We will investigate models of nominal exchange rates that share two properties: long-run purchasing power parity, and uncovered interest parity.

Long run purchasing power parity states that the real exchange rate is stationary. The real exchange rate is defined as: $q_t = s_t + p_t^* - p_t$.

By stationary, we mean covariance stationary, or integrated of order 0 – I(0). Such a stationary random variable has an unconditional mean, and it is expected to converge to its conditional mean:

$\lim_{k\to\infty}(E_t q_{t+k}) = \bar{q}$, where $\bar{q} = E(q_t)$.

There is a large industry that tests for the null that real exchange rates among major countries have a unit root – that is, I(1) – against the alternative of I(0). The literature is inconclusive. Perhaps it is fair to say there is a consensus that these real exchange rates are stationary but very persistent.

Uncovered interest parity:

The return on a one-period safe domestic nominal investment is i_t. A first-order approximation to the expected return on a foreign investment is:
$i_t^* + E_t s_{t+1} - s_t$

A foreign investment pays off both from the foreign interest rate, and the expected appreciation of the foreign currency.

The foreign investment is risky because there is exchange rate uncertainty. But uncovered interest parity is the theory that this risk is not "priced":

$$i_t = i_t^* + E_t s_{t+1} - s_t.$$

We will return to empirical tests of this theory later. There seems to be some evidence of violations of uncovered interest parity at horizons of a year or less. But we will assume it holds for now.

Now, rewrite uncovered interest parity as:

$i_t - i_t^* = E_t s_{t+1} - s_t$.

From each side of the equation, subtract home relative to foreign expected inflation, $E_t p_{t+1} - p_t - (E_t p_{t+1}^* - p_t^*)$. We get:

$$r_t - r_t^* = E_t q_{t+1} - q_t,$$

where $r_t \equiv i_t - (E_t p_{t+1} - p_t)$ and $r_t^* \equiv i_t^* - (E_t p_{t+1}^* - p_t^*)$. These are the ex ante real interest rates at home and abroad.

We will first explore the implications of "real interest parity". We can write:

$q_t = E_t q_{t+1} - (r_t - r_t^*)$

But

$q_{t+1} = E_{t+1} q_{t+2} - (r_{t+1} - r_{t+1}^*)$, so $E_t q_{t+1} = E_t q_{t+2} - E_t(r_{t+1} - r_{t+1}^*)$.

Substituting above, we have:

$q_t = -(r_t - r_t^*) - E_t(r_{t+1} - r_{t+1}^*) + E_t q_{t+2}$.

Continue this into the future:

$$q_t = -(r_t - r_t^*) - E_t(r_{t+1} - r_{t+1}^*) - ... - E_t(r_{t+k-1} - r_{t+k-11}^*) + E_t q_{t+k}.$$

Take the limit as $k \to \infty$, and we have:

$$q_t = \bar{q} - (r_t - r_t^*) - E_t(r_{t+1} - r_{t+1}^*) - ....$$

That is, the deviation of the real exchange rate from its unconditional mean is equal to minus the infinite sum of current and expected future home/foreign real interest differentials.

This relationship would hold in a purely neoclassical, flexible price model. But flexible-price monetary models tend not to put much emphasis on the determination of the real exchange rate.

Monetary models of the exchange rate with sticky nominal prices, in contrast, emphasize that monetary policy determines real interest rates in the short run. That is, monetary policy determines i_t and i_t^*. But in sticky price models, a change in the nominal interest rate – for example, a drop in i_t -- represents a decline in the real rate of interest.
That is, $i_t = r_t + E_t p_{t+1} - p_t$.
A monetary easing means i_t falls – this is common usage in the real world. But if monetary policy is easing, then expected inflation, $E_t p_{t+1} - p_t$, actually rises. How can i_t fall when $E_t p_{t+1} - p_t$ rises? Only if r_t falls, and falls more than the increase in $E_t p_{t+1} - p_t$.

The famous Dornbusch overshooting model of the exchange rate displays this mechanism. It adds two elements to real interest parity. First, there is a money demand equation, described by:
$m_t = p_t - \lambda i_t$.
There is an analogous equation in the foreign country:
$m_t^* = p_t^* - \lambda i_t^*$.

These give us that relative interest rates are determined by relative real money supplies:

$$i_t - i_t^* = -\frac{1}{\lambda}\left[m_t - p_t - \left(m_t^* - p_t^*\right)\right].$$

Dornbusch assumed that any money supply changes were permanent. That is, the money supply follows a random walk:

$$m_t = m_{t-1} + u_t$$
$$m_t^* = m_{t-1}^* + u_t^*$$

If prices were flexible, and the real interest rate were constant, we would have:

$$m_t = \tilde{p}_t - \lambda i_t = \tilde{p}_t - \lambda\left(E_t \tilde{p}_{t+1} - \tilde{p}_t\right).$$

When the money supply follows a random walk, the solution to this difference equation is simply $m_t = \tilde{p}_t$. In the foreign country, $m_t^* = \tilde{p}_t^*$

Then Dornbusch also assumed an ad hoc price adjustment scheme. He assumes that inflation reflects partial adjustment of the price towards its long-run value:
$p_{t+1} - p_t = \theta(\tilde{p}_t - p_t)$.
In the foreign country,
$p^*_{t+1} - p^*_t = \theta(\tilde{p}^*_t - p^*_t)$.

Put these together with our previous results, and we have

$$E_t p_{t+1} - p_t - (E_t p^*_{t+1} - p^*_t) = -\lambda\theta(i_t - i^*_t).$$

Then the relative real interest rate is given by:

$$r_t - r^*_t = (1+\lambda\theta)(i_t - i^*_t)$$

In the next period, $r_{t+1} - r^*_{t+1} = (1+\lambda\theta)(i_{t+1} - i^*_{t+1})$, so
$E_t(r_{t+1} - r^*_{t+1}) = (1+\lambda\theta)E_t(i_{t+1} - i^*_{t+1})$.

Now we can derive:

$$E_t i_{t+1} = -\frac{1}{\lambda} E_t (m_{t+1} - p_{t+1}) = -\frac{1}{\lambda}(m_t - E_t p_{t+1})$$

$$= -\frac{1}{\lambda}(m_t - p_t - (E_t p_{t+1} - p_t)) = -\frac{1}{\lambda}(-\lambda i_t + \lambda\theta i_t) = (1-\theta) i_t$$

Therefore,

$$E_t (r_{t+1} - r^*_{t+1}) = (1+\lambda\theta) E_t (i_{t+1} - i^*_{t+1}) = (1+\lambda\theta)(1-\theta)(i_t - i^*_t).$$

We can derive similarly,

$$E_t (r_{t+k} - r^*_{t+k}) = (1+\lambda\theta)(1-\theta)^k (i_t - i^*_t).$$

Hence,

$$q_t = \overline{q} - (r_t - r^*_t) - E_t (r_{t+1} - r^*_{t+1}) - \ldots$$

$$= -(1+\lambda\theta)(1 + (1-\theta) + (1-\theta)^2 + \ldots)(i_t - i^*_t)$$

$$= -\frac{1+\lambda\theta}{\theta}(i_t - i^*_t) = \frac{1+\lambda\theta}{\lambda\theta}\left[m_t - p_t - \left(m^*_t - p^*_t\right)\right]$$

Since p_t and p_t^* are predetermined, they are not affected by any shocks to the money supply. So the effect of a shock in the home money supply is to cause the real exchange rate to rise (a home real depreciation) by $\frac{1+\lambda\theta}{\lambda\theta}$ times the increase in money.

The real exchange rate is given by $q_t = s_t + p_t^* - p_t$. Since p_t and p_t^* are predetermined, the nominal exchange rate also rises by $\frac{1+\lambda\theta}{\lambda\theta}$.

In the long run, what happens to the nominal exchange rate? In the long run, prices adjust fully, so p_t ultimately rises one-for-one with an increase in m_t. The real exchange rate goes to its unconditional mean in the long run, so the exchange rate is given by $s_t = \bar{q} + p_t - p_t^*$. That is, in the long run, the nominal exchange rate goes up one-for-one with the increase in m_t.

The short-run depreciation, $\frac{1+\lambda\theta}{\lambda\theta}$, is greater than the long-run depreciation, = 1. That is, there is "overshooting".

Intuitively, why does this happen? Uncovered interest parity tells us: $i_t = i_t^* + E_t s_{t+1} - s_t$. An increase in m_t lowers i_t. But uncovered interest parity tells us that this means $E_t s_{t+1} - s_t$ then must fall. Since $E_t s_{t+1}$ rises, we must have s_t rise by even more.

That is, when i_t, we need an expectation of an appreciation of the currency (a drop in $E_t s_{t+1} - s_t$.) That means the immediate depreciation (increase in s_t) must be greater than the increase in the expected future value of the exchange rate, $E_t s_{t+1}$. This model is useful in explaining the high volatility of exchange rates.

We can write a solution for the nominal exchange rate:

$$s_t = \frac{1+\lambda\theta}{\lambda\theta}(m_t - m_t^*) - \frac{1}{\lambda\theta}(p_t - p_t^*)$$

We can derive a more general expression for the nominal exchange rate in the following way:

$$m_t - p_t - (m_t^* - p_t^*) = -\lambda(i_t - i_t^*).$$

Rewrite this as:

$$m_t - m_t^* + q_t - s_t = -\lambda E_t(s_{t+1} - s_t).$$

Now, rewrite this as:

$$s_t = \frac{1}{1+\lambda}(m_t - m_t^* + q_t) + \frac{\lambda}{1+\lambda}E_t s_{t+1}.$$

Iterate forward to get:

$$s_t = \frac{1}{1+\lambda}\left[m_t - m_t^* + q_t + \frac{\lambda}{1+\lambda}E_t(m_{t+1} - m_{t+1}^* + q_{t+1}) + \ldots + \left(\frac{\lambda}{1+\lambda}\right)^k E_t(m_{t+k} - m_{t+k}^* + q_{t+k})\right]$$

$$+\left(\frac{\lambda}{1+\lambda}\right)^{k+1} E_t s_{t+k+1}$$

Now we assume

$\lim_{k\to\infty}\left(\frac{\lambda}{1+\lambda}\right)^{k+1} E_t s_{t+k+1} = 0$. So, we can write:

$$s_t = \frac{1}{1+\lambda}\left[m_t - m_t^* + q_t + \frac{\lambda}{1+\lambda}E_t(m_{t+1} - m_{t+1}^* + q_{t+1}) + ...\right].$$

That is, the exchange rate is the expected present discounted value of x_t,

$$s_t = \frac{1}{1+\lambda}\sum_{j=0}^{\infty}\left(\frac{\lambda}{1+\lambda}\right)^j E_t x_{t+j}$$

where $x_t \equiv m_t - m_t^* + q_t$.

Note that x_t is an I(1) random variable, but it is not a random walk, even if $m_t - m_t^*$ is a random walk. In fact, we saw that if $m_t - m_t^*$ is a random walk, then $q_t = -\frac{1+\lambda\theta}{\theta}(i_t - i_t^*)$. But $E_t(i_{t+1} - i_{t+1}^*) = (1-\theta)(i_t - i_t^*)$, indicating that $i_t - i_t^*$ is an AR(1) random variable. This in turn implies that q_t is an AR(1). So, x_t is the sum of a pure random walk, $m_t - m_t^*$, and an AR(1), q_t.

The Engel-West theorem says that, nonetheless, s_t will nearly be a random walk when $\frac{\lambda}{1+\lambda}$ is "near" to one.

Note we can solve for s_t from the infinite sum:

$$s_t = \frac{1}{1+\lambda}\sum_{j=0}^{\infty}\left(\frac{\lambda}{1+\lambda}\right)^j E_t(m_{t+j} - m^*_{t+j}) + \frac{1}{1+\lambda}\sum_{j=0}^{\infty}\left(\frac{\lambda}{1+\lambda}\right)^j E_t q_{t+j}$$

$$= m_t - m_t^* + \frac{1}{1+\lambda}\frac{1}{1-\left(\frac{\lambda(1-\theta)}{1+\lambda}\right)} q_t$$

$$= m_t - m_t^* + \frac{1}{1+\lambda\theta} q_t$$

$$= m_t - m_t^* + \frac{1}{\lambda\theta}\left[m_t - p_t - \left(m_t^* - p_t^*\right)\right]$$

$$= \frac{\lambda\theta}{1+\lambda\theta}\left[m_t - m_t^*\right] - \frac{1}{\lambda\theta}\left[p_t - p_t^*\right]$$

Now consider the effect of expected inflation on exchange rates. From above:
$m_t - p_t - (m_t^* - p_t^*) = -\lambda(i_t - i_t^*)$.
We can rewrite this as:
$m_t - m_t^* - (p_t - s_t - p_t^*) - s_t = -\lambda(r_t - r_t^*) - \lambda(E_t(p_{t+1} - p_t) - E_t(p_{t+1}^* - p_t^*))$
$\Rightarrow \; m_t - m_t^* + q_t - s_t = -\lambda(r_t - r_t^*) - \lambda(E_t\pi_{t+1} - E_t\pi_{t+1}^*)$
where we $\pi_{t+1} = p_{t+1} - p_t$ and $\pi_{t+1}^* = p_{t+1}^* - p_t^*$.

We can rewrite as:
$s_t = m_t - m_t^* + q_t + \lambda(r_t - r_t^*) + \lambda(E_t\pi_{t+1} - E_t\pi_{t+1}^*)$.
Suppose expected home-to-foreign inflation, $E_t\pi_{t+1} - E_t\pi_{t+1}^*$, increases. Imagine that the current real interest differential is unaffected, and all future real interest differentials are unaffected so that q_t does not change. The increase in home relative to foreign inflation causes the home currency to depreciate (s_t increases.)

Intuitively, holding real interest rates constant, this change reduces the demand for home relative to foreign money, causing the home depreciation.

Now consider a model in which monetary policy reacts to current conditions. In particular, the policymaker uses the nominal interest rate as the policy instrument, and the instrument rule targets expected inflation and the output gap:
$i_t = \rho E_t \pi_{t+1} + \gamma \tilde{y}_t$ and $\quad i_t^* = \rho E_t \pi_{t+1}^* + \gamma \tilde{y}_t^*, \quad \rho > 1, \gamma > 0$.

Assume that a home depreciation increases domestic output relative to foreign output. Let $v_t - v_t^*$ be other factors that affect home/foreign output:
$\tilde{y}_t - \tilde{y}_t^* = v_t - v_t^* + \delta q_t$

Taking home relative to foreign interest rates, we get:
$i_t - i_t^* = \rho E_t(\pi_{t+1} - \pi_{t+1}^*) + \gamma(v_t - v_t^* + \delta q_t)$.

Using uncovered interest parity and the definition of the real exchange rate:

$$s_t = \frac{\delta\gamma}{1+\delta\gamma}(p_t - p_t^*) - \frac{\rho}{1+\delta\gamma}E_t(\pi_{t+1} - \pi_{t+1}^*) - \frac{\gamma}{1+\delta\gamma}(v_t - v_t^*) + \frac{1}{1+\delta\gamma}E_t s_{t+1}$$

Note that an increase in $E_t(\pi_{t+1} - \pi_{t+1}^*)$ leads to a home appreciation. Why?

Now define $z_t \equiv \frac{\delta\gamma}{1+\delta\gamma}(p_t - p_t^*) - \frac{\rho}{1+\delta\gamma}E_t(\pi_{t+1} - \pi_{t+1}^*) - \frac{\gamma}{1+\delta\gamma}(v_t - v_t^*)$.

We can solve the previous equation forward to get:

$$s_t = \sum_{j=0}^{\infty}\left(\frac{1}{1+\delta\gamma}\right)^j E_t z_{t+j}.$$

Although we have not fully specified the model, it is likely that $p_t - p_t^*$ is an I(1) random variable. Since the other variables in z_t are likely stationary, it follows that z_t is I(1). But there is nothing that says z_t is a random walk.

The Engel-West theorem says that, nonetheless, s_t will nearly be a random walk when $\frac{1}{1+\delta\gamma}$ is "near" to one.