

## Appendix

### AI. Model Specification

#### Consumers

The indexes of consumption of home and foreign goods are given by:

$$C_h = \left[ n^{-1/\lambda} \int_0^n C_h(i)^{\lambda-1/\lambda} di \right]^{\lambda/\lambda-1}; \quad C_f = \left[ (1-n)^{-1/\lambda} \int_n^1 C_f(i)^{\lambda-1/\lambda} di \right]^{\lambda/\lambda-1}.$$

The price indexes for home and foreign goods are given by:

$$P_h = \left[ \frac{1}{n} \int_0^n P_h(i)^{1-\lambda} di \right]^{1/1-\lambda}, \quad P_f = \left[ \frac{1}{1-n} \int_n^1 P_f(i)^{1-\lambda} di \right]^{1/1-\lambda}.$$

The intratemporal decisions by households can be characterized by the following relationships:

$$\begin{aligned} C_h(i) &= \frac{1}{n} \left[ \frac{P_h(i)}{P_h} \right]^{-\lambda} C_h, & C_f(i) &= \frac{1}{1-n} \left[ \frac{P_f(i)}{P_f} \right]^{-\lambda} C_f; \\ P_h C_h &= nPC, & P_f C_f &= (1-n)PC; \\ \int_0^n P_h(i) C_h(i) di &= P_h C_h, & \int_n^1 P_f(i) C_f(i) di &= P_f C_f. \end{aligned}$$

#### Firms

The optimization problem can be expressed as maximizing the expected present value of profits using the market nominal discount factor for the owners of the firm. Given there is no intertemporal aspect to the firms' optimization problems (see, Obstfeld and Rogoff (1998)), this reduces to maximizing in the PCP case:

$$E_{t-1} [d_t \pi_t(i)] = E_{t-1} \left[ d_t \left( (P_{ht}(i) - (W_t / \theta_t)(X_{ht}(i) + X_{ht}^*(i))) \right) \right],$$

where  $X_{ht}(i) = nC_{ht}(i)$  is total sales of firm  $i$  to home residents and  $X_{ht}^*(i) = (1-n)C_{ht}^*(i)$  is total sales to foreign residents.

In the PTM model, the firm chooses two different prices – one to charge residents of its own country, and one to charge residents of the other country. The typical home firm maximizes:

$$E_{t-1} \left[ d_t \left( P_{ht}(i) X_{ht}(i) + S_t P_{ht}^*(i) X_{ht}^*(i) - (W_t / \theta_t) (X_{ht}(i) + X_{ht}^*(i)) \right) \right].$$

Optimal prices in both cases are presented in Table 1 in the text.

### Money Demand Approximation

The money market equilibrium condition (1.2) may be approximated by first writing (1.2) as:

$$(A.1) \quad 1 - \left( \frac{P_t}{M_t} \right)^\varepsilon V_t C_t^\rho \chi = \beta E_t \frac{P_t C_t^\rho}{P_{t+1} C_{t+1}^\rho}$$

The right hand side may be expressed as

$$(A.2) \quad \beta E_t \frac{P_t C_t^\rho}{P_{t+1} C_{t+1}^\rho} = \beta \exp(p_t + \rho c_t - E_t p_{t+1} + \frac{\sigma_p^2}{2} - \rho E_t c_{t+1} + \rho^2 \frac{\sigma_c^2}{2} - \rho \sigma_{cp})$$

The left-hand side of (A.1) may be approximated around a steady state where the nominal interest rate

is constant, so that  $\beta E_t \frac{P_t C_t^\rho}{P_{t+1} C_{t+1}^\rho} = \frac{1}{1+i}$ . This gives

$$(A.3) \quad \ln \left( 1 - \left( \frac{P_t}{M_t} \right)^\varepsilon V_t C_t^\rho \chi \right) \approx \ln \left( \frac{1}{1+i} \right) + i(\varepsilon(m_t - p_t) - \rho c_t - \ln(V_t)) + i \ln \left( \frac{i}{1+i} \chi^{-1} \right)$$

Equating the log of (A.2) and (A.3) gives the money market clearing condition (1.3) of the text.

## AII. The PCP Model

### Solution for PCP Specification

Under PCP, there is full PPP at all times, so we have  $P_t = S_t P_t^*$  and  $C_t = C_t^*$ . By subtracting the foreign country money market clearing condition from the home condition, using PPP and the assumption of random walk money supply, we may derive a solution for the exchange rate given by

$$(A.4) \quad s_t = \Gamma_s + m_t - m_t^* - \frac{1}{\varepsilon} (\ln(V_t) - \ln(V_t^*)),$$

where  $\Gamma_s$  is a constant.

Now combine the home and foreign money market equilibrium conditions to obtain the equation:

$$(A.5) \quad \tilde{m}_t - \tilde{p}_t = \frac{\rho}{\varepsilon} c_t - \frac{\rho}{i\varepsilon} (E_t c_{t+1} - c_t) - \frac{1}{i\varepsilon} (E_t \tilde{p}_{t+1} - \tilde{p}_t) + \frac{1}{\varepsilon} (n \ln(V_t) + (1-n) \ln(V_t^*)) + \tilde{\Gamma}_m$$

where  $\tilde{\Gamma}_m$  is a constant, and  $\tilde{p}_t = np_{ht} + (1-n)p_{ft}^*$ ,  $\tilde{m}_t = nm_t + (1-n)m_t^*$ . Taking expectations dated period t-1, and solving for  $\tilde{p}_t$  gives:

$$(A.6) \quad \tilde{p}_t = -\Gamma'_m + \tilde{m}_{t-1} - \frac{1}{\varepsilon} (n \ln \theta_{t-1} + (1-n) \ln \theta_{t-1}^*) - \frac{1}{\varepsilon} (n \ln(V_{t-1}) + (1-n) \ln(V_{t-1}^*)),$$

where  $\Gamma'_m$  is a constant. Using (A.5) and (A.6), we may derive equation (3.2).

Now from (1.4) (and its foreign equivalent), and the pricing equations in Table 1, we may write implicitly the equation which determines the period t-1 expected value of consumption as

$$(A.7) \quad 1 = \frac{\lambda}{\lambda - 1} \eta \frac{\left( E_{t-1} \left( \frac{C_t S_t^{1-n}}{\theta_t} \right) \right)^n \left( E_{t-1} \left( \frac{C_t S_t^{-n}}{\theta_t^*} \right) \right)^{1-n}}{E_{t-1} (C_t^{1-\rho})}$$

Now using (A.7), imposing the fact that the solution for consumption and the exchange rate will be log-normal, and taking logs, we may solve for the value of expected consumption as

$$(A.8) \quad E_{t-1}c_t = -\frac{1}{\rho} \ln\left(\frac{\lambda\eta}{\lambda-1}\right) - \frac{(2-\rho)}{2} \sigma_c^2 - \frac{n(1-n)}{2\rho} \sigma_s^2 + \frac{(n \ln \theta_{t-1} + (1-n) \ln \theta_{t-1}^*)}{\rho} \\ - \frac{(n\sigma_u^2 + (1-n)\sigma_{u^*}^2)}{2\rho} + \frac{(n\sigma_{cu} + (1-n)\sigma_{cu^*})}{\rho} + n(1-n) \frac{(\sigma_{su} - \sigma_{su^*})}{\rho}$$

Note that

$$E_{t-1}C_t^{1-\rho} = (E_{t-1}C_t)^{1-\rho} \exp\left(-\frac{\rho(1-\rho)}{2} \sigma_c^2\right).$$

Using this relationship and equation (A.8) allows us to derive equation (3.3), because we have

$$(A.9) \quad E_{t-1}C_t^{1-\rho} = \exp(1-\rho)(E_{t-1}c_t + \frac{(1-\rho)}{2} \sigma_c^2)$$

### Proof of Proposition 1:

1) Since  $\frac{\lambda - (1-\rho)(\lambda-1)}{\lambda} > 0$ , maximizing  $E_{t-1}\hat{U}_t$  is equivalent from equation (3.3) to

$$\text{maximizing } \bar{U} \equiv -\frac{\sigma_c^2}{2} - \frac{n(1-n)}{2\rho} \sigma_s^2 - \frac{\tilde{\sigma}_u^2}{2\rho} + \frac{\tilde{\sigma}_{cu}}{\rho} + n(1-n) \frac{(\sigma_{su} - \sigma_{su^*})}{\rho}.$$

Using equations (3.1) and (3.2), in conjunction with the monetary rules (3.4) and (3.5), we can calculate the following variances and covariances:

$$\sigma_c^2 = [\phi(na_1 + (1-n)b_2) + \psi n]^2 \sigma_u^2 + [\phi(na_2 + (1-n)b_1) + (1-n)\psi]^2 \sigma_{u^*}^2 \\ + [\phi(na_3 + (1-n)b_4 - \frac{n}{\varepsilon})]^2 \sigma_v^2 + [\phi(na_4 + (1-n)b_3 - \frac{(1-n)}{\varepsilon})]^2 \sigma_{v^*}^2$$

$$\sigma_s^2 = (a_1 - b_2)^2 \sigma_u^2 + (a_2 - b_1)^2 \sigma_{u^*}^2 + (a_3 - b_4 - \frac{1}{\varepsilon})^2 \sigma_v^2 + (a_4 - b_3 + \frac{1}{\varepsilon})^2 \sigma_{v^*}^2$$

$$\tilde{\sigma}_{cu} = n[\phi(na_1 + (1-n)b_2) + \psi n] \sigma_u^2 + (1-n)[\phi(na_2 + (1-n)b_1) + (1-n)\psi] \sigma_{u^*}^2$$

$$\sigma_{su} = (a_1 - b_2) \sigma_u^2$$

$$\sigma_{su^*} = (a_2 - b_1)\sigma_{u^*}^2$$

We substitute these expressions into the objective function (3.3). Then, since the objective function for home and foreign governments is identical, the equilibrium policy choices are found simply by choosing  $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$  to maximize  $\bar{U}$  defined above. The resulting expressions in Proposition 1 can be derived by straightforward algebra.

2) We demonstrate here that the solutions for  $C_t, C_{ht}, C_{ft}, L_t$  and  $\frac{P_{ht}}{S_t P_{ft}}$  are identical to the solutions in the flexible-price model. Under the optimal policies, we find:

$$\sigma_c^2 = \frac{n^2 \sigma_u^2 + (1-n)^2 \sigma_{u^*}^2}{\rho^2} \quad \sigma_s^2 = \sigma_u^2 + \sigma_{u^*}^2$$

$$\sigma_{cu} = \frac{n}{\rho} \sigma_u^2 \quad \sigma_{cu^*} = \frac{1-n}{\rho} \sigma_{u^*}^2$$

$$\sigma_{su} = \sigma_u^2 \quad \sigma_{su^*} = -\sigma_{u^*}^2$$

Substituting into equation (A.8), we find

$$E_{t-1}c_t = -\frac{1}{\rho} \ln\left(\frac{\lambda\eta}{\lambda-1}\right) + \frac{(n \ln \theta_{t-1} + (1-n) \ln \theta_{t-1}^*)}{\rho}$$

Using this relationship, and the optimal monetary rules, equation (3.2) gives us

$$c_t = -\frac{1}{\rho} \ln\left(\frac{\lambda\eta}{\lambda-1}\right) + \frac{(n \ln \theta_t + (1-n) \ln \theta_t^*)}{\rho}$$

Taking the antilogs, we see that  $C_t$  is identical to the flex-price solution in equation (2.1).

Using equation (1.3) and its foreign counterpart, we find  $s_t = \ln \theta_t - \ln \theta_t^*$  or  $S_t = \frac{\theta_t}{\theta_t^*}$ . From

(1.4) and the pricing equations, we can write  $P_{ht} = \frac{\lambda}{\lambda-1} \frac{E_{t-1}\left(\frac{P_t C_t}{\theta_t}\right)}{E_{t-1}(C_t^{1-\rho})}$ . Using our solution for the

exchange rate, we have  $P_{ft}^* = \frac{\lambda}{\lambda-1} \frac{E_{t-1}\left(\frac{P_t^* C_t}{\theta_t^*}\right)}{E_{t-1}(C_t^{1-\rho})} = \frac{\lambda}{\lambda-1} \frac{E_{t-1}\left(\frac{P_t C_t}{S_t \theta_t^*}\right)}{E_{t-1}(C_t^{1-\rho})} = P_{ht}$ . So, the terms of trade are

given by  $\frac{P_{ht}}{S_t P_{ft}^*} = \frac{\theta_t^*}{\theta_t}$ , just as in equation (2.2) for the flex-price model.

In both the flexible price model and the PCP model, we can write

$$C_{ht} = \frac{n P_t C_t}{P_{ht}} = n \left( \frac{P_{ht}}{S_t P_{ft}^*} \right)^{n-1} C_t.$$

Since  $C_t$  and the terms of trade are identical in the two models, then so must be  $C_{ht}$ . A similar

argument shows  $C_{ft}$  is identical in the models. Since  $L_t = \frac{C_{ht}}{n \theta_t}$  in both models, it is also the same.

### AIII. The LCP Model

#### Solution for the LCP specification

Under the LCP specification, PPP does not generally hold, and so there is a failure of full risk-sharing. By using equation (1.4) and its foreign equivalent, the risk-sharing condition (1.5), and the pricing equations in Table 1, we may write the equation which implicitly defines expected home country consumption as

$$(A.10) \quad 1 = \frac{\lambda}{\lambda-1} \eta \frac{\left[ E_{t-1}\left(\frac{C_t}{\theta_t}\right) \right]^n \left[ E_{t-1}\left(\frac{C_t}{\theta_t^*}\right) \right]^{1-n}}{E_{t-1}(C_t^{1-\rho})}$$

Using the log-normality assumption, and taking logs, we may solve for expected consumption as

$$(A.11) \quad E_{t-1}c_t = -\frac{1}{\rho} \ln\left(\frac{\lambda\eta}{\lambda-1}\right) - \frac{(2-\rho)}{2} \sigma_c^2 + \frac{(n \ln \theta_{t-1} + (1-n) \ln \theta_{t-1}^*)}{\rho} - \frac{(n\sigma_u^2 + (1-n)\sigma_{u^*}^2)}{2\rho} + \frac{(n\sigma_{cu} + (1-n)\sigma_{cu^*})}{\rho}$$

A similar procedure establishes that

$$(A.12) \quad E_{t-1}c_t^* = -\frac{1}{\rho} \ln\left(\frac{\lambda\eta}{\lambda-1}\right) - \frac{(2-\rho)}{2} \sigma_{c^*}^2 + \frac{(n \ln \theta_{t-1} + (1-n) \ln \theta_{t-1}^*)}{\rho} - \frac{(n\sigma_u^2 + (1-n)\sigma_{u^*}^2)}{2\rho} + \frac{(n\sigma_{c^*u} + (1-n)\sigma_{c^*u^*})}{\rho}$$

From the home economy money market clearing condition (1.3), taking expectations based on  $t-1$  information, and using (A.11), we may solve for the home price level as

$$(A.13) \quad p_t = \Gamma_p + m_{t-1} - \frac{1}{\varepsilon} \ln(V_{t-1}) - \frac{1}{\varepsilon} (n \ln \theta_{t-1} + (1-n) \ln \theta_{t-1}^*)$$

Taking the difference of the money market condition (1.3) from its expected value, we get

$$\mu_t = \frac{\rho(1+i)}{i\varepsilon} (c_t - E_{t-1}c_t) - \frac{1}{i\varepsilon} (E_t(p_{t+1} + \rho c_{t+1}) - E_{t-1}(p_{t+1} + \rho c_{t+1})) + \frac{1}{\varepsilon} v_t.$$

Then using (A.11) and (A.13) we derive equation (4.1). Equation (4.2) is derived analogously.

We can use the risk-sharing condition (1.5), and the price equation (A.13) we get:

$$(A.14) \quad s_t = \rho\phi(m_t - m_t^* - \frac{1}{\varepsilon} (\ln(V_t) - \ln(V_t^*))) - \frac{i(\varepsilon-1)}{1+i} (p_t - p_t^*) + \Gamma_s,$$

where  $\Gamma_s$  is a constant term. First-differencing yields equation (4.3).

Equations (4.6) and (4.7) are arrived at by using equation (A.9) and its foreign counterpart along with equations (A.11) and (A.12).

### Proof of Proposition 2:

1) From equations (4.1) and (4.2) we can establish

$$\sigma_c^2 = (\phi a_1 + \psi n)^2 \sigma_u^2 + (\phi a_2 + \psi(1-n))^2 \sigma_{u^*}^2 + (\phi(a_3 - \frac{1}{\varepsilon}))^2 \sigma_v^2 + (\phi a_4)^2 \sigma_{v^*}^2$$

$$\tilde{\sigma}_{cu} = n(\phi a_1 + \psi m)\sigma_u^2 + (1-n)(\phi a_2 + \psi(1-n))\sigma_u^{*2}$$

$$\sigma_{c^*}^2 = (\phi b_2 + \psi m)^2 \sigma_u^2 + (\phi b_1 + \psi(1-n))^2 \sigma_u^{*2} + (\phi b_4)^2 \sigma_v^2 + (\phi(b_3 - \frac{1}{\varepsilon}))^2 \sigma_v^{*2}$$

$$\tilde{\sigma}_{c^*u} = n(\phi b_2 + \psi m)\sigma_u^2 + (1-n)(\phi b_1 + \psi(1-n))\sigma_u^{*2}$$

From these expressions, we see that  $E_{t-1}C_t^{*1-\rho}$  is not influenced by the policy choices of the home country. Since  $\frac{\lambda - n(\lambda - 1)(1 - \rho)}{\lambda} > 0$ , the home objective can be restated as maximizing

$$\bar{U} = -\frac{\sigma_c^2}{2} - \frac{\tilde{\sigma}_u^2}{2\rho} + \frac{\tilde{\sigma}_{cu}}{\rho}. \text{ Similar reasoning leads us to be able to rewrite the foreign objective as}$$

$$\text{maximizing } \bar{U}^* = -\frac{\sigma_{c^*}^2}{2} - \frac{\tilde{\sigma}_u^{*2}}{2\rho} + \frac{\tilde{\sigma}_{c^*u}}{\rho}. \text{ The first-order conditions for maximizing } \bar{U} \text{ and } \bar{U}^*$$

yield the solutions in Proposition 2.

2) Using this solution, we get  $\mu_t - \mu_t^* - \frac{1}{\varepsilon}(v_t - v_t^*) = 0$ . It then follows from equation (A.13) that

$p_t - p_t^*$  is a constant. Using these two facts, it follows immediately from (4.3) that the exchange rate is constant.

### Proof of Proposition 3:

The second part of proposition 2 establishes that the exchange rate is constant over time. Moreover, nominal prices cannot adjust within any period. Hence, under the LCP specification, the terms of trade cannot satisfy the flexible price equation (2.3). Furthermore, given that relative goods prices are constant within a period, from (1.7), total demand for each country's product must respond identically. Thus, the response of home and foreign output to a shock are identical. As a result, in response to a country specific productivity shock, employment cannot be equated across countries, as required by the flexible price equation (2.2). For both reasons, consumption of individual countries goods cannot satisfy the flexible price equations (2.4) and (2.5).

**Proof of Proposition 4:**

Under cooperation with welfare of home and foreign households treated equally, the objective function is to maximize

$$nE_{t-1}\hat{U}_t + (1-n)E_{t-1}\hat{U}_t^* = \left( \frac{n(1+(\lambda-1)\rho)}{\lambda(1-\rho)} \right) E_{t-1}C_t^{1-\rho} + \left( \frac{(1-n)(1+(\lambda-1)\rho)}{\lambda(1-\rho)} \right) E_{t-1}C_t^{*(1-\rho)}$$

Home monetary policy only influences  $E_{t-1}C_t^{1-\rho}$  and foreign monetary policy only influences

$E_{t-1}C_t^{*(1-\rho)}$ . Since  $1+(\lambda-1)\rho > 0$ , the optimal cooperative policy is to choose  $a_1, a_2, a_3, a_4$  to

$$\text{maximize } \bar{U} = -\frac{\sigma_c^2}{2} - \frac{\tilde{\sigma}_u^2}{2\rho} + \frac{\tilde{\sigma}_{cu}}{\rho} \text{ and to choose } b_1, b_2, b_3, b_4 \text{ to maximize } \bar{U}^* = -\frac{\sigma_{c^*}^2}{2} - \frac{\tilde{\sigma}_u^2}{2\rho} + \frac{\tilde{\sigma}_{c^*u}}{\rho}.$$

This is the same problem that is solved in the noncooperative equilibrium.

**Proof of Proposition 5:**

When foreign monetary policy is given as described under Proposition 2, then

$$\mu_t^* = \frac{1}{\varepsilon}(\tilde{u}_t + v_t^*).$$

From equation (4.3), then

$$(A.15) \quad \mu_t = \frac{1}{\varepsilon}(\tilde{u}_t + v_t) + \frac{1}{\rho\phi}(s_t - s_{t-1}) + \frac{i(\varepsilon-1)}{\rho\phi(1+i)}(p_t - p_{t-1} - (p_t^* - p_{t-1}^*)).$$

Using the exchange-rate policy rule (4.8), and substituting (A.15) into (4.1), we can calculate:

$$\sigma_c^2 = \frac{(\alpha_1 + n)^2}{\rho^2} \sigma_u^2 + \frac{(\alpha_2 + 1 - n)^2}{\rho^2} \sigma_{u^*}^2 + \frac{\alpha_3^2}{\rho^2} \sigma_v^2 + \frac{\alpha_4^2}{\rho^2} \sigma_{v^*}^2$$

$$\tilde{\sigma}_{cu} = \frac{n(\alpha_1 + n)}{\rho} \sigma_u^2 + \frac{(1-n)(\alpha_2 + 1 - n)}{\rho} \sigma_{u^*}^2$$

Home policymakers maximize  $\bar{U} = -\frac{\sigma_c^2}{2} - \frac{\tilde{\sigma}_u^2}{2\rho} + \frac{\tilde{\sigma}_{cu}}{\rho}$ . Straightforward algebra yields the optimal

policy choices  $\alpha_1^N = \alpha_2^N = \alpha_3^N = \alpha_4^N = 0$ .

The optimization problem for the foreign government is exactly as in the proof of Proposition 2. The first-order conditions for their policy choice are not even a function of the policy choices of the home country. So, their optimal policy choices are the same as in Proposition 2.

Substituting these optimal rules back into equation (4.3), we find  $\mu_t = \frac{1}{\varepsilon}(\tilde{u}_t + v_t)$ . This is the same money growth rate under problem P2. It follows that the equilibrium is identical to the one under the policy rules described in Proposition 2.

#### AIV. Endogenous Risk Sharing

We now analyze the case where monetary policy is set *before* private agents engage in optimal cross-country risk sharing. In order to examine this case, we need to introduce a more formal state-specific approach to the model. We use the following notation for the distribution of events. Say that at any time  $t$  event  $\vartheta_t$  is drawn from a finite number of possible events. The history of all events up to and including that at time  $t$  is denoted  $\vartheta^t = \{\vartheta_t, \vartheta_{t-1}, \vartheta_{t-2}, \dots\}$ , which occurs with date zero probability equal to  $\pi(\vartheta^t)$ .

#### Optimal Risk Sharing

Households have utility functions given as in section 2 of the text, where we define consumption of the home household at any history  $\vartheta^t$  as  $C(\vartheta^t)$ . The economy begins at date  $t = 0$ . But trading in a complete set of nominal state contingent assets can take place at date  $t = -1$ , before the realization of shocks for date 0. Let  $q(\vartheta^t)$  be defined as the price of delivery of one dollar of home currency at history  $\vartheta^t$ . Then the date  $-1$  budget constraint for the home individual is given by

$$(A16) \quad \sum_{t=0}^{\infty} \sum q(\vartheta^t) \left( P(\vartheta^t) C(\vartheta^t) + M(\vartheta^t) - W(\vartheta^t) L(\vartheta^t) - \Pi(\vartheta^t) - M(\vartheta^{t-1}) - T(\vartheta^t) \right) = 0,$$

where the second summation is taken over the finite set of possible histories at any date  $t$ . Here the notation  $P(\vartheta^t)$  is defined as the consumer price index at history  $\vartheta^t$ , etc.  $\Pi(\vartheta^t)$  is defined as total

profits earned by the home household from ownership of home firms at history  $\vartheta^t$ , and  $T(\vartheta^t)$  represents money transfers from the central bank at history  $\vartheta^t$ .

Likewise, the foreign household faces a date-1 budget constraint given by

$$(A17) \quad \sum_{t=0}^{\infty} \sum q(\vartheta^t) S(\vartheta^t) \left( P^*(\vartheta^t) C^*(\vartheta^t) + M^*(\vartheta^t) - W^*(\vartheta^t) L(\vartheta^t) - \Pi^*(\vartheta^t) - M^*(\vartheta^{t-1}) - T^*(\vartheta^t) \right) = 0.$$

Consumption of the home and foreign good bundles, and individual consumption of home and foreign goods are defined in the same way as before, except to allow for the state-contingent notation. Prices are set by firms maximizing the state contingent value of profits, defined for the home country as

$$\sum_{t=0}^{\infty} \sum q(\vartheta^t) \Pi(\vartheta^t)$$

where the individual profit functions are as defined above, both for the PCP and the LCP cases.

We first derive the conditions relating to optimal international risk-sharing across households. The home household chooses a sequence of consumption levels, money balances, and labor supply to maximize expected utility subject to (A16). The first order conditions may be written as

$$(A18) \quad \beta^t \pi(\vartheta^t) C(\vartheta^t)^{-\rho} = \gamma q(\vartheta^t) P(\vartheta^t)$$

$$(A19) \quad \beta^t \pi(\vartheta^t) \frac{1}{M(\vartheta^t)} = \gamma \left[ q(\vartheta^t) - \sum_{\vartheta^{t+1} \in \vartheta_c^t} q(\vartheta^{t+1}) \right]$$

$$(A20) \quad \beta^t \eta = \gamma q(\vartheta^t) W(\vartheta^t)$$

In these conditions, the variable  $\gamma$  represents the Lagrange multiplier on the home country budget constraint (A16)<sup>1</sup>. This is endogenously determined by the need to satisfy (A16), and will in turn determine, in a world equilibrium, the weight placed on home country consumption in the risk sharing

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<sup>1</sup> The term  $\sum_{\vartheta^{t+1} \in \vartheta_c^t} q(\vartheta^{t+1})$  denotes the price of a unit of currency in the subset of all histories  $\vartheta^{t+1}$  that are continuations of  $\vartheta^t$ . Then  $\frac{\sum_{\vartheta^{t+1} \in \vartheta_c^t} q(\vartheta^{t+1})}{q(\vartheta^t)}$  is an expression for the inverse of the nominal interest between period  $t$  and  $t+1$ .

rule. Note that this multiplier is state and time invariant. Because of the presence of complete markets, households only face one effective budget constraint.

The analogous relationship to equation (A18) for the foreign country is

$$(A21) \quad \beta^t \pi(\vartheta^t) C^*(\vartheta^t)^{-\rho} = \gamma^* q(\vartheta^t) S^*(\vartheta^t) P^*(\vartheta^t).$$

First, notice that (A18) and (A21) together imply that the following relationship holds between home and foreign consumption:

$$(A22) \quad \frac{\gamma}{\gamma^*} C(\vartheta^t)^\rho = \frac{S(\vartheta^t) P^*(\vartheta^t)}{P(\vartheta^t)} C^*(\vartheta^t)^\rho$$

which is just equation (1.5) of the text, where in the notation of the text,  $\Gamma = \frac{\gamma}{\gamma^*}$ . We may now

determine the value of  $\Gamma$  implied by an equilibrium in the market for state-contingent nominal assets, however. To do this, rearrange (A18) and (A22) in the following way

$$(A23) \quad q(\vartheta^t) = \frac{\beta^t \pi(\vartheta^t)}{\gamma P(\vartheta^t) C(\vartheta^t)^\rho} \quad q(\vartheta^t) = \frac{\beta^t \pi(\vartheta^t)}{\gamma^* S(\vartheta^t) P^*(\vartheta^t) C^*(\vartheta^t)^\rho}.$$

From the home country budget constraint, imposing the condition that  $M(\vartheta^t) = M(\vartheta^{t-1}) + T(\vartheta^t)$ , and the definition of profits, we have

$$(A24) \quad \sum_{t=0}^{\infty} \sum q(\vartheta^t) \left( P(\vartheta^t) C(\vartheta^t) - \int_0^n P_h(i, \vartheta^t) Y(i, \vartheta^t) di \right) = 0$$

By market clearing across home goods, and using the definition of the household consumption demands, we have that

$$(A25) \quad \int_0^n P_h(i, \vartheta^t) Y(i, \vartheta^t) di = n P(\vartheta^t) C(\vartheta^t) + (1-n) S(\vartheta^t) P^*(\vartheta^t) C^*(\vartheta^t).$$

Hence, using (A23), (A24), and (A25), we have

$$\sum_{t=0}^{\infty} \sum \frac{\beta^t \pi(\vartheta^t) C(\vartheta^t)^{1-\rho}}{\gamma} = \sum_{t=0}^{\infty} \sum \frac{\beta^t \pi(\vartheta^t) C^*(\vartheta^t)^{1-\rho}}{\gamma^*}$$

Thus, the sharing parameter in equation (A22) is therefore given by:

$$(A26) \quad \Gamma = \frac{\sum_{t=0}^{\infty} \sum \beta^t \pi(\vartheta^t) C(\vartheta^t)^{1-\rho}}{\sum_{t=0}^{\infty} \sum \beta^t \pi(\vartheta^t) C^*(\vartheta^t)^{1-\rho}}$$

### Endogenous Risk Sharing under PCP

Now using (A22) and (A26), we may establish the following result.

#### Result AIV.1

Under PCP, the equilibrium value of  $\Gamma$  is unity.

Proof: Under PCP, PPP holds, so we have  $C(\vartheta^t) = \Gamma^{\frac{1}{\rho}} C^*(\vartheta^t)$ . Substitute this into (A26) to get

$$\Gamma = \frac{\sum_{t=0}^{\infty} \sum \beta^t \pi(\vartheta^t) C^*(\vartheta^t)^{1-\rho}}{\sum_{t=0}^{\infty} \sum \beta^t \pi(\vartheta^t) C^*(\vartheta^t)^{1-\rho}} \Gamma^{-\frac{(1-\rho)}{\rho}}$$

The only solution to this equation must have  $\Gamma = 1$ .

Hence, as long as the law of one price holds, optimal risk sharing will equalize consumption across countries in all times and states. As a result, even if the optimal monetary rules are chosen *before* the opening of international financial markets, the optimal monetary policy under PCP coincides with that identified in the text.

It may seem very surprising that in the PCP case, we must have  $\Gamma = 1$ , irrespective of the policies chosen. The intuition, though, is simple. Because of the Cobb-Douglas utility function, and the law of one price, the world expenditure shares on each country's output are constant. This is true irrespective of the realization of productivity shocks, money demand shocks, or monetary policy.

Since wealth is determined the value of output sold,  $\Gamma$  is independent of productivity shocks, money demand shocks, or monetary policy.

### Endogenous Risk Sharing under LCP

We now focus on the LCP case. Under LCP, the approach used in Result 1 cannot be used, because PPP does not hold, and in general the real exchange rate will be state and time dependent. However, we can use the endogenous sharing rule given from (A26) in the construction of the objective function for the policy maker under LCP.

As we have noted, the detrended values of all variables  $X_t$  in the model are stationary, where the trend is given by the time  $-1$  expectation of the flexible price solution for  $X_t$ . We will use the notation  $\widehat{E}_{-1}$  to refer to the expectation conditional on  $-1$  information of the detrended variable. The model is “stationary” in the sense that  $\widehat{E}_{-1}X_t$  is constant for all  $t$ , for any variable  $X_t$ .

So  $C_t^{1-\rho} / E_{-1}(\widehat{C}_t^{1-\rho})$  is stationary, where  $\widehat{C}_t \equiv (\theta_t^n \theta_t^{*(1-n)})^{\frac{1}{\rho}}$ . Likewise,  $C_t^{*1-\rho} / E_{-1}(\widehat{C}_t^{1-\rho})$  is stationary. Dividing numerator and denominator of equation (A26) by  $E_{-1}(\widehat{C}_t^{1-\rho})$ , we have

$$\Gamma = \frac{\sum_{t=0}^{\infty} \sum \beta^t \pi(\vartheta^t) \frac{C(\vartheta^t)^{1-\rho}}{E_{-1}\widehat{C}_t^{1-\rho}}}{\sum_{t=0}^{\infty} \sum \beta^t \pi(\vartheta^t) \frac{C^*(\vartheta^t)^{1-\rho}}{E_{-1}\widehat{C}_t^{1-\rho}}} = E_{-1} \left( \frac{C_t^{1-\rho}}{E_{-1}\widehat{C}_t^{1-\rho}} \right) / E_{-1} \left( \frac{C_t^{*1-\rho}}{E_{-1}\widehat{C}_t^{1-\rho}} \right) = \frac{\widehat{E}_{-1}C_t^{1-\rho}}{\widehat{E}_{-1}C_t^{*1-\rho}}.$$

Then, using this expression in the optimal price setting rules, we may write the analogous expression to (A10) as

$$1 = \frac{\lambda}{\lambda-1} \eta \Gamma^{-n} \frac{\left[ E_{t-1} \left( \frac{C_t}{\theta_t} \right) \right]^n \left[ E_{t-1} \left( \frac{C_t}{\theta_t^*} \right) \right]^{1-n}}{E_{t-1}(C_t^{1-\rho})}.$$

Now transform these variables so they are stationary:

$$1 = \frac{\lambda}{\lambda-1} \Upsilon(t) \eta \Gamma^{-n} \frac{\left[ \widehat{E}_{-1} \left( \frac{C_t}{\theta_t} \right) \right]^n \left[ \widehat{E}_{-1} \left( \frac{C_t}{\theta_t^*} \right) \right]^{1-n}}{\widehat{E}_{-1}(C_t^{1-\rho})}, \text{ where } \Upsilon(t) = \exp\left(\frac{1}{2}tn(1-n)(\sigma_u^2 + \sigma_{u^*}^2)\right).$$

This implies that: 
$$1 = \frac{\lambda}{\lambda-1} \Upsilon(t) \eta \left[ \frac{\widehat{E}_{-1} \left( \frac{C_t}{\theta_t} \right)}{\widehat{E}_{-1} C_t^{1-\rho}} \right]^n \left[ \frac{\widehat{E}_{-1} \left( \frac{C_t}{\theta_t^*} \right)}{\widehat{E}_{-1} C_t^{*(1-\rho)}} \right]^{1-n}.$$

Likewise, for the foreign country, we have

$$1 = \frac{\lambda}{\lambda-1} \Upsilon(t) \eta \left[ \frac{\widehat{E}_{-1} \left( \frac{C_t^*}{\theta_t^*} \right)}{\widehat{E}_{-1} C_t^{*1-\rho}} \right]^n \left[ \frac{\widehat{E}_{-1} \left( \frac{C_t}{\theta_t} \right)}{\widehat{E}_{-1} C_t^{*(1-\rho)}} \right]^{1-n}.$$

Unlike the case of the text, where the optimal policy rule is chosen after international risk sharing has taken place, in this case, the mean level of log consumption in any country is determined partly by the volatility of consumption in the other country. The solutions for the mean log consumption are

$$(A.27) \quad \begin{aligned} E_{-1} c_t \propto & -[n(2-\rho) + \rho(1-n)] \frac{\sigma_c^2}{2} - [(1-\rho)(1-n)] \frac{\sigma_c^{*2}}{2} - \frac{(n\sigma_u^2 + (1-n)\sigma_{u^*}^2)}{2\rho} \\ & + [n + \rho(1-n)] \frac{(n\sigma_{cu} + (1-n)\sigma_{cu^*})}{\rho} + [(1-\rho)(1-n)] \frac{(n\sigma_{cu}^* + (1-n)\sigma_{cu^*}^*)}{\rho} \end{aligned}$$

A similar procedure establishes that

$$(A.28) \quad \begin{aligned} E_{-1} c_t^* \propto & -[(1-n)(2-\rho) + \rho n] \frac{\sigma_c^2}{2} - [(1-\rho)n] \frac{\sigma_c^{*2}}{2} - \frac{(n\sigma_u^2 + (1-n)\sigma_{u^*}^2)}{2\rho} \\ & + [(1-n) + \rho n] \frac{(n\sigma_{cu}^* + (1-n)\sigma_{cu^*}^*)}{\rho} + [(1-\rho)n] \frac{(n\sigma_{cu} + (1-n)\sigma_{cu^*})}{\rho} \end{aligned}$$

Now using (4.4) of the text to derive the expected level of employment in the case where the policymaker takes account of endogenous risk sharing, we have

$$\widehat{E}_{-1} L_t = \frac{n(\lambda-1)\widehat{E}_{-1} C_t^{1-\rho}}{\lambda\eta} + \frac{(1-n)(\lambda-1)\Gamma\widehat{E}_{-1} C_t^{*(1-\rho)}}{\lambda\eta} = \frac{(\lambda-1)\widehat{E}_{-1} C_t^{1-\rho}}{\lambda\eta}$$

Hence, expected employment is a function of domestic consumption alone. It follows immediately that the home (foreign) policy makers objective function depends only on  $\widehat{E}_{-1} C_t^{1-\rho}$  ( $\widehat{E}_{-1} C_t^{*(1-\rho)}$ ).

Moreover, from (A27), we may establish that

$$(A29) \quad \widehat{E}_{-1} C_t^{1-\rho} \propto \exp \left( (1-\rho) \left[ - \left( \frac{\sigma_c^2}{2} - \frac{\tilde{\sigma}_{cu}^2}{\rho} \right) (1 - (1-n)(1-\rho)) - \frac{\tilde{\sigma}_u^2}{2\rho} - [(1-\rho)(1-n)] \left( \frac{\sigma_c^{*2}}{2} - \frac{\tilde{\sigma}_{cu}^*}{\rho} \right) \right] \right)$$

Comparing this to (4.6) of the text, and recalling that home (foreign) monetary rules have no effect on the second moments of foreign (home) consumption, we can establish:

### **Result AIV.2**

When policy makers choose the optimal monetary rules before international risk sharing takes place, the optimal rules under LCP are identical to those in the case where the policy maker takes the degree of risk sharing as given.

Proof: Since the home policy maker does not influence  $\sigma_c^{*2}$  or  $\tilde{\sigma}_{c^*u}^2$ , her objective function from (A29) is identical to that in the case of the text, where risk-sharing is taken as given. Hence, the optimal monetary rules of Proposition 2 still apply. The same reasoning applies to the problem of the foreign policy maker.

## **V. Generality of Optimal Rules**

Here we address the question of the generality of the optimal monetary rules identified in the LCP case. In particular, we have assumed a set of monetary rules by which the change in log money supply is a linear function of the random shocks, and as a result unforecastable in each period (equations 3.4 and 3.5). We then derived the optimal monetary policies in the case of each type of price setting. Are these policies fully optimal? In other words, does there exist an alternative set of monetary policies not restricted in the manner of (3.4) and (3.5), which could give higher expected utility for all agents? We approach this problem in the following way. Imagine that there was a social planner who weighted all individuals equally, and who could choose allocations subject only to the constraints that

- a) nominal prices cannot adjust within the period of a shock,

- b) prices represent a markup over marginal costs, so that there exists a monopoly distortion in the economy leading the marginal utility of consumption to be higher than the marginal disutility of work effort, for each type of good.

Then could the planner do better than the allocations induced by the monetary rules of Propositions 1 and Propositions 2, in each pricing case? If the planner could not do better than this, there could not exist other monetary rules that dominate those of Proposition 1 and 2 in each case.

### PCP Model

We deal first with the PCP pricing case. Under PCP, it is possible for a monetary rule to replicate the flexible price equilibrium, as shown in Proposition 1. Therefore, a social planner who could choose allocations directly subject to constraints a) and b) above would in fact be constrained only by b), that is, by the presence of monopoly markups. Could the social planner do better than the allocations implied by Proposition 1? To determine this, look at the problem faced by the social planner in any given period. The planner will clearly find it efficient to equalize consumption of each type of home (foreign) good across goods and individuals. Thus, in equilibrium, the planner would maximize the utility function

$$(A30) \quad V_t = n \left( \frac{C_t^{1-\rho}}{1-\rho} - \frac{\eta\lambda}{(\lambda-1)} L_t \right) + (1-n) \left( \frac{C_t^{*(1-\rho)}}{1-\rho} - \frac{\eta\lambda}{(\lambda-1)} L_t^* \right)$$

subject to the constraints

$$C_t = \frac{C_{ht}^n C_{ft}^{1-n}}{n^n (1-n)^{1-n}}, \quad C_t^* = \frac{C_{ht}^{*n} C_{ft}^{*(1-n)}}{n^n (1-n)^{1-n}}$$

and

$$\theta L_t = n C_{ht} + (1-n) C_{ht}^*, \quad \theta^* L_t^* = n C_{ft} + (1-n) C_{ft}^*.$$

The adjustment of the parameters determining the disutility of labor supply in (A30) is a device to ensure that the planner is constrained by the presence of a monopoly markup in choosing allocations.

Now choosing  $C_{ht}, C_{ht}^*, C_{ft}, C_{ft}^*$  to maximize (A30), subject to these constraints, it is straightforward to show that the allocations that result from this problem are equivalent to those of the flexible price equilibrium (2.1) and (2.2), and therefore equivalent to those induced by the monetary policies followed in Proposition 2. Since this represents the solution of a planner's problem constrained only by a) and b), it follows that no monetary policy choice could do better than this.

The planner wishes to maximize lifetime utility, but the above argument applies to each period's utility. A planner choosing a set of monetary rules to maximize inter-temporal utility constrained only by a) and b) could not do better than the allocation of Proposition 1, because that allocation achieves the welfare in each period of a social planner unconstrained except by the presence of monopoly pricing (or, equivalently, this allocation is constrained efficient).

### **LCP Model**

Now we look at the LCP case. With LCP, the monetary policy rules of Proposition 2 do not induce the flexible price equilibrium, so we cannot ignore the presence of constraints b) in formulating the constrained social planning problem. In other words, the presence of sticky prices acts as a real constraint in the problem facing a social planner choosing allocations subject to the constraints imposed by monopoly and price rigidity.

How do we represent the constraints imposed by the presence of LCP? Equations (A10) describes the constraint on the distribution of home country consumption when prices are set using LCP, in the presence of optimal risk sharing. This constraint, and the analogous one for foreign consumption, gives rise to the expressions for expected log consumption in (A11) and (A12) for the home and foreign country. These expressions are derived without specifying the monetary policy rule at all. Therefore, they must be faced by any social planner constrained by LCP. Following the steps described in equations (4.4) and (4.5) of the text, it then follows that a social planner constrained by

LCP must face an objective function that may be described as (it is clear in this formulation that the optimum monetary rule will not depend on the presence of monopoly pricing)

$$(A31) \quad \begin{aligned} E_{-1}U_0 &= E_{-1} \left( \sum_{t=0}^{\infty} \beta^t \left( \frac{1}{1-\rho} C_t^{1-\rho} - \eta L_t \right) \right) \\ &\propto \frac{1}{1-\beta z} \frac{1}{1-\rho} \left[ n \exp((1-\rho) \left( -\frac{\sigma_c^2}{2} - \frac{\tilde{\sigma}_u^2}{2\rho} + \frac{\tilde{\sigma}_{cu}}{\rho} \right)) + (1-n) \exp((1-\rho) \left( -\frac{\sigma_{c^*}^2}{2} - \frac{\tilde{\sigma}_u^2}{2\rho} + \frac{\tilde{\sigma}_{c^*u}}{\rho} \right)) \right] \end{aligned}$$

where  $z \equiv \exp \left[ \frac{1}{2} \left( \frac{1-\rho}{\rho} \right)^2 (n^2 \sigma_u^2 + (1-n)^2 \sigma_u^{*2}) \right]$ , and  $\beta z < 1$ .

The planner must choose a monetary rule that induces a pair of functions  $c(u, u^*)$  and  $c^*(u, u^*)$  so as to maximize (A31). Since  $u$  and  $u^*$  are normally distributed, in a first best economy, log consumption will also be normal. We therefore impose that log consumption is normal. In that case, the functions  $c(u, u^*)$  and  $c^*(u, u^*)$  must be linear, so that

$$\begin{aligned} \sigma_c^2 &= a_{11}^2 \sigma_u^2 + a_{12}^2 \sigma_u^{*2}, & \sigma_{c^*}^2 &= a_{21}^2 \sigma_u^2 + a_{22}^2 \sigma_u^{*2} \\ \tilde{\sigma}_{cu} &= n a_{11} \sigma_u^2 + (1-n) a_{12} \sigma_u^{*2}, & \tilde{\sigma}_{c^*u} &= n a_{21} \sigma_u^2 + (1-n) a_{22} \sigma_u^{*2} \end{aligned}$$

where  $a_{ij}$ ,  $j=1,2$  are constants. Inspection of (A31) reveals that the planner will choose a distribution that *exactly replicates* the allocation induced by the monetary rules of Proposition 2. Since a planner constrained only by LCP will replicate the allocation of Proposition 2, it follows that Proposition 2 represents the optimum for any feasible monetary policy rule in which the log of consumption is normal.

This demonstrates that the allocations achieved under Proposition 1 in the PCP case and Proposition 2 in the LCP case are the same as the planner's optimum in each case, given the monopoly and sticky price constraint. Thus, the particular form of the policy function we have imposed – that money growth rates respond to current shocks – does not preclude reaching the constrained planner's optimum.