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CENTRAL LIMIT THEOREMS FOR DEPENDENT HETEROGENEOUS RANDOM VARIABLES

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This paper presents central limit theorems for triangular arrays of mixingale and near-epoch–dependent random variables. The central limit theorem for near-epoch–dependent random variables improves results from the literature in various respects. The approach is to define a suitable Bernstein blocking scheme and apply a martingale difference central limit theorem, which in combination with weak dependence conditions renders the result. The most important application of this central limit theorem is the improvement of the conditions that have to be imposed for asymptotic normality of minimization estimators.

1. INTRODUCTION

This paper presents central limit theorems for triangular arrays of mixingale and near-epoch–dependent random variables. The mixingale and near-epoch dependence assumptions ensure a form of asymptotically weak dependence. The central limit theorems permit heterogeneity of the distributions of the random variables for which the central limit theorem is to be proven. Moreover, it is explicitly allowed that the variances σ_{nt}^2 of the summands tend to zero as t tends to infinity. This case is usually referred to as the asymptotically degenerate case.

The technique of proof of the central limit theorem involves the definition of so-called "Bernstein sums," introduced in Bernstein (1927). This technique splits the summation into relatively big and relatively small blocks. The small blocks are positioned between the big blocks in order to create a form of asymptotic independence between the relatively big blocks; the sum of the small blocks is asymptotically negligible. More information about the history of central limit theorems for dependent random variables can be found in Hall and Heyde (1980, pp. 51–52).

Another approach to deriving central limit theorems for dependent random variables is the derivation of a functional central limit theorem. The

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central limit theorem is then a direct consequence of such a result—see, for example, Billingsley (1968, Theorem 20.1), McLeish (1977), Herrndorf (1984), and Wooldridge and White (1988).

This paper is inspired by those of Davidson (1992, 1993). In these papers, Davidson presented central limit theorems for near-epoch-dependent random variables. To the best of this author's knowledge, Davidson's central limit theorems are the best known for random variables that are near-epoch-dependent on a strong mixing triangular array of random variables. Another central limit theorem result for near-epoch-dependent random variables is found in Wooldridge and White (1988), who adapted a technique by Whithers (1981).

As Davidson (1993) noted, it is of theoretical interest to chart the outer limits of the class of processes for which the central limit theorem holds. Also, this paper will provide a relatively elegant and transparent proof of the central result that is largely self-contained and is, therefore, of pedagogical interest. Moreover and perhaps most importantly, central limit results such as those obtained here are of interest in establishing asymptotic normality results for minimization estimators, such as nonlinear least-squares estimators. Such results are established in Gallant and White (1988) and Pötscher and Prucha (1991a, 1991b). There is some room for improvement of their results by applying the result that is established in this paper. Finally, more ad hoc applications such as the one in Davidson (1993) can be thought of. Davidson (1993) considers application of his central limit theorem to the ordinary least-squares estimator where the regressor equals $t^{-\delta}$.. Davidson(1993) chose $\delta = \frac{1}{2}$ because his argument cannot handle the case $0 \leq \delta < \frac{1}{2}$; the theorem established here is able to establish the asymptotic normality of the ordinary least-squares estimator for such values of δ, as well.

The plan of this paper is as follows. In Section 2, we will state the definitions and the main results. The proofs of the lemmas and the theorems are gathered in the Appendix.

2. ASSUMPTIONS AND MAIN RESULTS

The first steps toward our central limit theorem will be made by making the *mixingale* assumption. The mixingale assumption is one of asymptotically weak dependence. The mixingale assumption has been introduced by McLeish (1975a) and has been extended to L_p-mixingales, $p \geq 1$, by Andrews (1988). The mixingale assumption typically is not strong enough for obtaining central limit theorems with easily verifiable conditions. Therefore, it is interesting to see to what point we can get without making a weak dependence assumption other than the mixingale assumption. A triangular mixingale array is defined as follows. Let $\mathcal{H}_{nt}$ denote an array of σ-fields that is increasing in t for each n, and let X_{nt} denote a triangular array of random variables defined on the probability space $(\Omega, \mathcal{F}, P)$. Let $\|X\|_p$ denote $(E|X|^p)^{1/p}$ for $p \geq 1$.

DEFINITION 1. *A triangular array* $\{X_{nt}, \mathcal{H}_{nt}\}$ *is called an* L_2*-mixingale if for* $m \geq 0$

$$\|X_{nt} - E(X_{nt}|\mathcal{H}_{n,t+m})\|_2 \leq a_{nt}\psi(m+1), \tag{1}$$

$$\|E(X_{nt}|\mathcal{H}_{n,t-m})\|_2 \leq a_{nt}\psi(m), \tag{2}$$

and $\psi(m) \to 0$ *as* $m \to \infty$.

The notation here and in the rest of the paper is chosen as in Davidson (1992, 1993). The a_{nt} sequence will be referred to as the mixingale magnitude indices. Furthermore, we will say that a sequence δ_m is of *size* $-\lambda$ if $\delta_m = O(m^{-\lambda-\varepsilon})$ for some $\varepsilon > 0$. $\{X_{nt}, \mathcal{H}_{nt}\}$ will be called "mixingale of size $-\frac{1}{2}$" if the $\psi(m)$ sequence of Definition 1 is of size $-\frac{1}{2}$. Note that weak and strong laws for mixingale random variables can be found in McLeish (1975a), Hansen (1991, 1992), Andrews (1988), and Davidson and de Jong (1995). Next, we will introduce our chunking scheme ("Bernstein sums") that we will use for establishing our result. Define l_n and b_n as positive nondecreasing integer-valued sequences such that $b_n \geq l_n + 1$, $l_n \to \infty$, $l_n \geq 1$, $b_n \leq n$, $b_n \to \infty$, $b_n/n \to 0$, and $l_n/b_n \to 0$ as $n \to \infty$. Let $r_n = [n/b_n]$. Define $Z_{ni} = \sum_{t=(i-1)b_n+l_n+1}^{ib_n} X_{nt}$, and assume that $EX_{nt} = 0$. Note that these assumptions imply that $n \geq 2$. Moreover, define

$$\mathcal{F}_{ni} = \sigma(\{V_{n,ib_n}, V_{n,ib_n-1}, \ldots\}), \tag{3}$$

where the V_{nt} are random variables defined on $(\Omega, \mathcal{F}, P)$ that will be specified later. For proving results such as central limit theorems or laws of large numbers, the V_{nt} are typically assumed to satisfy a mixing condition. Essentially, we will establish a central limit theorem for $\sum_{i=1}^{r_n} Z_{ni}$, and we will assume the remainder term to converge to zero. The basis of our result is provided by the following lemma. Note that in the sequel of this paper the symbol $\xrightarrow{p}$ denotes convergence in probability.

LEMMA 1. *If*

(a) $$\sum_{t=r_nb_n+1}^{n} X_{nt} \xrightarrow{p} 0, \tag{4}$$

(b) $$\sum_{i=1}^{r_n} \sum_{t=(i-1)b_n+1}^{(i-1)b_n+l_n} X_{nt} \xrightarrow{p} 0, \tag{5}$$

(c) $$\sum_{i=1}^{r_n} E(Z_{ni}|\mathcal{F}_{n,i-1}) \xrightarrow{p} 0, \tag{6}$$

(d) $$\sum_{i=1}^{r_n} (Z_{ni} - E(Z_{ni}|\mathcal{F}_{ni})) \xrightarrow{p} 0, \tag{7}$$

(e) $\sum_{i=1}^{r_n} (E(Z_{ni}|\mathcal{F}_{ni}) - E(Z_{ni}|\mathcal{F}_{n,i-1}))^2 \xrightarrow{p} 1$, *and* **(8)**

(f) $\sum_{i=1}^{r_n} E((E(Z_{ni}|\mathcal{F}_{ni}) - E(Z_{ni}|\mathcal{F}_{n,i-1}))^2 I(|E(Z_{ni}|\mathcal{F}_{ni}) - E(Z_{ni}|\mathcal{F}_{n,i-1})| > \varepsilon)) \to 0$ **(9)**

as $n \to \infty$ *for all* $\varepsilon > 0$,

then $\sum_{t=1}^{n} X_{nt} \to N(0,1)$ *in distribution.*

To show that the first four conditions of Lemma 1 hold under general conditions, we use the following lemma. It is essentially an argument from McLeish (1975a), but the proof can also be found in Hall and Heyde (1980, Lemma 2.1).

LEMMA 2. *If* $\{X_{nt}, \mathcal{K}_{nt}\}$ *is a triangular* L_2*-mixingale array of size* $-\frac{1}{2}$ *with mixingale magnitude indices* a_{nt}, *then* $E(\sum_{t=1}^{n} X_{nt})^2 = O(\sum_{t=1}^{n} a_{nt}^2)$.

Using Lemma 2, the conditions of Lemma 1 can be turned into the conditions of a central limit theorem for L_2-mixingales. The conditions needed are stated in the following assumption.

Assumption 1.

(a) $\{X_{nt}, \mathcal{K}_{nt}\}$ is a triangular L_2-mixingale array of size $-\frac{1}{2}$ with mixingale magnitude indices a_{nt} such that X_{nt}^2/a_{nt}^2 is uniformly integrable.
(b) Defining $M_{ni} = \max_{(i-1)b_n+1 \le t \le ib_n} a_{nt}$ and $M_{n,r_n+1} = \max_{r_n b_n+1 \le t \le n} a_{nt}$,

$$\max_{1 \le i \le r_n+1} M_{ni} = o(b_n^{-1/2}) \quad \textbf{(10)}$$

and

$$\sum_{i=1}^{r_n} M_{ni}^2 = O(b_n^{-1}). \quad \textbf{(11)}$$

(c) $\sum_{i=1}^{r_n} Z_{ni}^2 \xrightarrow{p} 1.$ **(12)**

THEOREM 1. *Under the conditions of Assumption* 1, $\sum_{t=1}^{n} X_{nt} \to N(0,1)$ *in distribution.*

The preceding theorem was the most general that the author could obtain without imposing dependence requirements that were stronger than the L_2-mixingale requirement. Because a martingale difference sequence is a special case of a mixingale sequence, and because a conditional variance condition is a well-known element of central limit theorems for martingale sequences, it is natural that some conditional variance-type condition like Assumption 1(c) is imposed in a central limit theorem for mixingales also. Moreover, note that the size requirement on the mixingale assumption cannot easily be improved upon. This can be illustrated by noting that it is possible to construct a stationary MA(∞) process Y_t that is an L_2-mixingale sequence such that $\psi(m) = O(m^{-1/2})$, but $E(n^{-1/2}\sum_{t=1}^{n} Y_t)^2 \to \infty$ as $n \to \infty$.

The most well-known concepts of weak dependence are the various mixing concepts. Econometric literature usually considers uniform (ϕ-) and strong (α-)mixing random variables. For the definition of these dependence concepts, see, for example, Gallant and White (1988, p. 23) or Pötscher and Prucha (1991a, p. 164). Next, we define *near-epoch-dependent* random variables. The concept of near-epoch dependence is discussed in Gallant and White (1988). See also Pötscher and Prucha (1991a) for a treatment of this and similar dependence concepts. The definition of near-epoch dependence is as follows.

DEFINITION 2. *A triangular array of random variables* X_{nt} *is called* L_2*-near-epoch-dependent on a triangular array of random variables* V_{nt} *if for* $m \ge 0$

$$\|X_{nt} - E(X_{nt}|V_{n,t-m}, \ldots, V_{n,t+m})\|_2 \le d_{nt}\nu(m) \quad \textbf{(13)}$$

and $\nu(m) \to 0$ *as* $m \to \infty$.

To obtain results such as laws of large numbers or central limit theorems, we typically have to assume that the random variables V_{nt} satisfy a weak dependence concept such as independence or a form of mixing. Under integrability conditions, triangular arrays of random variables that are near-epoch-dependent on a triangular array V_{nt} of strong or uniform mixing random variables are mixingales, in view of inequalities such as those in Andrews (1988), which apparently have been established for the first time in McLeish (1975a).

Using the techniques from Davidson (1992, 1993) and some new arguments, we can verify the conditions of Theorem 1 for near-epoch-dependent random variables from the following set of assumptions.

Assumption 2.

(a) X_{nt} is a mean zero random variable such that $\|\sum_{t=1}^{n} X_{nt}\|_2 = 1$.
(b) There exists a positive constant array c_{nt} such that $\{X_{nt}/c_{nt}\}$ is L_r-bounded for $r > 2$ uniformly in t and n.
(c) X_{nt} is L_2-near-epoch-dependent of size $-\frac{1}{2}$ on an α-mixing array of size $-r/(r-2)$, or X_{nt} is L_2-near-epoch-dependent of size $-\frac{1}{2}$ on a ϕ-mixing array of size $-r/(2(r-1))$, and d_{nt}/c_{nt} is bounded uniformly in t and n.
(d) Defining $M_{ni} = \max_{(i-1)b_n+1 \le t \le ib_n} c_{nt}$ and $M_{n,r_n+1} = \max_{r_n b_n+1 \le t \le n} c_{nt}$, we have

$$\max_{1 \le i \le r_n+1} M_{ni} = o(b_n^{-1/2}) \quad \textbf{(14)}$$

and

$$\sum_{i=1}^{r_n} M_{ni}^2 = O(b_n^{-1}). \quad \textbf{(15)}$$

THEOREM 2. *Under Assumption* 2, $\sum_{t=1}^{n} X_{nt} \to N(0,1)$ *in distribution.*

Note that from the proof it can be seen that in the case of ϕ-mixing we can also allow for $r = 2$ if the assumption of uniform integrability of X_{nt}^2/c_{nt}^2 is added to Assumption 2(b). Theorem 2 strictly improves upon Theorem 2.1 of Davidson (1993) by relaxing the trade-off between dependence and degenerateness that is imposed through his Assumption A3′ and the requirement from his equation (2.5). Also, the size requirement on the ν sequence is relaxed; Davidson(1992, 1993) and Wooldridge and White (1988) require the ν sequence to be of size -1, whereas we only impose the ν sequence to be of size $-\frac{1}{2}$. Moreover, Davidson (1992, 1993) requires that the summands X_{nt} satisfy an adaptability requirement (see Corrigendum of Davidson, 1993), whereas we do not impose such a requirement. Also, we were able to derive results for the uniform mixing case that improve upon the strong mixing results in terms of the condition that is imposed on the mixing sequence. Note that the condition of uniform boundedness of d_{nt}/c_{nt} in t and n is imposed in Davidson (1992, 1993) also, although it is not mentioned as a condition in the theorem.

Because Theorem 2 is proven from the central limit theorem for mixingales of Theorem 1, and because the class of mixingale random variables is very wide, for less general forms of weak dependence the results presented here may not be the best possible ones (see, e.g., Doukhan, Massart, and Rio, 1994, for better results for strong mixing random variables).

Finally, the following corollary considers the case where c_{nt} takes a simple form (cf. Lemma 2.1 of Davidson, 1993). Similarly to the previous theorem, by setting $\beta = \gamma = 0$, this corollary strictly improves upon Theorem 3.6 of Davidson (1992) in view of the requirement that $\nu(m)$ is of size $-\frac{1}{2}$, instead of Davidson's requirement that this sequence is of size -1, as well as providing a result for the uniform mixing case. Also note that no restriction on the sign of β and γ is imposed here.

COROLLARY 1. *Suppose that* $c_{nt}^2 = O(t^{\beta} n^{-1-\gamma})$. *If conditions* (a)–(c) *of Assumption* 2 *hold, and* $\beta \le \gamma$, *then* $\sum_{t=1}^{n} X_{nt} \to N(0,1)$ *in distribution.*

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APPENDIX

Proof of Lemma 1. Note that

$$\sum_{t=1}^{n} X_{nt} = \sum_{i=1}^{r_n} \sum_{t=(i-1)b_n+l_n+1}^{ib_n} X_{nt} + \sum_{i=1}^{r_n} \sum_{t=(i-1)b_n+1}^{(i-1)b_n+l_n} X_{nt} + \sum_{t=r_n b_n+1}^{n} X_{nt}. \tag{A.1}$$

To prove the central limit theorem, we will show

$$\sum_{i=1}^{r_n} Z_{ni} = \sum_{i=1}^{r_n} \sum_{t=(i-1)b_n+l_n+1}^{ib_n} X_{nt} \to N(0,1) \tag{A.2}$$

in distribution while, according to the conditions of both equations (4) and (5), other terms are assumed to converge to zero in probability. Next, note that

$$\sum_{i=1}^{r_n} Z_{ni} = \sum_{i=1}^{r_n} (Z_{ni} - E(Z_{ni}|\mathcal{F}_{ni})) + \sum_{i=1}^{r_n} (E(Z_{ni}|\mathcal{F}_{ni}) - E(Z_{ni}|\mathcal{F}_{n,i-1})) + \sum_{i=1}^{r_n} E(Z_{ni}|\mathcal{F}_{n,i-1}). \quad \textbf{(A.3)}$$

Note that the first and third term converge to zero in probability by the assumptions of equations (6) and (7), and note that

$$E(Z_{ni}|\mathcal{F}_{ni}) - E(Z_{ni}|\mathcal{F}_{n,i-1}) \quad \textbf{(A.4)}$$

is a martingale difference triangular array with respect to the $\mathcal{F}_{n,i-1}$. Therefore, a martingale difference central limit theorem can be applied to the second term on the right-hand side of equation (18). We applied Theorem 3.1 of Davidson (1992), with his conditions (b) and (d′) and the condition from his Corollary 3.3. Condition (d′) is satisfied because of the martingale property of our summands. This resulted in the last two conditions of the theorem. ■

Proof of Theorem 1. We verify the assumptions of equations (4)–(9) of Lemma 1.

Verification of (4). This assumption is easily verified by noting that, by Lemma 2,

$$E\left(\sum_{t=r_n b_n+1}^{n} X_{nt}\right)^2 = O\left(\sum_{t=r_n b_n+1}^{n} a_{nt}^2\right) = O(b_n M_{n,r_n+1}^2) = o(1) \quad \textbf{(A.5)}$$

by assumption.

Verification of (5). Let T denote the set $\{t : t \in \bigcup_{i=1}^{r_n} [(i-1)b_n + 1, (i-1)b_n + l_n]\}$. Then, note that by Lemma 2

$$E\left(\sum_{i=1}^{r_n} \sum_{t=(i-1)b_n+1}^{(i-1)b_n+l_n} X_{nt}\right)^2 = E\left(\sum_{t\in T} X_{nt}\right)^2 = O\left(\sum_{i=1}^{r_n} \sum_{t=(i-1)b_n+1}^{(i-1)b_n+l_n} a_{nt}^2\right)$$

$$= O\left(\sum_{i=1}^{r_n} M_{ni}^2 l_n\right) = O(l_n/b_n) = o(1) \quad \textbf{(A.6)}$$

by assumption.

Verification of (6). Note that

$$\sum_{i=1}^{r_n} E(Z_{ni}|\mathcal{F}_{n,i-1}) = \sum_{i=1}^{r_n} \sum_{t=(i-1)b_n+l_n+1}^{ib_n} E(X_{nt}|\mathcal{F}_{n,i-1}) \equiv \sum_{t\in T} E(X_{nt}|\mathcal{F}_{n,i-1}) \quad \textbf{(A.7)}$$

if we define $T = \{t : t \in \bigcup_{i=1}^{r_n} [(i-1)b_n + l_n + 1, ib_n]\}$. Next, note that $\{E(X_{nt}|\mathcal{F}_{n,i-1}), \mathcal{H}_{nt}\}$ is an L_2-mixingale because

$$\|E(E(X_{nt}|\mathcal{F}_{n,i-1})|\mathcal{H}_{n,t-m})\|_2 \le a_{nt}\psi(m) \quad \textbf{(A.8)}$$

by the mixingale definition but also

$$\|E(E(X_{nt}|\mathcal{F}_{n,i-1})|\mathcal{H}_{n,t-m})\|_2 \le \|E(X_{nt}|\mathcal{H}_{n,t-l_n})\|_2 \le a_{nt}\psi(l_n) \quad \textbf{(A.9)}$$

because $\mathcal{F}_{n,i-1} \subseteq \mathcal{H}_{n,t-l_n}$. Moreover, the second condition for a mixingale also holds because

$$\|E(X_{nt}|\mathcal{F}_{n,i-1}) - E(E(X_{nt}|\mathcal{F}_{n,i-1})|\mathcal{H}_{n,t+m})\|_2 \le a_{nt}\psi(m) \quad \textbf{(A.10)}$$

and

$$\|E(X_{nt}|\mathcal{F}_{n,i-1}) - E(E(X_{nt}|\mathcal{F}_{n,i-1})|\mathcal{H}_{n,t+m})\|_2 \le 2a_{nt}\psi(l_n). \quad \textbf{(A.11)}$$

Therefore, $\{E(X_{nt}|\mathcal{F}_{n,i-1}), \mathcal{H}_{nt}\}$ is also an L_2-mixingale with mixingale sequence $\psi(m)^{1-\eta}$ and mixingale magnitude index numbers of order $a_{nt}\psi(l_n)^\eta$, for all $\eta \in (0,1)$. By the definition of size of a mixingale, it is possible to choose a tiny $\eta > 0$ such that $E(X_{nt}|\mathcal{F}_{n,i-1})$ is an L_2-mixingale of size $-1/2$ with index numbers of order $a_{nt}\psi(l_n)^\eta$. Therefore, by Lemma 2,

$$E\left(\sum_{t\in T} E(X_{nt}|\mathcal{F}_{n,i-1})\right)^2 = O\left(\sum_{t\in T} a_{nt}^2\psi(l_n)^{2\eta}\right) = O(\psi(l_n)^{2\eta}) = o(1) \quad \textbf{(A.12)}$$

by assumption.

Verification of (7). This proof is analogous to the verification of (6).

Verification of (8). We will show that

$$\lim_{n\to\infty} E\left|\sum_{i=1}^{r_n} Z_{ni}^2 - \sum_{i=1}^{r_n} (E(Z_{ni}|\mathcal{F}_{ni}) - E(Z_{ni}|\mathcal{F}_{n,i-1}))^2\right| = 0. \quad \textbf{(A.13)}$$

If we are able to show this, the condition of equation (8) of Lemma 1 follows from the condition $\sum_{i=1}^{r_n} Z_{ni}^2 \xrightarrow{p} 1$, as is imposed in Theorem 1. Note that, similarly to the verification of (6),

$$\|E(Z_{ni}|\mathcal{F}_{n,i-1})\|_2 = \left\|\sum_{t=(i-1)b_n+l_n+1}^{ib_n} E(X_{nt}|\mathcal{F}_{n,i-1})\right\|_2$$

$$= O\left(\left(\sum_{t=(i-1)b_n+l_n+1}^{ib_n} a_{nt}^2\psi(l_n)^{2\eta}\right)^{1/2}\right), \quad \textbf{(A.14)}$$

where the last equation follows from Lemma 2, and $\eta \in (0,1)$ is chosen such that $\psi(m)^{1-\eta}$ is of size $-\frac{1}{2}$. Similarly,

$$\|Z_{ni} - E(Z_{ni}|\mathcal{F}_{n,i})\|_2 = O\left(\left(\sum_{t=(i-1)b_n+l_n+1}^{ib_n} a_{nt}^2\psi(l_n)^{2\eta}\right)^{1/2}\right). \quad \textbf{(A.15)}$$

Moreover, another consequence of Lemma 2 is

$$\|Z_{ni}\|_2 = O\left(\left(\sum_{t=(i-1)b_n+l_n+1}^{ib_n} a_{nt}^2\right)^{1/2}\right). \quad \textbf{(A.16)}$$

Therefore,

$$\left\|\sum_{i=1}^{r_n} Z_{ni}^2 - (E(Z_{ni}|\mathcal{F}_{ni}) - E(Z_{ni}|\mathcal{F}_{n,i-1}))^2\right\|_1$$

$$\le 3\sum_{i=1}^{r_n} \|Z_{ni} - E(Z_{ni}|\mathcal{F}_{ni}) + E(Z_{ni}|\mathcal{F}_{n,i-1})\|_2 \|Z_{ni}\|_2$$

$$= O\left(\sum_{i=1}^{r_n}\left(\sum_{t=(i-1)b_n+l_n+1}^{ib_n} a_{nt}^2\psi(l_n)^{2\eta}\right)^{1/2}\left(\sum_{t=(i-1)b_n+l_n+1}^{ib_n} a_{nt}^2\right)^{1/2}\right) = o(1) \quad \textbf{(A.17)}$$

by assumption.

Verification of (9). Note that

$$W_{ni} \equiv E(Z_{ni}|\mathcal{F}_{ni}) - E(Z_{ni}|\mathcal{F}_{n,i-1}) = \sum_{t=(i-1)b_n+l_n+1}^{ib_n} E(X_{nt}|\mathcal{F}_{ni}) - E(X_{nt}|\mathcal{F}_{n,i-1}), \tag{A.18}$$

and it is easily verified that

$$\{E(X_{nt}|\mathcal{F}_{ni}) - E(X_{nt}|\mathcal{F}_{n,i-1}), \mathcal{H}_{nt}\} \tag{A.19}$$

is a triangular L_2-mixingale array of size $-\frac{1}{2}$ with mixingale magnitude indices $2a_{nt}$. Similarly to Davidson (1992, 1993), define $\nu_{ni}^2 = \sum_{t=(i-1)b_n+l_n+1}^{ib_n} a_{nt}^2$, and note the distinction between the $\nu(m)$ sequence and the ν_{ni}^2 array. Then, similar to Davidson (1993),

$$\sum_{i=1}^{r_n} EW_{ni}^2 I(|W_{ni}| > \varepsilon) \le \max_{1\le i\le r_n} E(W_{ni}^2/\nu_{ni}^2) I(|W_{ni}| > \varepsilon) \sum_{i=1}^{r_n} \nu_{ni}^2$$
$$\le \max_{1\le i\le r_n} E(W_{ni}^2/\nu_{ni}^2) I(|W_{ni}|/\nu_{ni} > \varepsilon/\nu_{ni}) \sum_{i=1}^{r_n} M_{ni}^2 b_n$$
$$= O(\max_{1\le i\le r_n} E(W_{ni}^2/\nu_{ni}^2) I(|W_{ni}|/\nu_{ni} > \varepsilon/\nu_{ni})). \tag{A.20}$$

Next, note that the latter term converges to zero for any $\varepsilon > 0$ if W_{ni}^2/ν_{ni}^2 is uniformly integrable and $\lim_{n\to\infty} \max_{1\le i\le r_n} \nu_{ni} = 0$. The second property is verified by noting that

$$\max_{1\le i\le r_n} \sum_{t=(i-1)b_n+l_n+1}^{ib_n} a_{nt}^2 \le b_n \max_{1\le i\le r_n} M_{ni}^2 = o(1) \tag{A.21}$$

by assumption. Uniform integrability of W_{ni}^2/ν_{ni}^2 is a consequence of Lemma 3.2 of Davidson (1992) and the discussion that follows this lemma, and the mixingale property of $\{E(X_{nt}|\mathcal{F}_{ni}) - E(X_{nt}|\mathcal{F}_{n,i-1}), \mathcal{H}_{nt}\}$. ■

The proof of Theorem 2 will be preceded by the following lemmas.

LEMMA 3. *If $\{X_{nt}, \mathcal{H}_{nt}\}$ is a triangular mixingale array, then*

$$EX_{nt}X_{n,t+m} \le \sum_{j=0}^{\infty} (EE(X_{nt}|\mathcal{H}_{n,t-j})^2 - EE(X_{nt}|\mathcal{H}_{n,t-j-1})^2)^{1/2}$$
$$\times (EE(X_{n,t+m}|\mathcal{H}_{n,t-j})^2 - EE(X_{n,t+m}|\mathcal{H}_{n,t-j-1})^2)^{1/2}$$
$$+ \sum_{j=1}^{\infty} (E(X_{nt} - E(X_{nt}|\mathcal{H}_{n,t+j-1}))^2 - E(X_{nt} - E(X_{nt}|\mathcal{H}_{n,t+j}))^2)^{1/2}$$
$$\times (E(X_{n,t+m} - E(X_{n,t+m}|\mathcal{H}_{n,t+j-1}))^2$$
$$- E(X_{n,t+m} - E(X_{n,t+m}|\mathcal{H}_{n,t+j}))^2)^{1/2}. \tag{A.22}$$

Proof. Note that $X_{nt} = \sum_{j=-\infty}^{\infty} Y_{ntj}$ almost surely, where $Y_{ntj} = E(X_{nt}|\mathcal{H}_{n,t-j}) - E(X_{nt}|\mathcal{H}_{n,t-j-1})$, and note that $EY_{ntj}Y_{n,t+m,i+m} = 0$ if $j \ne i$. Therefore,

$$EX_{nt}X_{n,t+m} = \sum_{j=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} EY_{ntj}Y_{n,t+m,i+m}$$
$$= \sum_{j=-\infty}^{\infty} EY_{ntj}Y_{n,t+m,j+m} \le \sum_{j=-\infty}^{\infty} \|Y_{ntj}\|_2 \|Y_{n,t+m,j+m}\|_2$$
$$= \sum_{j=0}^{\infty} (EE(X_{nt}|\mathcal{H}_{n,t-j})^2 - EE(X_{nt}|\mathcal{H}_{n,t-j-1})^2)^{1/2}$$
$$\times (EE(X_{n,t+m}|\mathcal{H}_{n,t-j})^2 - EE(X_{n,t+m}|\mathcal{H}_{n,t-j-1})^2)^{1/2}$$
$$+ \sum_{j=1}^{\infty} (E(X_{nt} - E(X_{nt}|\mathcal{H}_{n,t+j-1}))^2 - E(X_{nt} - E(X_{nt}|\mathcal{H}_{n,t+j}))^2)^{1/2}$$
$$\times (E(X_{nt+m} - E(X_{n,t+m}|\mathcal{H}_{n,t+j-1}))^2$$
$$- E(X_{nt+m} - E(X_{n,t+m}|\mathcal{H}_{n,t+j}))^2)^{1/2}, \tag{A.23}$$

where the second equality uses the earlier noted property of Y_{ntj} and the inequality is Cauchy-Schwartz. ■

LEMMA 4. *If $\{X_{nt}, \mathcal{H}_{nt}\}$ is a triangular mixingale array of size $-\frac{1}{2}$ with mixingale magnitude indices c_{nt} such that $\sum_{t=1}^{n} c_{nt}^2 = O(1)$, then for l_n and b_n as before*

$$\lim_{n\to\infty} \left| \sum_{i=1}^{r_n} \sum_{k=i+1}^{r_n} \sum_{t=(i-1)b_n+l_n+1}^{ib_n} \sum_{s=(k-1)b_n+l_n+1}^{kb_n} EX_{nt}X_{ns} \right| = 0. \tag{A.24}$$

Proof. By Lemma 3, and because $m = s - t \ge l_n$ in the following formula,

$$\left| \sum_{i=1}^{r_n} \sum_{k=i+1}^{r_n} \sum_{t=(i-1)b_n+l_n+1}^{ib_n} \sum_{s=(k-1)b_n+l_n+1}^{kb_n} EX_{nt}X_{ns} \right|$$
$$\le \sum_{i=1}^{r_n} \sum_{k=i+1}^{r_n} \sum_{t=(i-1)b_n+l_n+1}^{ib_n} \sum_{m=(k-1)b_n+l_n+1-t}^{kb_n-t} I(m \ge l_n)|EX_{nt}X_{n,t+m}|$$
$$\le \sum_{t=1}^{n} \sum_{m=0}^{n-t} I(m \ge l_n) \sum_{j=0}^{\infty} (EE(X_{nt}|\mathcal{H}_{n,t-j})^2 - EE(X_{nt}|\mathcal{H}_{n,t-j-1})^2)^{1/2}$$
$$\times (EE(X_{n,t+m}|\mathcal{H}_{n,t-j})^2 - EE(X_{n,t+m}|\mathcal{H}_{n,t-j-1})^2)^{1/2}$$
$$+ \sum_{t=1}^{n} \sum_{m=0}^{n-t} I(m \ge l_n) \sum_{j=1}^{\infty} (E(X_{nt} - E(X_{nt}|\mathcal{H}_{n,t+j-1}))^2$$
$$- E(X_{nt} - E(X_{nt}|\mathcal{H}_{n,t+j})^2))^{1/2}$$
$$\times (E(X_{n,t+m} - E(X_{n,t+m}|\mathcal{H}_{n,t+j-1}))^2 - E(X_{n,t+m} - E(X_{n,t+m}|\mathcal{H}_{n,t+j}))^2)^{1/2}$$
$$\equiv T_1 + T_2. \tag{A.25}$$

Next, we will show that the first term T_1 converges to zero. The second term can be shown to converge to zero in a similar way. For the first term, we have

$$T_1 = \sum_{j=0}^{\infty}\sum_{t=1}^{n} (EE(X_{nt}|\mathcal{H}_{n,t-j})^2 - EE(X_{nt}|\mathcal{H}_{n,t-j-1})^2)^{1/2}$$
$$\times \sum_{m=0}^{n-t} I(m \ge l_n)(EE(X_{n,t+m}|\mathcal{H}_{n,t-j})^2 - EE(X_{n,t+m}|\mathcal{H}_{n,t-j-1})^2)^{1/2}$$
$$\le \sum_{j=0}^{\infty}\left(\sum_{t=1}^{n} EE(X_{nt}|\mathcal{H}_{n,t-j})^2 - EE(X_{nt}|\mathcal{H}_{n,t-j-1})^2\right)^{1/2}$$
$$\times \left(\sum_{t=1}^{n}\left(\sum_{m=0}^{n-t} I(m \ge l_n)(EE(X_{n,t+m}|\mathcal{H}_{n,t-j})^2 - EE(X_{n,t+m}|\mathcal{H}_{n,t-j-1})^2)^{1/2}\right)^2\right)^{1/2}$$
$$= O\Bigg(\sum_{j=0}^{\infty}\left(\sum_{t=1}^{n} EE(X_{nt}|\mathcal{H}_{n,t-j})^2 - EE(X_{nt}|\mathcal{H}_{n,t-j-1})^2\right)^{1/2}$$
$$\times \Bigg(\sum_{s=1}^{n}\sum_{m=0}^{s-1} I(m \ge l_n)\gamma_m^{-1}(EE(X_{ns}|\mathcal{H}_{n,s-j-m})^2$$
$$- EE(X_{ns}|\mathcal{H}_{n,s-j-m-1})^2)\Bigg)^{1/2}\Bigg), \tag{A.26}$$

where $\gamma_m = m^{-1}(\log(m))^{-2}$ for $|m| > 1$ and $\gamma_m = 1$ for $|m| \le 1$. The first inequality is Cauchy-Schwartz of the type $\sum_{t=1}^{n} |h_t|^{1/2}|g_t| \le (\sum_{t=1}^{n}|h_t|)^{1/2}(\sum_{t=1}^{n}|g_t|^2)^{1/2}$, and the last equation follows from an application of the Cauchy-Schwartz inequality of the form $(\sum_{m=0}^{n-t}|h_m|^{1/2})^2 = O(\sum_{m=0}^{n-t}\gamma_m^{-1}|h_m|)$. Also, we used the substitution $t + m = s$, and the equality $\sum_{t=1}^{n}\sum_{m=0}^{n-t} h_{t+m,m} = \sum_{s=1}^{n}\sum_{m=0}^{s-1} h_{sm}$. We continue the argument as follows:

$$T_1 = O\Bigg(\sum_{j=0}^{\infty}\left(\sum_{t=1}^{n} EE(X_{nt}|\mathcal{H}_{n,t-j})^2 - EE(X_{nt}|\mathcal{H}_{n,t-j-1})^2\right)^{1/2}$$
$$\times \left(\sum_{s=1}^{n} c_{ns}^2\left(l_n(\log(l_n))^2\psi(l_n)^2 + \sum_{m=l_n}^{\infty}(\log(m))^2\psi(m)^2\right)\right)^{1/2}\Bigg)$$
$$= O\left(\left(\sum_{t=1}^{n} c_{nt}^2\right)\left(l_n(\log(l_n))^2\psi(l_n)^2 + \sum_{m=l_n}^{\infty}(\log(m))^2\psi(m)^2\right)^{1/2}\right) = o(1). \tag{A.27}$$

The first equality follows from rearranging of the summation of the type

$$\sum_{m=0}^{s-1} I(m \ge l_n)\gamma_m^{-1}(|h_m| - |h_{m+1}|) \le \gamma_{l_n}^{-1}|h_{l_n}| + \sum_{m=l_n}^{\infty}(\gamma_{m+1}^{-1} - \gamma_m^{-1})|h_{m+1}| \tag{A.28}$$

and from the fact that the $\mathcal{H}_{nt}$ are nondecreasing and the mixingale definition, and the last equality again follows from the inequality $(\sum_{j=0}^{\infty}|h_j|^{1/2})^2 = O(\sum_{j=0}^{\infty}|h_j|\gamma_j^{-1})$ and rearranging of the summation. The conclusion of equation (A.27) follows by the size requirement imposed on the $\psi(m)$ sequence and the conditions on the c_{nt} sequence. ∎

LEMMA 5. *Under the conditions of Theorem 2,* $\sum_{i=1}^{r_n}(Z_{ni}^2 - EZ_{ni}^2) \xrightarrow{p} 0$.

Proof. Define

$$h_K(x) = xI(|x| \le K) + KI(x > K) - KI(x < -K), \tag{A.29}$$

and note that $|h_K(x) - h_K(y)| \le |x - y|$, and define $\tilde{Z}_{ni} = h_{K_\delta \nu_{ni}}(Z_{ni})$, where K_δ is chosen later on and ν_{ni}^2 is as in the proof of Theorem 1. Then,

$$\left\|\sum_{i=1}^{r_n} Z_{ni}^2 - \tilde{Z}_{ni}^2\right\|_1 \le 2E\sum_{i=1}^{r_n} Z_{ni}^2 I(Z_{ni}^2 > K_\delta^2\nu_{ni}^2)$$
$$\le 2\max_{n\ge 1}\max_{1\le i\le r_n} E(Z_{ni}/\nu_{ni})^2 I(Z_{ni}^2/\nu_{ni}^2 > K_\delta^2)\sum_{i=1}^{r_n}\nu_{ni}^2 \le \delta \tag{A.30}$$

by uniform integrability of Z_{ni}^2/ν_{ni}^2 (see the verification of (11) in the proof of Lemma 1), the fact that $\sum_{i=1}^{r_n}\nu_{ni}^2 = O(1)$ by assumption, and a sufficiently large choice of K_δ. The assertion of the lemma is therefore proven if we can show

$$\lim_{n\to\infty}\left\|\sum_{i=1}^{r_n}\tilde{Z}_{ni}^2 - E\tilde{Z}_{ni}^2\right\|_1 = 0 \tag{A.31}$$

for any $\delta > 0$. Next, note that $\tilde{Z}_{ni}^2 - E\tilde{Z}_{ni}^2$ is L_2-near-epoch-dependent with respect to the $\mathcal{V}_{n,i-m}^{n,i+m} = \sigma(\{V_{n,(i-m-1)b_n+l_n+1}, \dots, V_{n,(i+m)b_n}\})$ because for $m \ge 1$

$$\|\tilde{Z}_{ni}^2 - E(\tilde{Z}_{ni}^2|\mathcal{V}_{n,i-m}^{n,i+m})\|_2$$
$$= \|h_{K_\delta\nu_{ni}}(Z_{ni})^2 - E(h_{K_\delta\nu_{ni}}(Z_{ni})^2|\mathcal{V}_{n,i-m}^{n,i+m})\|_2$$
$$\le \|h_{K_\delta\nu_{ni}}(Z_{ni})^2 - h_{K_\delta\nu_{ni}}(E(Z_{ni}|\mathcal{V}_{n,i-m}^{n,i+m}))^2\|_2$$
$$\le \|h_{K_\delta\nu_{ni}}(Z_{ni}) - h_{K_\delta\nu_{ni}}(E(Z_{ni}|\mathcal{V}_{n,i-m}^{n,i+m}))\|_2 2K_\delta\nu_{ni}$$
$$= O\left(\nu_{ni}\sum_{t=(i-1)b_n+l_n+1}^{ib_n}\|X_{nt} - E(X_{nt}|\mathcal{V}_{n,i-m}^{n,i+m})\|_2\right)$$
$$= O\left(\nu_{ni}\sum_{t=(i-1)b_n+l_n+1}^{ib_n}\|X_{nt} - E(X_{nt}|V_{n,t-ml_n}, \dots, V_{n,t+mln})\|_2\right)$$
$$= O\left(\nu_{ni}\sum_{t=(i-1)b_n+l_n+1}^{ib_n} c_{nt}\nu(ml_n)\right) = O(b_n M_{ni}^2 m^{-1/2-\varepsilon}), \tag{A.32}$$

where the first inequality follows because the conditional expectation is the best $\mathcal{V}_{n,i-m}^{n,i+m}$-measurable L_2-approximation, the third equality again follows because conditional expectation is the best $\mathcal{V}_{n,i-m}^{n,i+m}$-measurable L_2-approximation and because

$$\sigma(\{V_{n,t-ml_n}, \dots, V_{n,t+min}\}) \subseteq \mathcal{V}_{n,i-m}^{n,i+m},$$

the fourth equality uses the near-epoch dependence definition, and the last equality follows because we define l_n such that it satisfies $l_n^{-1/2-\varepsilon} = O(b_n^{-1/2})$. If $m = 0$, note that

$$\|\tilde{Z}_{ni}^2 - E(\tilde{Z}_{ni}^2|\mathcal{V}_{ni}^{ni})\|_2 \le 2K_\delta\nu_{ni}\|Z_{ni}\|_2 = O(b_n M_{ni}^2) \tag{A.33}$$

by Lemma 2. Next, note that $\tilde{Z}_{ni}^2$ is an L_2-mixingale of size $-\frac{1}{2}$ because for $m \ge 1$

$$\|E\tilde{Z}_{ni}^2 - E(\tilde{Z}_{ni}^2|\mathcal{F}_{n,i-2m})\|_2$$
$$\le \|E\tilde{Z}_{ni}^2 - E(\tilde{Z}_{ni}^2|\mathcal{V}_{n,i-m}^{n,i+m})\|_2 + \|E(E(\tilde{Z}_{ni}^2|\mathcal{V}_{n,i-m}^{n,i+m}) - E\tilde{Z}_{ni}^2|\mathcal{F}_{n,i-2m})\|_2$$
$$= O(b_n M_{ni}^2 m^{-1/2-\varepsilon} + \alpha(ml_n)^{1/2-1/r} b_n M_{ni}^2) = O(b_n M_{ni}^2 m^{-1/2-\varepsilon}), \tag{A.34}$$

where the first equality follows from the fact that $E(\tilde{Z}_{ni}^2|\mathcal{V}_{n,i-m}^{n,i+m}) - E\tilde{Z}_{ni}^2$ is α-mixing and from Theorem 17.5 of Davidson (1994) and by noting that the index ranges of the V_{nt} in $\mathcal{V}_{n,i-m}^{n,i+m}$ and $\mathcal{F}_{n,i-2m}$ are separated by at least ml_n index values. If X_{nt} is uniform mixing, the last two lines should read, again using Theorem 17.5 of Davidson (1994),

$$= O(b_n M_{ni}^2 m^{-1/2-\varepsilon} + \phi(ml_n)^{1-1/r} b_n M_{ni}^2)$$
$$= O(b_n M_{ni}^2 m^{-1/2-\varepsilon} + m^{-1/2-\varepsilon} l_n^{-1/2-\varepsilon} b_n M_{ni}^2) = O(b_n M_{ni}^2 m^{-1/2-\varepsilon}). \quad \textbf{(A.35)}$$

For $m = 0$,

$$\| E(\tilde{Z}_{ni}^2 - E\tilde{Z}_{ni}^2 | \mathcal{F}_{n,i}) \|_2 = O(b_n M_{ni}^2) \quad \textbf{(A.36)}$$

as before. Moreover, the second condition of the mixingale definition also holds because

$$\| \tilde{Z}_{ni}^2 - E(\tilde{Z}_{ni}^2 | \mathcal{F}_{n,i+m}) \|_2 \le \| \tilde{Z}_{ni}^2 - E(\tilde{Z}_{ni}^2 | \mathcal{V}_{n,i-m}^{i+m}) \|_2 = O(b_n M_{ni}^2 m^{-1/2-\varepsilon}) \quad \textbf{(A.37)}$$

because conditional expectation is the best $\mathcal{F}_{n,i+m}$-measurable L_2-approximation and because $\mathcal{V}_{n,i-m}^{n,i+m} \subseteq \mathcal{F}_{n,i+m}$. Therefore, under the conditions of the theorem, $\tilde{Z}_{ni}^2$ is an L_2-mixingale of size $-\frac{1}{2}$ with mixingale magnitude indices of order $b_n M_{ni}^2$. Finally, note that by Lemma 2

$$\left\| \sum_{i=1}^{r_n} \tilde{Z}_{ni}^2 - E\tilde{Z}_{ni}^2 \right\|_2^2 = O\left(b_n^2 \sum_{i=1}^{r_n} M_{ni}^4 \right) = O\left(\left(b_n \sum_{i=1}^{r_n} M_{ni}^2 \right) (b_n \max_{1 \le i \le r_n} M_{ni}^2) \right) = o(1) \quad \textbf{(A.38)}$$

by assumption. ∎

Proof of Theorem 2. Note that by Theorem 17.5 of Davidson (1994) X_{nt} is a triangular L_2-mixingale array with mixingale magnitude indices c_{nt} and mixingale numbers $\alpha([m/2])^{1/2-1/r} + \nu([m/2])$ or $\phi([m/2])^{1-1/r} + \nu([m/2])$ depending on whether uniform or strong mixing triangular arrays are considered. By the assumptions on the $\nu(m)$, $\alpha(m)$, and $\phi(m)$ sequence, X_{nt} is an L_2-mixingale of size $-\frac{1}{2}$. Therefore, the first condition of Theorem 1 holds under the conditions stated. The second condition is verified as follows. Similarly to Davidson (1992, 1993), write

$$\sum_{i=1}^{r_n} Z_{ni}^2 - 1 = \left(\sum_{i=1}^{r_n} Z_{ni}^2 - EZ_{ni}^2 \right) + \left(\sum_{i=1}^{r_n} EZ_{ni}^2 - E\left(\sum_{t=1}^{n} X_{nt} \right)^2 \right) \equiv A_n - B_n. \quad \textbf{(A.39)}$$

It suffices to show $B_n \to 0$ because by Lemma 5 $A_n \xrightarrow{p} 0$. Note that

$$E\left(\sum_{t=1}^{n} X_{nt} \right)^2 - E\left(\sum_{t=1}^{r_n b_n} X_{nt} \right)^2 = E\left(\sum_{t=r_n b_n+1}^{n} X_{nt} \right) \left(\sum_{t=1}^{n} X_{nt} + \sum_{t=1}^{r_n b_n} X_{nt} \right)$$
$$\le \left\| \sum_{t=r_n b_n+1}^{n} X_{nt} \right\|_2 \left\| \sum_{t=1}^{n} X_{nt} + \sum_{t=1}^{r_n b_n} X_{nt} \right\|_2 = o(1) \quad \textbf{(A.40)}$$

by the arguments from the proof of Theorem 1. Next, note that

$$\left\| \sum_{i=1}^{r_n} \sum_{t=(i-1)b_n+1}^{(i-1)b_n+l_n} X_t \right\|_2 = o(1)$$

by the arguments of the proof of Theorem 1. Therefore, the proof of the result $B_n \to 0$ is complete if it is shown that

$$\left| \sum_{i=1}^{r_n} EZ_{ni}^2 - E\left(\sum_{i=1}^{r_n} \sum_{t=(i-1)b_n+l_n+1}^{ib_n} X_{nt} \right)^2 \right|$$
$$= 2 \left| \sum_{i=1}^{r_n} \sum_{k=i+1}^{r_n} \sum_{t=(i-1)b_n+l_n+1}^{ib_n} \sum_{s=(k-1)b_n+l_n+1}^{kb_n} EX_{nt} X_{ns} \right| \to 0. \quad \textbf{(A.41)}$$

Lemma 4 ensures the latter result. Note that the requirement of Lemma 4 on the c_{nt} sequence is met because

$$\sum_{t=1}^{n} c_{nt}^2 \le b_n \left(\sum_{i=1}^{r_n} M_{ni}^2 + M_{n,r_n+1}^2 \right) = O(1)$$

by assumption. ∎

Proof of Corollary 1. Note that for $1 \le i \le r_n$

$$M_{ni} = O\left(\max_{(i-1)b_n+1 \le t \le ib_n} t^{\beta/2} n^{-1/2-\gamma/2} \right) = O(i^{\beta/2} b_n^{\beta/2} n^{-1/2-\gamma/2}). \quad \textbf{(A.42)}$$

Therefore, if $\beta > 0$,

$$\max_{1 \le i \le r_n} M_{ni} = O(n^{-\gamma/2+\beta/2-1/2}) = O(n^{-1/2}) = o(b_n^{-1/2}) \quad \textbf{(A.43)}$$

by assumption and, if $\beta < 0$,

$$\max_{1 \le i \le r_n} M_{ni} = O(b_n^{-1/2}(b_n^{\beta/2+1/2} n^{-1/2-\gamma/2})) = o(b_n^{-1/2}) \quad \textbf{(A.44)}$$

because $\beta \le \gamma$ and $b_n/n = o(1)$. Moreover,

$$\sum_{i=1}^{r_n} M_{ni}^2 = O\left(\sum_{i=1}^{r_n} i^\beta b_n^\beta n^{-1-\gamma} \right) = O(r_n^{1+\beta} b_n^\beta n^{-1-\gamma}) = O(n^{\beta-\gamma} b_n^{-1}) = O(b_n^{-1}) \quad \textbf{(A.45)}$$

because $\beta - \gamma \le 0$. ∎