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Tests for Cointegration in Models with Regime and Trend Shifts

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I. INTRODUCTION

Recently Gregory and Hansen (1996) developed residual-based tests for cointegration that are valid against an alternative hypothesis that there may be one break in the cointegrating vector. The tests are extensions of the *ADF*, Z_α, and Z_t tests for cointegration and are non-informative with respect to the timing of the break. The null hypothesis is the same as conventional tests (no cointegration). Gregory and Hansen (1996) considered three alternative models: (i) level shift; (ii) level shift with trend; and (iii) regime shift (both level shift and slope coefficients can change).

In this paper we introduce a more general model that permits a trend shift as well as a regime shift and provide the critical values appropriate for testing this alternative. This more general alternative may prove more interesting in some applications and is a natural extension of Perron (1989) and Zivot and Andrews (1992) who concentrated on the possibility of a shift in the trend in the context of unit root testing. We provide no formal proof here of the tests for this model but refer the interested reader to an appendix in Gregory and Hansen (1996) as a guide to how this might be accomplished. Since there are no closed-formed solutions for the limiting distributions, critical values for the tests are calculated by simulation methods. We follow MacKinnon (1991) and estimate response surfaces to approximate the appropriate critical values.

The organization of this paper is as follows. In Section II we review the standard cointegrating model and the model with regime and trend shift. In Section III we present the tests, and report the critical values and close in Section IV with some brief remarks.

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II. REGIME AND TREND SHIFT

In this section we outline the standard single-equation cointegration model and generalize it to allow for both a regime and trend shift under the alternative hypothesis. The observed data are $y_t=(y_{1,t}, y_{2t})$, where y_{1t} is real-valued and y_{2t} is an m-vector. The standard model of cointegration with a trend and no structural change is

$$y_{1t}=\mu+\beta t+\alpha^{\mathrm{T}}y2t+e_t, \qquad t=1, \ldots, n, \tag{2.1}$$

where y_{2t} is $I(1)$ and e_t is $I(0)$.

The motivation for these tests is that there may be occasions in which the researcher may wish to test that cointegration holds over some (fairly long) period of time, but then shifts to a new 'long-run' relationship. We treat the timing of this shift as unknown. The most general kind of structural change considered in Gregory and Hansen (1996) permits changes in the intercept μ and/or changes to the slope coefficients α but not the trend coefficient β.

To model the structural change, we define the dummy variable

$$\varphi_{t\tau}=\begin{cases}0, & \text{if } t\leq[n\tau]\\ 1, & \text{if } t>[n\tau]\end{cases}$$

where the unknown parameter $\tau\in(0, 1)$ denotes the (relative) timing of the change point, and [ ] denotes integer part. The regime and trend shift alternative is

$$y_{1t}=\mu_1+\mu_2\varphi_{t\tau}+\beta_1 t+\beta_2 t\varphi_{t\tau}+\alpha_1^{\mathrm{T}}y_{2t}+\alpha_2^{\mathrm{T}}y_{2t}\varphi_{t\tau}+e_t, \qquad t=1, \ldots, n. \tag{2.2}$$

In this case μ_1, α_1, and β_1 are the intercept, slope coefficients and trend coefficient respectively before the regime shift and μ_2, α_2, and β_2 are the corresponding changes after the break.

III. TESTING THE NULL OF NO COINTEGRATION

It is common in time series regression to test the null of no cointegration against the alternative in equation (2.1). A potential pitfall to this strategy is when there is some regime shift as in equation (2.2), the distributional theory to evaluate the residual-based tests is not the same. For instance, Gregory and Hansen (1996) and Gregory, Nason and Watt (1996) have shown that the rejection frequency of the ADF test falls dramatically in the presence of a break in the cointegrating vector. To test against alternative (2.2), we define the innovation vector $u_t=\Delta y_t$, its cumulative process $S_t=\sum_{i=1}^{t}u_i$, (so $y_t=y_0+S_t$), and its long-run variance $\Omega=\lim_n n^{-1}ES_nS_n^{\mathrm{T}}$. *When* u_t is covariance stationary, Ω is proportional to the spectral density matrix evaluated at the zero frequency. The null hypothesis of no cointegration is that equation (2.1) holds with $e_t \equiv I(1)$. This implies that $\Omega > 0$.

The distributional details can be found in Gregory and Hansen (1996) and we will simply outline the construction of the tests. We compute the cointegration test statistic for each possible regime shift $\tau \in T$, and take the smallest value (the large negative value) across all possible break points. In principle the set T can be any compact subset of (0, 1). In practice, it will need to be small enough so that all of the statistics discussed here can be calculated. A standard choice in the literature is $T = (0.15, 0.85)$. Although T contains an uncountable number of points, all the statistics that we consider are step functions on T, taking jumps only on the points $\{(i/n), i \text{ } integer\}$. For computational purposes, the test statistic is computed for each break point in the interval $([0.15n], [0.85n])$.

For each τ, estimate equation (2.2) by OLS, yielding the residual $\hat{e}_{t\tau}$. The subscript τ on the residuals denotes the fact that the residual sequence depends on the choice of change point τ. From these residuals, calculate the first-order serial correlation coefficient

$$\hat{\rho}_\tau = \frac{\sum_{t=1}^{n-1} \hat{e}_{t\tau}\hat{e}_{t+1\tau}}{\sum_{t=1}^{n-1} \hat{e}_{t\tau}^2}.$$

The Phillips (1987) test statistics are formed using a bias-corrected version of the first-order serial correlation coefficient. Define the second-stage residuals

$$\hat{v}_{t\tau} = \hat{e}_{t\tau} - \hat{\rho}_\tau \hat{e}_{t-1\tau}.$$

The correction involves the following estimate of a weighted sum of autocovariances

$$\hat{\lambda}_\tau = \sum_{j=1}^{M} w\left(\frac{j}{M}\right) \hat{\gamma}_\tau(j),$$

where

$$\hat{\gamma}_\tau(j) = \frac{1}{n} \sum_{t=j+1}^{n} \hat{v}_{t-j\tau} \hat{v}_{t\tau}.$$

and $M = M(n)$ is the bandwidth number selected so that $M \to \infty$. The kernel weights $w(\cdot)$ need to satisfy the standard conditions for spectral density estimators. The estimate of the long-run variance of $\hat{v}_{t\tau}$ is

$$\hat{\sigma}_\tau^2 = \hat{\gamma}_\tau(0) + 2\hat{\lambda}_\tau.$$

The bias-corrected first-order serial correlation coefficient estimate is given by

$$\hat{\rho}_\tau^* = \frac{\sum_{t=1}^{n-1} (\hat{e}_{t\tau}\hat{e}_{t+1\tau} - \hat{\lambda}_\tau)}{\sum_{t=1}^{n-1} \hat{e}_{t\tau}^2}.$$

The Phillips test statistics can be written as

$$Z_\alpha(\tau) = n(\hat{\rho}_\tau^* - 1)$$

$$Z_t(\tau) = \frac{(\hat{\rho}_\tau^* - 1)}{\hat{s}_\tau}, \qquad \hat{s}_\tau^2 = \frac{\hat{\sigma}_\tau^2}{\sum_{t=1}^{n-1} \hat{e}_{t\tau}^2}.$$

The final statistic we discuss is the augmented Dickey–Fuller (ADF) statistic. This is calculated by regressing $\Delta\hat{e}_{t\tau}$ upon $\hat{e}_{t-1\tau}$ and $\Delta\hat{e}_{t-1\tau}, \ldots, \Delta\hat{e}_{t-K\tau}$ for some suitably chosen lag truncation K. The ADF statistic is the t-statistic for the regressor $\hat{e}_{t-1\tau}$. We denote this by

$$ADF(\tau) = tstat(\hat{e}_{t-1\tau}).$$

The test statistics are the *smallest* values of the above statistics, across all values of $\tau \in T$. We focus on the smallest values since small values of the test statistics provide evidence against the null hypothesis. These test statistics are

$$Z_\alpha^* = \inf_{\tau \in T} Z_\alpha(\tau) \tag{3.1}$$

$$Z_t^* = \inf_{\tau \in T} Z_t(\tau) \tag{3.2}$$

$$ADF^* = \inf_{\tau \in T} ADF(\tau). \tag{3.3}$$

Since there are no closed-form expressions we approximate the limiting distribution of the tests using simulation methods. Specifically, we calculate critial values for equations (3.1)–(3.3) following a procedure based upon fitting a response surface suggested by MacKinnon (1991). This approach is especially useful in situations where calculations are recursive over the sample and approximating distributions of tests statistics with large samples is problematic.

Using 10,000 replications for each sample sizes $n = 50$, 100, 150, 200, 250, and 300 we obtain critical values, $Crt(n, p, m)$, where p is the precent quantile and m is the number of regressors in the equation (excluding a constant and/or trend). We then estimate by ordinary least squares for each p and m the response surface

$$\text{Crt}(n, p, m) = \psi_0 + \psi_1 n^{-1} + error.$$

The asymptotic critical value is taken to be the OLS estimate $\hat{\psi}_0$. In Table 1 we present results for $p = 0.01$, 0.025, 0.05, 0.10, and 0.975 and $m = 1$, 2, 3, and 4.

TABLE 1
Approximate Asymptotic Critical Values for Regime and Trend Shift

	Level	*0.01*	*0.025*	*0.05*	*0.10*	*0.975*
$m=1$	ADF^*, Z_t^*	−6.02	−5.72	−5.50	−5.24	−3.30
	Z_α^*	−69.37	−63.23	−58.58	−53.31	−21.99
$m=2$	ADF^*, Z_t^*	−6.45	−6.17	−5.96	−5.72	−3.76
	Z_α^*	−79.65	−73.26	−68.43	−63.10	−28.13
$m=3$	ADF^*, Z_t^*	−6.89	−6.65	−6.32	−6.16	−4.17
	Z_α^*	−90.84	−84.33	−78.87	−72.75	−34.26
$m=4$	ADF^*, Z_t^*	−7.31	−7.06	−6.84	−6.58	−4.57
	Z_α^*	−100.69	−94.00	−88.47	−82.30	−40.99

Note:
These critical values are based on the response surface

$$Crt = \psi_0 + \psi_1 n^{-1} + error,$$

where Crt is the critical value obtained from 10,000 replications at sample size $n=50$, 100, 150, 200, 250 and 300. The asymptotic critical value is the ordinary least squares estimate (OLS) of ψ_0. Z_α^*, Z_t^* and ADF^* are the test statistics defined in equations (3.1)–(3.3) respectively.

V. CONCLUSION

Testing for cointegration in situations in which the researcher believes that there may be a structural break in the long-run relation follows directly from the unit root versus trend break literature. The trend break alternative is highlighted in this debate and we include this alternative with critical values for our residual-based tests for cointegration. Of course, the real value of such tests can only be established in empirical studies.

We remind practitioners applying these tests that the null hypothesis, like the usual residual-based cointegration, is no cointegration. A rejection of the null hypothesis does not imply that there is a break in the cointegrating vector since the tests will have power against a time-invariant cointegrating relation. To discern between these kinds of alternatives, we suggest using one of the structural change tests (with a null hypothesis of cointegration) proposed in Hansen (1992).

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