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The trade balance and real exchange rate under currency substitution

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It is demonstrated that the Calvo and Rodriguez (1977) results hold up in a utility maximizing model, contrary to the claim of Liviatan (1981). It is argued that Liviatan's conclusions are based on a mistaken assumption. Having rescued Calvo–Rodriguez from Liviatan, it is then shown that the Calvo–Rodriguez results are dependent on two special assumptions—that there is only one foreign asset traded and its real return is zero. The paper also shows that there is an avenue for foreign inflation transmission under perfectly floating exchange rates even when goods prices are perfectly flexible. However, the direction of response of real variables is in general ambiguous.

Calvo and Rodriguez (1977) demonstrate in a model of a small economy in which residents hold foreign currency balances that an increase in domestic money growth will cause an immediate real depreciation of the currency. The more rapid money growth increases inflation in domestic currency prices, thus increasing the real demand for foreign money. The country can acquire this money only through a trade surplus, which requires the real depreciation.

This paper makes three points, in increasing order of importance.

First, Liviatan (1981) provides a parallel model to Calvo and Rodriguez in which agents' behavior is optimal (rather than described by *ad hoc* equations.) Liviatan concludes that an increase in domestic money growth leads to a real appreciation domestically, and thus the Calvo–Rodriguez result is not correct in a model in which agents optimize. This conclusion depends critically on an assumption about the nature of the utility of money function. It is argued here that Liviatan's justification of his assumption is mistaken. Were the alternative assumption made, the Calvo–Rodriguez results would still hold in the optimizing framework (as Liviatan acknowledges).

Second, one might be tempted to conclude from the Calvo–Rodriguez analysis that an increase in foreign inflation would lead to a real appreciation of the domestic currency. It is true that real demand for foreign money should fall, but some diminution of these balances will occur over time as foreign prices rise even if nominal holdings of foreign money remain constant. The optimal depletion of foreign money may not require a trade balance deficit or a real appreciation.

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Third, if residents can trade assets other than foreign money internationally there is no necessary relation between changes in domestic or foreign inflation and the rate of accumulation of real foreign assets. If the rates of inflation change on the two monies, leading to a change in real demand for foreign money, residents may exchange foreign money for some other traded asset—a capital account transaction that does not require any trade balance change. In the simple model presented in Section III of this paper, foreign money and bonds with a fixed real return are traded. Under the particular assumptions on utility, a change in domestic money growth or foreign inflation has no effect on the net real foreign asset position.

Although currency substitution is often thought of in terms of certain less developed countries that can accurately be characterized by the absence of capital mobility, in fact currency substitution is a more widespread phenomenon. It is not uncommon for individuals and firms in industrialized countries to hold foreign currency balances (particularly dollars) for international trade transactions. For example, there may be a liquidity premium on short-term Eurocurrency balances. The models of this paper serve to demonstrate that while currency substitution may provide a channel through which inflation has real effects internationally, the nature of such effects is ambiguous and is likely to be difficult to identify empirically.

Section I presents the consumer's choice problem and discusses the Liviatan assumption. Section II fleshes out the model, and examines the effects of domestic and foreign inflation in the absence of capital mobility. The third section adds trade in bonds. Section IV concludes.

I. The Model

The model considered here is essentially the same as that of Liviatan (1981) except that foreign inflation is non-zero and there are traded foreign bonds.[1]

The country is small, and there are two goods: non-traded (or home) and traded. Their prices are given by: P_H = domestic currency price of home goods; P_T = domestic currency price of traded goods; P_T^* = foreign currency price of traded goods (exogenous). Both goods are consumed. c_H = consumption of home goods; c_T = domestic consumption of traded goods.

Residents hold foreign and domestic money. (Foreigners do not hold domestic money.) M = nominal domestic balances; F = nominal holdings of foreign money by domestic residents. Utility is a function of real balances. Here we define real balances as nominal balances deflated by the traded goods price.[2] $m \equiv M/P_T$ = real domestic balances; $f \equiv F/P_T^*$ = real balances of foreign currency.

Consumers are assumed to be identical within the country and will be represented by a single representative consumer. Instantaneous utility is assumed to be separable.[3]

$$U = u(c_T, c_H) + v(m, f).$$

It is assumed that the appropriate Inada conditions on the utility functions hold so that positive amounts of both goods are always consumed and positive amounts of both moneys are held.

The presence of real money balances in the utility function is a familiar device that allows individuals to hold money even when it is dominated in its risk and rate of return characteristics. For example, in the model of Section III, foreign bonds dominate foreign money, because they pay a higher real rate of return (and there is

no risk because of the absence of uncertainty in this model). If foreign money were judged on the same basis as other assets, it would not be held in an optimal portfolio. However, money is desirable because it facilitates transactions. This feature of money can be more closely approximated (as, for example, in the cash-in-advance models), but to make the points of this paper it is not necessary to incur the extra complexity that this would entail.

An asset that pays a given real return is also traded. b = holdings of foreign bonds (a negative value of b represents net debt to foreigners) defined in units of the traded good; r = fixed rate of return in terms of traded goods (exogenous). Total financial asset holdings are given by

$$\langle 1 \rangle \qquad a = m + f + b.$$

We will assume that outputs of home and traded goods are fixed at some positive levels. y_T = output of traded goods; y_H = output of home goods. Seignorage is transferred lump-sum to consumers. τ = monetary transfer.

There are four sources of accumulation of assets: interest earnings on b; inflation losses on m and f, where $\pi = \dot{P}_T/P_T$ = rate of domestic inflation,[4] $\pi^* = \dot{P}_T^*/P_T^*$ = rate of foreign inflation; monetary transfers; and the trade balance, where $y_T - c_T + (y_H - c_H)/p$ = trade balance, and $p = P_T/P_H$ = real exchange rate. We have

$$\langle 2 \rangle \qquad \dot{a} = ra - (\pi + r)m - (\pi^* + r)f + \tau + y_T - c_T + (y_H - c_H)/p.$$

Consumers have a constant discount rate equal to the world interest rate, r.[5] They maximize

$$\int_0^\infty [u(c_T, c_H) + v(m, f)]e^{-rt}dt$$

subject to $\langle 1 \rangle$ and $\langle 2 \rangle$. We also impose the lifetime budget constraint

$$\lim_{t \to \infty} a_t e^{-rt} \geqslant 0.$$

This constraint is imposed from outside the model, because in its absence it would always be optimal for domestic residents to borrow an arbitrarily large amount from abroad at any time, and borrow more in the future to meet interest obligations.

The Hamiltonian for the optimization problem described above is given by

$$H = u(c_T, c_H) + v(m, f) + q(ra - (\pi + r)m - (\pi^* + r)f + \tau + y_T - c_T + (y_H - c_H)/p) + \lambda(a - m - f - b).$$

The first-order conditions yield

$$\langle 3a \rangle \qquad u_T(c_T, c_H) = pu_H(c_T, c_H),$$

$$\langle 3b \rangle \qquad v_f/u_T = r + \pi^* - \dot{u}_T/u_T,$$

$$\langle 3c \rangle \qquad v_m/u_T = r + \pi - \dot{u}_T/u_T,$$

$$\langle 3d \rangle \qquad \dot{u}_T = 0.$$

In the above equations $\dot{u}_T$ refers to the time derivative of u_T.

Letting $\dot{u}_T = 0$,[6] and totally differentiating equations ⟨3b⟩ and ⟨3c⟩, we get

⟨4a⟩ $$v_{ff}\,df + v_{fm}dm = (r+\pi^*)u_{TT}dc_T + (r+\pi^*)u_{TH}dc_H + u_T d(r+\pi^*).$$

⟨4b⟩ $$v_{fm}df + v_{mm}dm = (r+\pi)u_{TT}dc_T + (r+\pi)u_{TH}dc_H + u_T d(r+\pi).$$

Solving for df and dm, using ⟨3b⟩ and ⟨3c⟩, we can derive money demand functions:

$$df = (1/D)[A(u_{TT}/u_T)dc_T + A(u_{HT}/u_T)dc_H + v_{mm}u_T d(r+\pi^*) - v_{fm}u_T d(r+\pi)],$$

$$dm = (1/D)[B(u_{TT}/u_T)dc_T + B(u_{HT}/u_T)dc_H - v_{fm}u_T d(r+\pi^*) + v_{ff}u_T d(r+\pi)].$$

Concavity of v ensures

⟨5a⟩ $$D \equiv v_{mm}v_{ff} - v_{fm}^2 > 0.$$

Assuming m and f are normal goods, we have

⟨5b⟩ $$A \equiv v_f v_{mm} - v_m v_{fm} < 0,$$

⟨5c⟩ $$B \equiv v_m v_{ff} - v_f v_{fm} < 0.$$

Concavity of u yields

⟨5d⟩ $$u_{TT} < 0.$$

From assumptions ⟨5⟩, it follows that both domestic and foreign money demand are positively related to consumption of the traded good. The relation of changes in money demands to changes in consumption of home goods depends on the sign of u_{HT}.[7] Demand for foreign money is negatively related to the foreign rate of inflation, and domestic money demand is negatively related to the domestic rate of inflation. If $v_{fm} < 0$, then as domestic inflation rises, demand for foreign money increases, and as foreign inflation rises, demand for domestic money increases. It is this shift from one money to the other that is the essence of the economic behavior that we wish to describe, so we will assume $v_{fm} < 0$.

The assumption on the sign of v_{fm} is crucial to Liviatan's criticism of Calvo and Rodriguez (CR). He assumes that this derivative has a positive sign, and then shows that the CR result (which states that higher domestic money growth leads immediately to a real depreciation) is reversed. Were this derivative negative, the CR result would hold up.

Calvo and Rodriguez assume that the expected rate of inflation (π from above) of traded goods prices in terms of the domestic currency is a function of the ratio of holdings of domestic money to holdings of foreign money.

⟨6a⟩ $$\pi = g(m/f)$$

(Calvo and Rodriguez do not consider foreign inflation.) That ratio increases when the expected rate of inflation falls. We have demonstrated above that a negative sign on v_{fm} is consistent with this property.[8]

Rather than focus on the property of ⟨6a⟩ that m/f falls as π rises, Liviatan chooses to concentrate on the fact that ⟨6a⟩ also implies m/f does not change if π does not change. He states (p. 1226), 'For a given π, m and f should be positively related.'[9] He then draws attention to equation ⟨3c⟩, (with $\dot{u}_T = 0$) and concludes, 'We see immediately that a positive relationship between m and f implies $v_{fm} > 0$.' A good guess about what Liviatan means by this statement is that, if in ⟨3c⟩

everything except for m and f are held constant (*i.e.*, c_T, c_H, and $\pi + r$), then in order for the ratio m/f to stay constant while m and f change, v_{fm} must be positive. Mathematically, imposing that m/f is constant, so that $dm = (m/f)df$, and imposing that dc_T, dc_H and $d(\pi + r)$ equal zero, from equation ⟨4b⟩ we must have

$$\langle 6b \rangle \qquad (v_{fm} + (m/f)v_{mm})df = 0.$$

Since v_{mm} is negative, Liviatan would conclude that v_{fm} is positive.

The problem with this argument is that if equations ⟨3⟩ hold, and the nominal interest rates and consumption of the goods cannot change, then there is no way for demand for either money to change. It is not the terms in parentheses in ⟨6b⟩ that must be zero, but instead df itself must vanish. The simplest way to see this is to perform the same exercise on ⟨4a⟩ that was performed on ⟨4b⟩. We get

$$\langle 6c \rangle \qquad (v_{ff} + (m/f)v_{fm})df = 0.$$

For ⟨6b⟩ and ⟨6c⟩ to hold simultaneously, df must be zero.[10] So, in Liviatan's exercise, the m/f ratio is kept constant because m and f themselves do not change. It says nothing about the correct sign for v_{fm}.

Liviatan claims to be looking for 'a portfolio allocation condition analogous to condition ⟨6a⟩ in the CR model.' It is argued above that this condition is achieved by allowing v_{fm} to be negative, but perhaps the most compelling argument that Liviatan chose the wrong sign according to this guideline is that his conclusions are exactly the opposite of CR.

II. The Model with No Foreign Bonds

In this section the model is fleshed out by describing the behavior of government and deriving the general equilibrium given the production side of the economy. We take up the case in which there are no foreign bonds (which is the model studied by Calvo and Rodriguez). We emphasize the role of the foreign inflation tax in determining the response of the real exchange rate to a change in the rate of foreign inflation.

Output of home goods and traded goods is assumed to be fixed. Home goods market equilibrium requires that consumption of home goods be constant at the level of output of non-traded goods.

$$y_H = c_H = \bar{c}_H.$$

Totally differentiating ⟨3a⟩, and setting $dc_H = 0$, we see that under the assumption that goods are normal in consumption, the real exchange rate is negatively related to consumption of the traded good.

$$\langle 7 \rangle \qquad dp/dc_T = (u_H u_{TT} - u_T u_{HT})/u_H^2 < 0.$$

An increase in consumption of traded goods necessarily implies a worsening of the trade balance, which is accompanied by a real appreciation.

Domestic money growth is assumed to be constant.

$$\dot{M}/M = \mu.$$

Hence,

$$\langle 8 \rangle \qquad \dot{m} = \mu m - \pi m.$$

Real transfers are given by

$$\tau = \dot{M}/P_T = \mu m.$$

In the absence of bonds,

$$\dot{a} = \dot{m} + \dot{f},$$

so

$$\langle 9 \rangle \qquad \dot{f} = y_T - c_T - \pi^* f.$$

If bonds are not traded ($b=0$), equations ⟨3a⟩, ⟨3b⟩, and ⟨3c⟩ hold, but not ⟨3d⟩. We can rewrite ⟨3b⟩ as

$$\langle 10 \rangle \qquad u_{TT}\dot{c}_T = (r+\pi^*)u_T - v_f.$$

Using ⟨3b⟩ and ⟨3c⟩ to get an expression for π in terms of m, f and c_T,

$$\langle 11 \rangle \qquad \pi = \pi^* + (v_m - v_f)/u_T,$$

we see that ⟨8⟩, ⟨9⟩, and ⟨10⟩ are a three-equation dynamic system in m, f and c_T.

The steady state is given by

$$\bar{\pi} = \mu,^{11}$$

and

$$\langle 12a \rangle \qquad \bar{v}_f - (r+\pi^*)\bar{u}_T = 0.$$

$$\langle 12b \rangle \qquad \bar{v}_m - (r+\mu)\bar{u}_T = 0.$$

$$\langle 12c \rangle \qquad y_T - \bar{c}_T - \pi^*\bar{f} = 0.$$

The appendix shows $d\bar{f}/d\mu > 0$, and $d\bar{f}/d\pi^* < 0$. The appendix linearizes the system in the neighborhood of the steady state. It shows that there is one negative and two positive roots to the dynamic system, so it is saddle stable near the steady state. In order to satisfy the transversality conditions for the optimization problem, we must stay on the saddle path.

Hence, in the neighborhood of the steady state

$$\dot{f} = \theta(f - \bar{f}), \qquad \text{where } \theta = \text{negative root} < 0.$$

From equation ⟨9⟩

$$\theta(f - \bar{f}) = y_T - c_T - \pi^* f.$$

Since f cannot change on impact (there is no other asset to trade for foreign money),

$$dc_T/d\mu = \theta(d\bar{f}/d\mu) < 0.$$

So, c_T falls immediately when domestic money growth increases and p rises—there is a real depreciation. This is precisely the result of Calvo and Rodriguez, and it occurs for the same reason as in their model. When domestic money growth increases, residents want to shift out of domestic money and into foreign money in anticipation of greater domestic inflation. Foreign money balances can be acquired only gradually by running a current account surplus, so a real depreciation is necessary.

One difference this model has from that of Calvo–Rodriguez is that here there is not 'superneutrality' in the long run. In the CR model, the real exchange rate eventually returns to its initial level after an increase in μ. In this model,

consumption of traded goods falls even in steady state when domestic money growth increases, so that there is a long-run real depreciation. Consumption of traded goods must be lower in the long run because equation ⟨12c⟩ requires that steady-state consumption of traded goods equals the level of production of traded goods less the real losses from holding foreign money caused by foreign inflation. Greater steady-state holdings of foreign money balances implies greater real losses on those balances, hence lower consumption. This feature is missing from Calvo–Rodriguez because they implicitly assume foreign inflation is zero.

On the other hand, note that

$$dc_T/d\pi^* = -\bar{f} + \theta(d\bar{f}/d\pi^*).$$

While it is true that in response to an increase in foreign inflation accumulation of real foreign money must fall ($\dot{f}$ declines because $\theta(d\bar{f}/d\pi^*) > 0$), the trade balance may not have to improve. The increase in the inflation tax, $\pi^* f$, may be large enough to achieve the desired real decumulation of foreign balances, so c_T need not drop. Hence, a real appreciation is not necessary. The appendix constructs an example to show that this derivative could be of either sign.

So, there is a sort of asymmetry in the response of the real exchange rate. When domestic money growth increases, we have confirmed the Calvo–Rodriguez conclusion that there must be a real depreciation. However, when foreign inflation rises there need not be a real appreciation domestically.

III. The Model with Bonds

Here we consider the model with traded bonds. In response to a change in domestic money growth or foreign inflation, there need not be a change in the rate of accumulation of foreign assets, z,

$$z = f + b.$$

For example, if domestic inflation rises, making foreign balances more attractive, they can be acquired through a capital account transaction in which bonds are traded for foreign money. So, $\dot{z}$ need not change and under the special assumptions of the model here (that the discount rate equals the interest rate), $\dot{z}$ will not change. This is in contrast to the model of the previous section, in which nominal and real foreign money balances could not jump.

When b is traded, there is a zero root in the dynamic system. Given any shock c_T, m, f, and b must immediately jump to their new steady-state levels. The zero root comes from equation ⟨3d⟩. When this equation holds, we now have from equations ⟨3b⟩ and ⟨3c⟩

$$v_f - (r + \pi^*)u_T = 0, \tag{13}$$

$$v_m - (r + \pi)u_T = 0. \tag{14}$$

With $\dot{c}_T = 0$, the dynamics of the system are given by equation ⟨8⟩ and

$$\dot{z} = rz - (\pi^* + r)f + y_T - c_T. \tag{15}$$

The appendix shows that this modified system has two positive roots, and is therefore unstable. To satisfy the transversality conditions, m and z must adjust to their long-run values immediately. Since z cannot jump, it always equals its initial level.

Since $\dot{z}=0$,

$$\langle 16\rangle \qquad y_T - c_T - (\pi^* + r)f + rz = 0.$$

Equations ⟨13⟩, ⟨14⟩, and ⟨16⟩ determine the value of c_T, f and m. With z always equal to its initial level, the response of these variables to changes in μ and π^* are very similar to the steady-state responses in the model without bonds—as can be seen by comparing ⟨13⟩, ⟨14⟩, and ⟨16⟩ to equations ⟨12⟩.

First let us note what is by now obvious, that $d\dot{z}/d\mu = 0$. Contrary to the result of Calvo and Rodriguez, when there is capital mobility an increase in domestic inflation need not lead to a higher rate of accumulation of foreign assets. Foreign money can be acquired in exchange for foreign bonds. Similarly, $d\dot{z}/d\pi^* = 0$.

However, from ⟨16⟩, note that

$$dc_T/d\mu = -(\pi^* + r)df/d\mu < 0.$$

Hence, consumption falls and there is a real depreciation when domestic money growth increases—as in Calvo and Rodriguez. However the reason for this reaction is completely different than in CR. In their model (and the model of the previous section) the need to run a current account surplus in order to acquire more foreign money leads to the real depreciation. Here, there is a loss of real income on the portfolio of foreign assets, which requires a trade balance improvement (which in turn necessitates a real depreciation) to maintain the net external position of the country. Since $-\pi^* f + rb$ falls (because b is traded for f, so b goes down and f goes up), there is a tendency for $\dot{z}$ to fall. However, because consumers' discount rate equals the interest rate, they do not wish to change their overall debt position. So to keep $\dot{z}=0$, the trade balance must move into surplus and c_T must fall.

We also have

$$dc_T/d\pi^* = -d(\pi^* + r)f/d\pi^*.$$

The appendix shows the sign of this derivative is related to the elasticity of money demand. If $(\pi^* + r)f$ rises, c_T must fall and p rise. If $(\pi^* + r)f$ falls, c_T must rise and p fall. Again, the response of the real exchange rate is completely determined by the change in the total real earnings on foreign assets.

So, in the model of this section, changes in domestic money growth or foreign inflation have no effect on domestic saving or the net foreign asset position of the country—in contrast to dynamic models of currency substitution in which only foreign money can be traded (such as Calvo and Rodriguez, 1977; Liviatan, 1981; Daniel, 1985). This is because real foreign balances can be altered by a shift in portfolios (one asset can be traded for another) rather than through accumulation over time (trading of goods for assets). The impact of such changes on the real exchange rate depends entirely on how real earnings from assets are affected—an aspect that is completely absent from the CR and Liviatan studies.

IV. Conclusion

In general, in the presence of currency substitution, or whenever residents hold foreign nominal assets that are not fully indexed for changes in foreign inflation, the response of the real exchange rate to a change in domestic or foreign inflation is not clear. It will depend on how the rate of accumulation of real foreign assets is affected, and it will depend on how the total real earnings on foreign assets changes.

It is useful to summarize how the Calvo and Rodriguez (1977) results depend upon the two key assumptions that there is only one traded asset, and it has a zero real rate of return. Their main conclusion—that an increase in domestic money growth causes an initial real depreciation—holds up if there is a non-zero rate of return on the traded asset, as long as there is only one traded asset. However, this conclusion very much depends on the assumption of only one traded asset (whether or not its real return is non-zero). The result that the long-run real exchange rate is unaffected by a permanent increase in the domestic money growth rate holds only when the foreign asset has a zero real return—even when there is only one foreign asset.

These models demonstrate that if there are assets whose real return is not neutral with respect to changes in inflation, then there is an avenue for foreign inflation transmission under flexible exchange rates even when goods prices are perfectly flexible. However, even the direction of response of real variables to these changes depends on such things as the nature of saving behavior, the degree of capital mobility and the elasticity of money demand.

Appendix

Steady State of Model without Bonds

The steady state of the model without bonds is given by equations ⟨12⟩. Totally differentiating, we have

$$\bar{v}_{ff}df+\bar{v}_{fm}dm-(r+\pi^*)\bar{u}_{TT}dc_T = \bar{u}_T d\pi^*,$$
$$\bar{v}_{fm}df+\bar{v}_{mm}dm-(r+\mu)\bar{u}_{TT}dc_T = \bar{u}_T d\mu,$$
$$-\pi^* df-dc_T = \bar{f}d\pi^*.$$

Solving, we obtain

$$\begin{bmatrix} d\bar{f} \\ d\bar{m} \\ d\bar{c}_T \end{bmatrix} = (1/E)\begin{bmatrix} -\bar{v}_{mm} & \bar{v}_{fm} & A\bar{u}_{TT}/\bar{u}_T \\ \bar{v}_{fm}+(\pi^*\bar{v}_m\bar{u}_{TT})/\bar{u}_T & -\bar{v}_{ff}-(\pi^*\bar{v}_f\bar{u}_{TT})/\bar{u}_T & B\bar{u}_{TT}/\bar{u}_T \\ \pi^*\bar{v}_{mm} & -\pi^*\bar{v}_{fm} & D \end{bmatrix}\begin{bmatrix} \bar{u}_T d\pi^* \\ \bar{u}_T d\mu \\ \bar{f}d\pi^* \end{bmatrix},$$

where,

$$E = -D-A\pi^*\bar{u}_{TT}/\bar{u}_T < 0,$$

and A, B, and D are defined in the text in equations ⟨5⟩.

In particular

$$d\bar{f}/d\mu = \bar{u}_T\bar{v}_{fm}/E > 0,$$
$$d\bar{f}/d\pi^* = (-\bar{u}_T\bar{v}_{mm}+A\bar{f}\bar{u}_{TT}/\bar{u}_T)/E < 0.$$

The Dynamic Model without Bonds

It is useful to define the elasticity of foreign money demand with respect to π^*. From the money demand equations above

$$\varepsilon \equiv -(\pi^*/f)df/d\pi^* = -v_{mm}u_T\pi^*/fD.$$

Equations ⟨8⟩, ⟨9⟩, and ⟨10⟩ can be linearized as

$$\begin{bmatrix} \dot{m} \\ \dot{f} \\ \dot{c} \end{bmatrix} \cong \begin{bmatrix} -\bar{m}(\bar{v}_{mm}-\bar{v}_{fm})/\bar{u}_T & \bar{m}(\bar{v}_{ff}-\bar{v}_{fm})/\bar{u}_T & \bar{m}\bar{u}_{TT}(v\bar{c}_m-\bar{v}_f)/\bar{u}_T^2 \\ 0 & -\pi^* & -1 \\ -\bar{v}_{fm}/\bar{u}_{TT} & -\bar{v}_{ff}/\bar{u}_{TT} & r+\pi^* \end{bmatrix} \begin{bmatrix} m-\bar{m} \\ f-\bar{f} \\ c-\bar{c} \end{bmatrix}.$$

The determinant of the matrix of coefficients is given by

$$\bar{m}D/\bar{u}_T\bar{u}_{TT}+A\pi^*\bar{m}/\bar{u}_T^2 < 0.$$

The product of the eigenvalues equals this determinant. So, there are either three negative roots or one. The globally stable system with three negative roots cannot be ruled out. However, the sum of the eigenvalues equals the trace. The trace is given by

$$\begin{aligned} r-\bar{m}(\bar{v}_{mm}-\bar{v}_{fm})/\bar{u}_T &= r-\bar{m}(\bar{v}_f\bar{v}_{mm}(\bar{v}_m/\bar{u}_t)-\bar{v}_m\bar{v}_{fm}(\bar{v}_f/\bar{u}_t))/\bar{v}_f\bar{v}_m \\ &= r-\bar{m}(\bar{v}_f\bar{v}_{mm}(r+\mu)-\bar{v}_m\bar{v}_{fm}(r+\pi^*))/\bar{v}_f\bar{v}_m. \end{aligned}$$

If the foreign rate of inflation is initially close to the domestic rate of inflation ($\mu=\pi^*$), then the trace is positive (assuming the moneys are normal goods) and there must be only one negative root, so the system is saddle stable in the neighborhood of the steady state.

The text derives an expression for $dc_T/d\pi^*$ in the model with no bonds:

$$dc_T/d\pi^* = -\bar{f}+\theta(d\bar{f}/d\pi^*).$$

Here we construct an example to show this derivative could be positive or negative, so it is truly ambiguous. Note that the derivative could be written as

$$dc_T/d\pi^* = -d\pi^*\bar{f}/d\pi^*+(\theta+\pi^*)d\bar{f}/d\pi^* = \bar{f}(\bar{\varepsilon}-1)+(\theta+\pi^*)d\bar{f}/d\pi^*.$$

Inspection of the dynamic system immediately reveals that as u_{TT} approaches infinity, the negative eigenvalue approaches $-\pi^*$.[12] Under this assumption, the matrix of the dynamic system would be upper triangular, so the eigenvalues would equal the diagonal elements. If so, the term $(\theta+\pi^*)d\bar{f}/d\pi^*$ approaches zero (noting that $d\bar{f}/d\pi^*$ goes to a finite number—$\bar{u}_T\bar{v}_{mm}/D$). However, the elasticity does not involve u_{TT}, and so it can be either greater or less than one. Hence, in this example $dc_T/d\pi^*$ can be either positive or negative.

Dynamics of System with Bonds

Using equations ⟨13⟩ and ⟨14⟩, the dynamic system given by equations ⟨8⟩, ⟨15⟩, and ⟨3d⟩ can be linearized as

$$\begin{bmatrix} \dot{m} \\ \dot{z} \\ \dot{c}_T \end{bmatrix} \cong \begin{bmatrix} -\bar{m}D/\bar{v}_{ff}\bar{u}_T & 0 & B\bar{u}_{TT}\bar{m}/\bar{v}_{ff}\bar{u}_T^2 \\ \bar{v}_f\bar{v}_{fm}/\bar{u}_T\bar{v}_{ff} & r & -(1+\bar{v}_f^2\bar{u}_{TT}/\bar{u}_T^2\bar{v}_{ff}) \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} m-\bar{m} \\ z-\bar{z} \\ c_T-\bar{c}_T \end{bmatrix}.$$

The roots of the system are 0, r and $-\bar{m}D/\bar{v}_{ff}\bar{u}_T$ (>0), so it is unstable.

The system of equations ⟨13⟩, ⟨14⟩, and ⟨16⟩ determine the values of c_T, f, and m.

Totally differentiating, we have

$$v_{ff}df + v_{fm}dm - (r+\pi^*)u_{TT}dc_T = u_T d\pi^*,$$
$$v_{fm}df + v_{mm}dm - (r+\mu)u_{TT}dc_T = u_T d\mu,$$
$$-(r+\pi^*)df - dc_T = f d\pi^*.$$

Solving, we obtain

$$\begin{bmatrix} df \\ dm \\ dc_T \end{bmatrix} = (1/F)\begin{bmatrix} -v_{mm} & v_{fm} & Au_{TT}/u_T \\ v_{fm} + ((\pi^*+r)v_m u_{TT})/u_T & -v_{ff} - ((\pi^*+r)v_f u_{TT})/u_T & Bu_{TT}/u_T \\ (r+\pi^*)v_{mm} & (r+\pi^*)v_{fm} & D \end{bmatrix}\begin{bmatrix} u_T d\pi^* \\ u_T d\mu \\ f d\pi^* \end{bmatrix},$$

where

$$F = -D - A(r+\pi^*)u_{TT}/u_T < 0,$$

and A, B, and D are defined in the text in equations ⟨5⟩.

In particular

$$df/d\mu = u_T v_{fm}/F > 0,$$
$$df/d\pi^* = (-u_T v_{mm} + Afu_{TT}/u_T)/F < 0,$$
$$dc_T/d\pi^* = (u_T(r+\pi^*)v_{mm} + fD)/F = (fD/F)(1-((r+\pi^*)/\pi^*)\varepsilon).$$

Notes

1. The set-up is similar to the descriptive (non-optimizing) frameworks of Calvo and Rodriguez (1977) and Daniel (1985). It shares some of the characteristics of the optimizing models of Calvo (1985), Feenstra (1985), Sen (1986), and Rogers (1987).
2. Liviatan (1981) deflates home and foreign money by the home goods price, but notes that the choice of deflators is not critical for the issues examined. In an earlier version of this paper (Engel, 1986), I deflated domestic money by home goods prices and foreign money by traded goods prices. The results presented here are qualitatively the same under those assumptions. Sen (1986) deflates both currencies by a price index.
3. Calvo (1985) considers the case in which utility over goods and money is not separable.
4. $\dot{x}$ is the time derivative of x throughout this paper.
5. The discount rate is assumed to be constant in order to keep the illustrative model of this paper simple. It also leads to a useful borderline case examined in Section III in which changes in domestic and foreign inflation have no effect on real foreign asset accumulation. Other settings in which individuals have finite lifespans or in which the rate of time preference is not constant may lead to more interesting dynamics, but for the purposes of this paper the example pursued here is adequate and convenient. Given the constancy of the discount rate, its equality to r is needed to ensure that wealth does not go to zero or that wealth accumulation is not unbounded.
6. This condition holds, of course, in the model with bonds as given by equation ⟨3d⟩. It is also true of the model in steady state when bonds are not traded. This is the model considered by Liviatan (1981) and examined in Section II.
7. Since, as we will see in Section II, consumption of home goods never changes, this term is not important.
8. Strictly speaking, a negative sign for v_{fm} is not necessary for the CR property to hold. If v_{fm} were a small enough positive number, then even though demand for foreign money might move perversely in response to a change in domestic inflation, the ratio might not.
9. Liviatan's set-up and notation are slightly different than that here, so I have changed the notation from that in the original quotes to match that of this paper.
10. If $m/f = (v_{ff}/v_{mm})^{1/2}$, then df need not equal zero.

11. $\bar{x}$ represents the steady-state value of x.
12. Here we make the same assumption as when we showed the trace was positive—that $\mu = \pi^*$ initially, so that $(v_{mm} - v_{fm})$ is negative when the goods are normal. Therefore,the first diagonal element is positive.

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