side variables. The posterior distribution from which $\tilde{\phi}$ is to be generated is given by:

$$\tilde{\phi} \sim N((A + \tilde{W}'\tilde{W})^{-1}(\alpha A + \tilde{W}'\tilde{X}), (A + \tilde{W}'\tilde{W})^{-1}). \tag{A28}$$

We adopt $\alpha = [0\ 0]'$ and $A^{-1} = 4$ for our application. In generating $\tilde{\phi}$ from the above posterior distribution, we adopt rejection sampling, so that the roots of $\phi(L) = 1 - \phi_1 L - \phi_2 L^2$ lie outside the unit circle.

REFERENCES

[1] Kim, Chang-Jin, Charles R. Nelson, and Richard Startz, 1997, "Testing for Mean Reversion in Heteroskedastic Data Based on Gibbs-Sampling-Augmented Randomization," Forthcoming, *Journal of Empirical Finance*.