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COMPLEX UNIT ROOTS AND BUSINESS CYCLES: ARE THEY REAL?

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In this paper the asymptotic properties of ARMA processes with complex-conjugate unit roots in the AR lag polynomial are studied. These processes behave quite differently from regular unit root processes (with a single root equal to one). In particular, the asymptotic properties of a standardized version of the *periodogram* for such processes are analyzed, and a nonparametric test of the complex unit root hypothesis against the stationarity hypothesis is derived. This test is applied to the annual change of the monthly number of unemployed in the United States to see whether this time series has complex unit roots in the business cycle frequencies.

1. INTRODUCTION

As is well known, AR processes with roots on the complex unit circle are nonstationary, and are actually more interesting than AR processes with a real valued unit root, because these processes display a persistent cyclical behavior. Thus, if there exist persistent business cycles, it seems that the data-generating process involved is more compatible with an AR(MA) process with complex-conjugate unit roots than with a real unit root and/or roots outside the complex unit circle.

The current literature on nonseasonal unit root processes focuses almost entirely on the case of real unit roots (equal to one). Notable exceptions are Ahtola and Tiao (1987a, 1987b), Chan and Wei (1988), and Gregoir (1999c), who derive the limiting distribution of least squares estimates of AR processes with complex-conjugate unit roots, with inference based on parameter estimates. Moreover, Gregoir (1999a, 1999b) studies covariance stationary vector moving average (VMA) processes where the determinant of the lag polynomial matrix involved has multiple real and/or complex unit roots. These processes give rise to a form of cointegration.

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In this paper, however, we will take a different route. Rather than focussing on estimation and parameter testing, we will derive a nonparametric test for multiple (but distinct) pairs of complex-conjugate unit roots in the AR lag polynomial of an ARMA process, without estimating the parameters involved, on the basis of the properties of the periodogram. This test will be applied to U.S. unemployment time series data[1] to see whether this series has complex unit roots in the business cycle frequencies.

Most of the proofs involve tedious but elementary trigonometric computations. These proofs are given in a separate Appendix.[2] Only the proofs of Theorems 1, 2, and 3 will be presented in an included Appendix.

2. AR(2) PROCESSES WITH COMPLEX UNIT ROOTS

2.1. Introduction

Consider the AR(2) process

$$y_t = 2\cos(\phi)y_{t-1} - y_{t-2} + \mu + u_t, \tag{1}$$

where u_t is independent and identically distributed (i.i.d.) $(0,\sigma^2)$ with $E|u_t|^{2+\delta} < \infty$ for some $\delta > 0$, μ is a constant, and $\phi \in (0,\pi)$. Throughout this paper we assume that y_t is observable for $t = 1,\dots,n$. The AR lag polynomial $\Phi(L) = 1 - 2\cos(\phi)L + L^2$ can be written as $\Phi(L) = (1 - \exp(i\phi)L)(1 - \exp(-i\phi)L)$; hence $\Phi(L)$ has two roots on the complex unit circle, $\exp(i\phi) = \cos(\phi) + i\sin(\phi)$ and its complex conjugate $\exp(-i\phi) = \cos(\phi) - i\sin(\phi)$, provided that $\sin(\phi) \neq 0$. The latter condition will be assumed throughout the paper, because otherwise either $\cos(\phi) = 1$, which implies that y_t is $I(2)$, or $\cos(\phi) = -1$, which implies that $y_t + y_{t-1}$ is $I(1)$.

Note that (1) generates a persistent cycle of $2\pi/\phi$ periods. If $\phi \in (\pi,2\pi)$, the cycle length is less than two periods. Such short cycles are unlikely to occur in macroeconomic time series, and if they occur, they are difficult, if not impossible, to distinguish from random variation. This is the reason for only considering the case $\phi \in (0,\pi)$.

It can be shown along the lines in Chan and Wei (1988) and Gregoir (1999a, 1999b, 1999c) that the solution of (1) is of the form

$$y_t = \frac{1}{\sin(\phi)} S_t(\phi)u_t + d_t \tag{2}$$

for $t \geq 1$, where

$$S_t(\phi)u_t = \sum_{j=1}^{t} \sin(\phi(t+1-j))u_j \tag{3}$$

and d_t is a deterministic process of the form

$$d_t = a\cos(\phi t) + b\sin(\phi t) + c, \tag{4}$$

with a, b, and c real valued time invariant (random) variables depending on initial conditions.[3]

Moreover, it is a standard calculus exercise to show that

$$S_t(\phi)u_t = (\cos(\phi t), \sin(\phi t))\begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix} \times \begin{pmatrix} -\sum_{j=1}^{t} u_j \sin(\phi j) \\ \sum_{j=1}^{t} u_j \cos(\phi j) \end{pmatrix}.$$

Furthermore, denoting[4]

$$W_{1,n}^*(x) = -\frac{\sqrt{2}}{\sigma\sqrt{n}}\sum_{j=1}^{[xn]} u_j \sin(\phi j), \qquad W_{2,n}^*(x) = \frac{\sqrt{2}}{\sigma\sqrt{n}}\sum_{j=1}^{[xn]} u_j \cos(\phi j) \tag{5}$$

for $x \in [0,1]$, it follows from Chan and Wei (1988, Theorem 2.2)[5] that jointly[6]

$$W_{1,n}^* \Rightarrow W_1 \quad \text{and} \quad W_{2,n}^* \Rightarrow W_2,$$

where W_1 and W_2 are independent standard Wiener processes. See Billingsley (1968). The same applies to

$$\begin{pmatrix} W_{1,n}(x) \\ W_{2,n}(x) \end{pmatrix} = Q_0 \begin{pmatrix} W_{1,n}^*(x) \\ W_{2,n}^*(x) \end{pmatrix}, \tag{6}$$

where

$$Q_0 = \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix}, \tag{7}$$

because the matrix Q_0 is orthogonal. Consequently, we have the following lemma.

LEMMA 1. *Under data-generating process (1),*

$$y_t/\sqrt{n} = \frac{\sigma}{\sin(\phi)\sqrt{2}}\left(\cos(\phi t)W_{1,n}(t/n) + \sin(\phi t)W_{2,n}(t/n)\right) + O_p(1/\sqrt{n}), \tag{8}$$

where

$$\begin{pmatrix} W_{1,n} \\ W_{2,n} \end{pmatrix} \Rightarrow \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}$$

on $[0,1]$, *with* W_1 *and* W_2 *independent standard Wiener processes. Moreover, the* $O_p(1/\sqrt{n})$ *remainder term is uniform in* $t = 1,\dots,n$.

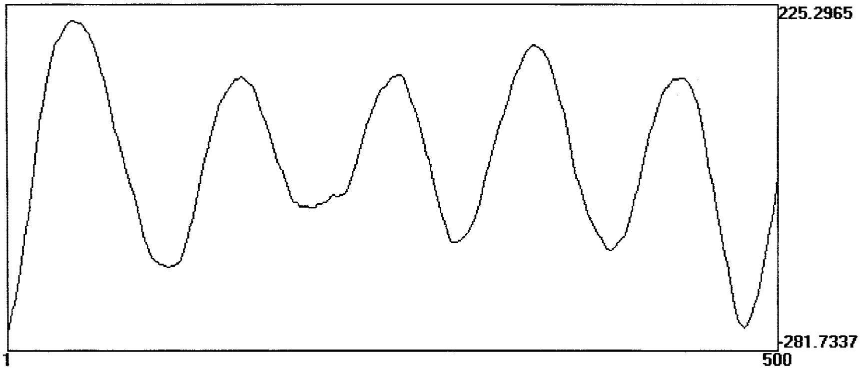

FIGURE 1. AR(2) process with complex unit roots and a cycle of 100 periods.

Thus, $y_t/\sqrt{n}$ takes the form of a linear function of $\sin(\phi t)$ and $\cos(\phi t)$, with random coefficients $W_{1,n}(t/n)$ and $W_{2,n}(t/n)$, respectively, plus a vanishing remainder term. Consequently, the series y_t will display a rather smooth cyclical pattern, with a cycle of $2\pi/\phi$ periods. A typical example is the artificial time series displayed in Figure 1. This time series is generated by $y_t = 1.9960534 y_{t-1} - y_{t-2} + u_t$, with u_t i.i.d. $N(0,1)$, for $t = 1,\ldots,500$. This series has a cycle of 100 periods.

2.2. Relaxing the i.i.d. Error Assumption

The assumption that the errors u_t in (1) are i.i.d. is not essential. We may replace it by the following assumption.

Assumption 1. Let (1) hold, with u_t a zero-mean stationary $MA(\infty)$ process: $u_t = \eta(L)\varepsilon_t$, where ε_t is i.i.d. (0,1), $E(|\varepsilon_t|^{2+\delta}) < \infty$ for some $\delta > 0$, $\eta(L) = \sum_{j=0}^{\infty}\eta_j L^j = \theta_1(L)/\theta_2(L)$ is a rational lag polynomial with all the roots of $\theta_2(L)$ outside the complex unit circle, and $\theta_1(e^{i\phi}) \neq 0$.[7]

Using the decomposition

$$\begin{aligned} u_t &= \eta(e^{i\phi})\varepsilon_t + (e^{i\phi} - L)\,\frac{\eta(L) - \eta(e^{i\phi})}{e^{i\phi} - L}\,\varepsilon_t \\ &= \eta(e^{i\phi})\varepsilon_t + e^{i\phi} w_t - w_{t-1}, \end{aligned} \quad \textbf{(9)}$$

say, and denoting

$$Q_1 = \frac{1}{|\eta(e^{i\phi})|}\begin{pmatrix} -\mathrm{Re}(\eta(e^{i\phi})) & \mathrm{Im}(\eta(e^{i\phi})) \\ \mathrm{Im}(\eta(e^{i\phi})) & \mathrm{Re}(\eta(e^{i\phi})) \end{pmatrix},$$

it is not hard to verify that the following lemma bolds.

LEMMA 2. *Let Assumption 1 hold. Redefine σ as*

$$\sigma = |\eta(e^{i\phi})| \tag{10}$$

and redefine $W_{1,n}$ and $W_{2,n}$ as

$$\begin{pmatrix} W_{1,n}(x) \\ W_{2,n}(x) \end{pmatrix} = Q_0 Q_1 \begin{pmatrix} W_{1,n}^{**}(x) \\ W_{2,n}^{**}(x) \end{pmatrix},$$

where

$$W_{1,n}^{**}(x) = -\frac{\sqrt{2}}{\sqrt{n}} \sum_{j=1}^{[xn]} \varepsilon_j \sin(\phi j), \qquad W_{2,n}^{**}(x) = \frac{\sqrt{2}}{\sqrt{n}} \sum_{j=1}^{[xn]} \varepsilon_j \cos(\phi j). \tag{11}$$

Then the result of Lemma 1 goes through.

2.3. Differencing and Other Lag Operators

The argument in the previous section also implies that, e.g., differencing of y_t does not eliminate the cycle, because the difference operator $1 - L$ changes $\eta(L)$ to $\eta_*(L) = (1 - L)\eta(L)$, which still satisfies Assumption 1. The same applies to any other polynomial lag operator $\tau(L)$ with $\tau(e^{i\phi}) \neq 0$. Denoting

$$Q_2 = \frac{1}{|\tau(e^{i\phi})|} \begin{pmatrix} \mathrm{Re}(\tau(e^{i\phi})) & \mathrm{Im}(\tau(e^{i\phi})) \\ -\mathrm{Im}(\tau(e^{i\phi})) & \mathrm{Re}(\tau(e^{i\phi})) \end{pmatrix},$$

we have the following lemma.

LEMMA 3. *Let $\tau(L)$ be a lag polynomial satisfying $\tau(e^{i\phi}) \neq 0$. Under Assumption 1,*

$$\tau(L)y_t/\sqrt{n} = \frac{\sigma|\tau(e^{i\phi})|}{\sin(\phi)\sqrt{2}} (\cos(\phi t)\widetilde{W}_{1,n}(t/n) + \sin(\phi t)\widetilde{W}_{2,n}(t/n)) + O_p(1/\sqrt{n}),$$

where $\sigma = |\eta(e^{i\phi})|$ and

$$\begin{pmatrix} \widetilde{W}_{1,n}(x) \\ \widetilde{W}_{2,n}(x) \end{pmatrix} = Q_0 Q_3 Q_2 \begin{pmatrix} W_{1,n}^{**}(x) \\ W_{2,n}^{**}(x) \end{pmatrix} \Rightarrow \begin{pmatrix} W_1(x) \\ W_2(x) \end{pmatrix}$$

on $[0,1]$, with $W_{1,n}^{**}$ and $W_{2,n}^{**}$ defined by (11) and W_1 and W_2 the same as before.

The significance of this result is that we can eliminate a real unit root one or a linear trend, and seasonal unit roots, by applying the appropriate difference operator, without affecting possible complex roots corresponding to the business cycle frequency.

Strictly speaking, the result in Lemma 3 also applies to the double differencing operator $\tau(L) = (1 - L)^2 = 1 - 2L + L^2 = \Delta^2$. However, in practice this lag operator would wipe out a complex unit root in $\Delta^2 y_t$ if the complex unit root involved corresponds to a business cycle frequency. For example, the AR(2) lag polynomial of the process y_t displayed in Figure 1 is $1 - 1.9960534L + L^2$, which is numerically too close to $1 - 2L + L^2$ to be distinguishable; hence the AR and MA lag polynomials of the resulting ARMA(2,2) process $\Delta^2 y_t$ will approximately cancel out, causing $\Delta^2 y_t$ to look like a white noise process.

Another consequence of this argument is that it will be virtually impossible to test for complex unit root in the business cycle frequency by using a parametric test on the basis of the results of Ahtola and Tiao (1987a, 1987b), Chan and Wei (1988), and Gregoir (1999c): It will in practice be impossible to distinguish in the auxiliary regression $y_t = \beta_1 y_{t-1} + \beta_2 y_{t-2} + u_t$ the null hypothesis $\beta_1 = 2\cos(\phi)$, $\beta_2 = -1$ from the $I(2)$ hypothesis $\beta_1 = 2$, $\beta_2 = -1$.

3. FREQUENCY ANALYSIS

The periodogram $\rho_n(\xi)$, say, of a time series y_t is defined by

$$\rho_n(\xi) = \frac{2}{n}\left[\left(\sum_{t=1}^{n} y_t \cos(\xi t)\right)^2 + \left(\sum_{t=1}^{n} y_t \sin(\xi t)\right)^2\right] \tag{12}$$

for $\xi \in (0, \pi)$ and odd n. See Fuller (1976, Ch. 7).

If y_t is a stationary linear process, say,

$$y_t = \mu + \eta(L)\varepsilon_t, \text{ where } \eta(L) \text{ and } \varepsilon_t \text{ are the same as in Assumption 1,} \tag{13}$$

then for fixed $\xi \in (0, \pi)$,

$$\rho_n(\xi) \Rightarrow 2\pi f(\xi)\chi_2^2, \tag{14}$$

where $f(\xi)$ is the spectral density of y_t. See Fuller (1976, Theorem 7.1.2, p. 280). As is not hard to verify, this result is due to the fact that under the stationarity hypothesis (13),

$$\rho_n(\xi) \Rightarrow |\eta(e^{i\xi})|^2(W_{1,\xi}(1)^2 + W_{2,\xi}(1)^2),$$

pointwise in $\xi \in (0, \pi)$, where $W_{1,\xi}$ and $W_{2,\xi}$ are independent standard Wiener processes depending on ξ, which are also independent across the ξ's.

The main idea in this paper is to use the standardized periodogram $\hat{\rho}(\xi) = \rho_n(\xi)/\hat{\sigma}_y^2$, where $\hat{\sigma}_y^2$ is the sample variance of the y_t's, as the basis for a nonparametric test of the complex unit root hypothesis against the stationarity hypothesis, because in the complex unit root case the properties of $\hat{\rho}(\xi)$ are quite different from the stationary case. This is illustrated in Figures 2 and 3. Figure 2 displays the periodogram of the complex unit root process plotted in Figure 1. Figure 3 displays the periodogram of the stationary Gaussian AR(2) process $y_t = 1.411423y_{t-1} - 0.5y_{t-2} + u_t$, $t = 1, \ldots, 500$, where the u_t's are

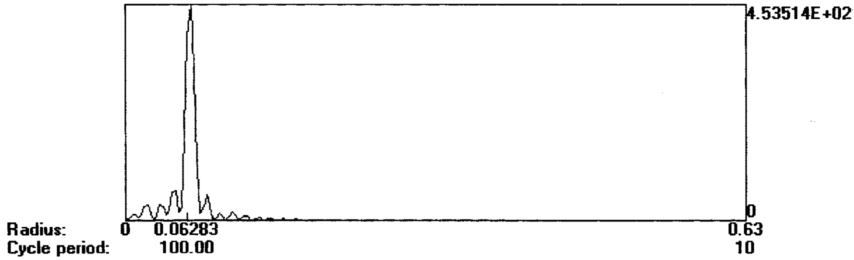

FIGURE 2. Periodogram of the complex unit root process plotted in Figure 1.

i.i.d. $N(0,1)$. The lag polynomial of this AR(2) process has complex roots outside the unit circle, corresponding to a (vanishing) cycle of 100 periods.

We see that the two periodograms are very distinct, both in shape and in scale. In particular, the periodogram of the stationary process has many more, and more widely spread, peaks than the periodogram of the complex unit root process, and the peaks are much lower than in the latter case.

The following theorem, which is proved in the Appendix, explains the differences between these two cases.

THEOREM 1. *Consider the standardized periodogram*

$$\hat{\rho}(\xi) = \frac{2}{n\hat{\sigma}_y^2}\left(\left(\sum_{t=1}^{n} y_t \cos(\xi t)\right)^2 + \left(\sum_{t=1}^{n} y_t \sin(\xi t)\right)^2\right), \qquad \xi \in (0,\pi),$$

where $\hat{\sigma}_y^2$ is the sample variance. Let

$$B_1 = \frac{\left(\int_0^1 W_1(x)dx\right)^2 + \left(\int_0^1 W_2(x)dx\right)^2}{\int_0^1 W_1(x)^2 dx + \int_0^1 W_2(x)^2 dx}.$$

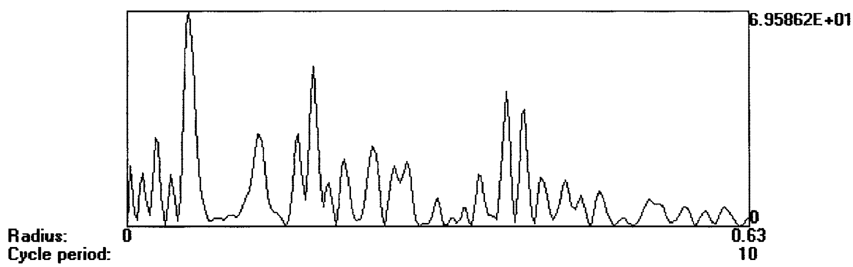

FIGURE 3. Periodogram of a stationary AR(2) process with complex roots and a cycle of 100 periods.

Under Assumption 1,

$$\frac{\hat{\rho}(\phi)}{n} \Rightarrow B_1, \tag{15}$$

whereas pointwise in $\xi \in (0,\phi) \cup (\phi,\pi)$,

$$\hat{\rho}(\xi) = O_p(1). \tag{16}$$

Under the stationarity hypothesis (13),

$$\hat{\rho}(\xi) \Rightarrow \frac{|\eta(e^{i\xi})|^2}{\mathrm{var}(y_t)} (W_{1,\xi}(1)^2 + W_{2,\xi}(1)^2) \tag{17}$$

pointwise in $\xi \in (0,\pi)$, *where* $\{(W_{1,\xi}, W_{2,\xi}), \xi \in (0,\pi)\}$ *is a collection of independent bivariate standard Wiener processes.*

Theorem 1 implies that in the complex unit root case $\hat{\rho}(\xi)/n$ has a sharp spike at $\xi = \phi$, with height asymptotically distributed as B_1, and asymptotically zero elsewhere, whereas (14) implies that under the stationarity hypothesis, $\hat{\rho}(\xi)$ is bounded away from zero, and asymptotically bounded from above by independent χ_2^2 random variables, times $|\eta(e^{i\xi})|^2/\mathrm{var}(y_t)$, pointwise in $\xi \in (0,\pi)$.

4. MULTIPLE CYCLES

4.1. The State Space Case

The periodograms of macroeconomic time series often display multiple peaks in the business cycle frequencies. If k of these peaks are due to complex unit roots, then one way of modeling the process involved is as an ARMA$(2k,\infty)$ process with all the roots of the AR lag polynomial on the complex unit circle. However, as is already clear from Figure 1, the plots of such processes are very smooth, much smoother than for most economic time series. Therefore, in the first instance we propose to model these time series as a state space model of aggregates of ARMA processes with different single pairs of complex-conjugate unit roots, plus a stationary ARMA process representing the noise. The ARMA$(2k,\infty)$ case will be considered in the next section.

Assumption 2. The data-generating process is $y_t = \sum_{j=0}^{k} y_{j,t}$, where $y_{0,t} = \mu_0 + \eta_0(L)\varepsilon_{0,t}$ satisfies the conditions in (13), and for $j = 1,\dots,k$, $y_{j,t} = 2\cos(\phi_j)y_{j,t-1} - y_{j,t-2} + \mu_j + \eta_j(L)\varepsilon_{j,t}$, with $0 < \phi_1 < \cdots < \phi_k < \pi$. The lag polynomials $\eta_j(L)$ are rational: $\eta_j(L) = \eta_{1,j}(L)/\eta_{2,j}(L)$, with $\eta_{2,j}(L)$ having all its roots outside the unit circle, and the $(\varepsilon_{1,t},\dots,\varepsilon_{k,t})$'s are i.i.d. $(0,I)$, with $E(|\varepsilon_{j,t}|^{2+\delta}) < \infty$ for some $\delta > 0$.

Admittedly, the assumption that the $\varepsilon_{j,t}$'s are contemporaneously uncorrelated is quite restrictive, but it is needed to derive nuisance-free asymptotic null distributions of the tests we are going to propose.

The process $y_{0,t}$ will only play a role under the alternative hypothesis of stationarity, which corresponds to the case $k = 0$.

Except for parts (19) and (20), the following results follow straightforwardly from Theorem 1.

THEOREM 2. *Let* $\omega_j = (1/\sqrt{2})\sigma_j/\sin(\phi_j)$, *where* $\sigma_j = |\eta_j(\exp(i\phi_j))|$. *Under Assumption 2,*

$$y_t/\sqrt{n} = \sum_{m=1}^{k} \omega_m(\cos(\phi_m t)W_{1,m,n}(t/n) + \sin(\phi_m t)W_{2,m,n}(t/n)) + O_p(1/\sqrt{n}), \tag{18}$$

where jointly $(W_{1,j,n}, W_{2,j,n})' \Rightarrow W_j = (W_{1,j}, W_{2,j})'$, *with* $W_1, \ldots, W_k$ *independent bivariate standard Wiener processes. Consequently,*

$$\frac{\hat{\rho}(\phi_j)}{n} \Rightarrow \psi_k(\phi_j) = \frac{\omega_j^2\left(\left(\int_0^1 W_{1,j}(x)dx\right)^2 + \left(\int_0^1 W_{2,j}(x)dx\right)^2\right)}{\sum_{m=1}^{k}\omega_m^2\left(\int_0^1 W_{1,m}(x)^2dx + \int_0^1 W_{2,m}(x)^2dx\right)}$$

jointly for $j = 1, \ldots, k$. *Moreover,*

$$\frac{\hat{\rho}(\xi)}{n} = O_p(1/\sqrt{n}),$$

pointwise in $\xi \in (0,\pi)\backslash\{\phi_1, \ldots, \phi_k\}$.

The result of Theorem 2 cannot be used directly to design a test for complex unit roots, because of the presence of the nuisance parameters ω_j. However, it is possible to construct a test statistic for which the asymptotic null distribution has a nuisance-free lower bound.

THEOREM 3. *Under Assumption 2,*

$$\max_{j=1,\ldots,k}\psi_k(\phi_j) \geq B_k, \qquad \min_{j=1,\ldots,k}\psi_k(\phi_j) \leq B_k, \tag{19}$$

where

$$B_k = \left(\sum_{m=1}^{k}\frac{\int_0^1 W_{1,m}(x)^2dx + \int_0^1 W_{2,m}(x)^2dx}{\left(\int_0^1 W_{1,m}(x)dx\right)^2 + \left(\int_0^1 W_{2,m}(x)dx\right)^2}\right)^{-1}. \tag{20}$$

Theorem 3 suggests testing the complex unit root hypothesis:

$$H_0: \text{Assumption 2 holds for given } k \text{ and } \phi_1 = \phi_{0,1}, \ldots, \phi_k = \phi_{0,k}, \tag{21}$$

TABLE 1. Values of $\underline{\beta}_k(\alpha)$

k	$\alpha = 0.05$	$\alpha = 0.10$
1	0.1403	0.2411
2	0.0667	0.1146
3	0.0441	0.0732
4	0.0313	0.0519
5	0.0249	0.0409
6	0.0210	0.0337
7	0.0177	0.0287
8	0.0154	0.0250
9	0.0137	0.0222
10	0.0120	0.0196

by using the test statistic[8]

$$\hat{B}_k = \max_{j=1,\ldots,k} \hat{\rho}(\phi_{0,j})/n \qquad \textbf{(22)}$$

with $\alpha \times 100\%$ critical values $\underline{\beta}_k(\alpha)$, say, based on the lower bound B_k of the asymptotic null distribution of $\hat{B}_k$: $P(B_k \leq \underline{\beta}_k(\alpha)) = \alpha$. In Table 1 we present the critical values $\underline{\beta}_k(\alpha)$ for $k = 1,\ldots,10$ and $\alpha = 0.05$, 0.10, which have been computed by Monte Carlo simulation.[9]

Given that k and $\phi_{0,1},\ldots,\phi_{0,k}$ are specified in advance, this test is consistent against the stationarity hypothesis and also against the hypothesis that none of the given values of $\phi_{0,1},\ldots,\phi_{0,k}$ correspond to the ones in Assumption 2.

4.2. The ARMA(2*k*,∞) Case

Consider the ARMA$(2k,\infty)$ model with k pairs of complex-conjugate unit roots.

Assumption 3. $\left[\prod_{j=1}^{k}(1 - 2\cos(\phi_{k+1-j})L + L^2)\right]y_t = \mu + \eta(L)\varepsilon_t$, where $\eta(L)$ and ε_t are the same as in Assumption 1 and $0 < \phi_1 < \cdots < \phi_k < \pi$.

Let $u_t = \eta(L)\varepsilon_t$. It follows similarly to (2) that

$$y_t = \frac{S_t(\phi_k)S_t(\phi_{k-1})\ldots S_t(\phi_1)u_t}{\prod_{j=0}^{k-1}\sin(\phi_{k-j})} + d_t,$$

where $S_t(\phi)u_t$ is defined by (3), for each pair ϕ_1, ϕ_2,

$$S_t(\phi_2)S_t(\phi_1)u_t = \sum_{j=1}^{t}\sin(\phi_2(t+1-j))S_j(\phi_1)u_j,$$

and d_t is a deterministic process of the type (4).

Next, let

$$C_t(\phi)u_t = \sum_{j=1}^{t} \cos(\phi_1(t+1-j))u_t$$

and let for each pair ϕ_1, ϕ_2,

$$S_t(\phi_2)C_t(\phi_1)u_t = \sum_{j=1}^{t} \sin(\phi_2(t+1-j))C_j(\phi_1)u_j.$$

Then we have the following lemma.

LEMMA 4. *If* $\phi_1 \neq \phi_2$ *then*

$$\begin{aligned} S_t(\phi_2)S_t(\phi_1)u_t = {} & (\gamma_1(\phi_2,\phi_1) - \delta_1(\phi_2,\phi_1))(C_t(\phi_2)u_t - C_t(\phi_1)u_t) \\ & - \gamma_2(\phi_2,\phi_1)(S_t(\phi_2)u_t - S_t(\phi_1)u_t) \\ & + \delta_2(\phi_2,\phi_1)(S_t(\phi_2)u_t + S_t(\phi_1)u_t) \end{aligned}$$

and

$$\begin{aligned} S_t(\phi_2)C_t(\phi_1)u_t = {} & (\gamma_2(\phi_2,\phi_1) + \delta_2(\phi_2,\phi_1))(C_t(\phi_2)u_t - C_t(\phi_1)u_t) \\ & + \gamma_1(\phi_2,\phi_1)(S_t(\phi_2)u_t - S_t(\phi_1)u_t) \\ & + \delta_1(\phi_2,\phi_1)(S_t(\phi_2)u_t + S_t(\phi_1)u_t), \end{aligned}$$

where

$$\gamma_1(\phi_2,\phi_1) = \frac{1}{2}\frac{\cos(\phi_2)-\cos(\phi_1)}{(\cos(\phi_2)-\cos(\phi_1))^2+(\sin(\phi_2)-\sin(\phi_1))^2},$$

$$\gamma_2(\phi_2,\phi_1) = \frac{1}{2}\frac{\sin(\phi_2)-\sin(\phi_1)}{(\cos(\phi_2)-\cos(\phi_1))^2+(\sin(\phi_2)-\sin(\phi_1))^2},$$

$$\delta_1(\phi_2,\phi_1) = \frac{1}{2}\frac{\cos(\phi_2)-\cos(\phi_1)}{(\cos(\phi_2)-\cos(\phi_1))^2+(\sin(\phi_2)+\sin(\phi_1))^2}$$

$$\delta_2(\phi_2,\phi_1) = \frac{1}{2}\frac{\sin(\phi_2)+\sin(\phi_1)}{(\cos(\phi_2)-\cos(\phi_1))^2+(\sin(\phi_2)+\sin(\phi_1))^2}.$$

The proof of Lemma 4 is quite tedious but involves only elementary trigonometric operations and will therefore be given in the separate Appendix to this paper.

Lemma 4 implies that y_t can be written as

$$y_t = \sum_{j=1}^{k} \gamma_j S_t(\phi_j)u_t + \sum_{j=1}^{k} \delta_j C_t(\phi_j)u_t + d_t,$$

where the γ_j's and δ_j's are constants depending on the ϕ_j's. Moreover, similarly to Lemma 1 it follows that there exist orthogonal 2×2 matrices $Q_1, \ldots, Q_k$ and constants κ_j such that

$$\gamma_m S_t(\phi_m)u_t + \delta_m C_t(\phi_m)u_t$$
$$= \kappa_m(\cos(\phi t), \sin(\phi t))Q_m \begin{pmatrix} -\sum_{j=1}^{t} u_j \sin(\phi_m j) \\ \sum_{j=1}^{t} u_j \cos(\phi_m j) \end{pmatrix}.$$

Furthermore, it follows from Lemma 2 that there exist orthogonal 2×2 matrices $R_1, \ldots, R_k$ such that

$$\begin{pmatrix} -\sum_{j=1}^{t} u_j \sin(\phi_m j) \\ \sum_{j=1}^{t} u_j \cos(\phi_m j) \end{pmatrix} = |\eta(\exp(i\phi_m))| R_m \begin{pmatrix} -\sum_{j=1}^{t} \varepsilon_j \sin(\phi_m j) \\ \sum_{j=1}^{t} \varepsilon_j \cos(\phi_m j) \end{pmatrix}.$$

Therefore, defining

$$\begin{pmatrix} W_{1,m}(x) \\ W_{2,m}(x) \end{pmatrix} = Q_m R_m \begin{pmatrix} -(\sqrt{2}/\sqrt{n}) \sum_{j=1}^{t} \varepsilon_j \sin(\phi_m j) \\ (\sqrt{2}/\sqrt{n}) \sum_{j=1}^{t} \varepsilon_j \cos(\phi_m j) \end{pmatrix},$$

it follows that there exist constants ω_j such that (18) carries over.

THEOREM 4. *Apart from the definition of the constants ω_j, Theorems 2 and 3 hold under Assumption 3 also.*

This result also holds if we combine Assumptions 2 and 3, i.e., in the following assumption.

Assumption 4. Let $y_t = \sum_{j=0}^{K} y_{j,t}$, where $y_{0,t}$ is the same as in Assumption 2, and for $j = 1, \ldots, K$, $[\prod_{j=1}^{k}(1 - 2\cos(\phi_{k+1-j})L + L^2)]y_{j,t} = \mu_j + \eta_j(L)\varepsilon_{j,t}$, where $\eta_j(L)$ and $\varepsilon_{j,t}$ are the same as in Assumption 2 and $0 < \phi_1 < \cdots < \phi_k < \pi$.

THEOREM 5. *Theorem 4 carries over under Assumption 4.*

Thus, in this case the processes $y_{j,t}$, $j = 1, \ldots, K$, have common complex-conjugate unit roots. The condition in Assumption 2 that the $\varepsilon_{j,t}$'s are uncorrelated across the j's is now no longer needed, because if the variance matrix of $(\varepsilon_{1,t}, \ldots, \varepsilon_{K,t})'$ is Σ, say, we may without loss of generality replace $(y_{1,t}, \ldots, y_{K,t})'$ by $Q'(y_{1,t}, \ldots, y_{K,t})'$, where Q is the $K \times K_*$ matrix of eigenvectors of Σ corresponding to the K_* positive eigenvalues. Thus, without loss of generality we may assume that $\Sigma = I$.

5. ARE BUSINESS CYCLES DUE TO COMPLEX UNIT ROOTS?

In conducting the tests for complex unit roots, it is tempting to formulate the null hypothesis (21) by looking at the periodogram of the time series involved and selecting the frequencies $\phi_{0,1},\dots,\phi_{0,k}$ corresponding to the k highest peaks. However, this is akin to pretesting and will affect the actual size and power of the test. The correct way of conducting the test is to formulate the null hypothesis prior to looking at the data. But all information about business cycles is based on empirical investigations (see, e.g., Diebold and Rudebusch, 1999, and the references therein), so that even if we would choose $\phi_{0,1},\dots,\phi_{0,k}$ corresponding to the NBER business cycle dates and durations listed in Diebold and Rudebusch (1999, Table 2.1, p. 39), prior to looking at the periodogram, we would indirectly commit a pretesting type of sin also. In testing for seasonal unit roots this problem does not occur, of course, but it is virtually impossible to avoid when testing for complex unit roots in the business cycle frequencies. In our empirical application we will therefore ignore this problem and look at the periodogram first to determine potential complex unit root frequencies.

The time series we analyze is the monthly number of civilian unemployed in the United States, times 1,000, from 1948.01 to 1999.12. To eliminate possible seasonal unit roots, and to eliminate a possible unit root one also, we have transformed the series to annual changes. The plot of the transformed series is displayed in Figure 4.

The standardized periodogram $\hat{\rho}(\xi)$ is displayed in Figure 5. The first peak (with a little dip in the top) corresponds to a cycle duration between 104 and 133 months. The second, and highest, peak corresponds to a cycle of 65 months, and the four next highest peaks correspond to cycles of 50, 43, 33, and 28 months, respectively. These cycle durations are quite close to the post–World War II NBER business cycle (trough to trough) durations listed in Diebold and

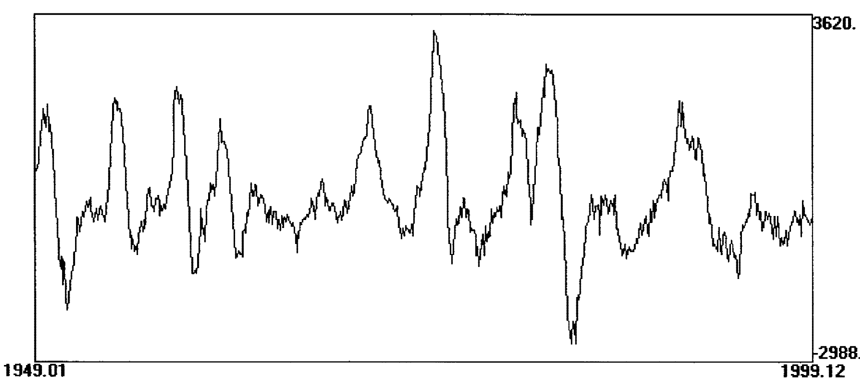

FIGURE 4. The data.

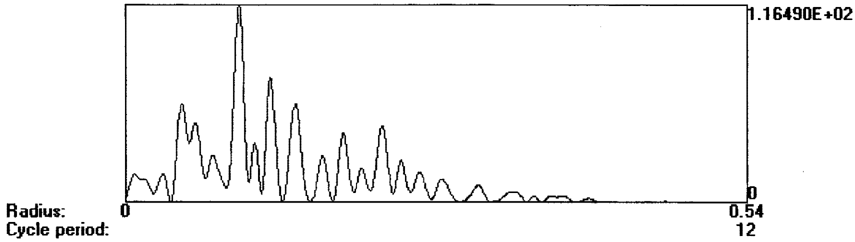

FIGURE 5. Standardized periodogram of the data.

Rudebusch (1999, Table 2.1, p. 39). The longest postwar NBER cycle duration is 117 months, which corresponds to the little dip in the top of the first peak.

We now test the null hypothesis that this series has six pairs of complex-conjugate unit roots, with frequencies corresponding to cycles of 117, 65, 50, 43, 33, and 28 months. The results are shown in Table 2.

Clearly, the complex unit root hypothesis involved is not rejected. However, the critical values are based on lower bounds, which become increasingly conservative with the number of pairs of complex-conjugate unit roots. Only for $k = 1$ are the asymptotic critical values exact. If we would test the null hypothesis that there is only one pair of complex-conjugate unit roots, then it follows from Table 1 that the hypothesis corresponding to the cycle of 65 months is accepted at the 5% significance level but rejected at the 10% level (the p-value involved is 0.0645), whereas all the five other cycles tested in Table 2 are rejected at the 5% level. Thus the question whether the NBER cycles of 117, 50, 43, 33, and 28 months are due to complex unit roots remains unanswered. Only for the NBER cycle of 65 months is there some weak evidence of a complex unit root.

TABLE 2. Null hypothesis and test results

j	$\phi_{0,j}$	Cycle	$\hat{\rho}(\phi_{0,j})/n$
1	0.05370	117	0.05284
2	0.09666	65	0.16829
3	0.12566	50	0.10728
4	0.14612	43	0.08585
5	0.19040	33	0.05284
6	0.22440	28	0.05703

Test statistic $= \max_{j-1,\ldots,6} \hat{\rho}(\phi_{0,j})/n = 0.16829$; 10% critical region = (0, 0.0337); 5% critical region = (0, 0.0210); p-value ≈ 1.

TABLE 3. Tests of the stationary AR(p) hypothesis

p	$\hat{A}_{k,p}$	p	$\hat{A}_{k,p}$
1	87.921	9	129.097
2	91.020	10	133.758
3	104.142	11	129.373
4	112.324	12	138.025
5	119.759	24	158.656
6	123.815	36	183.863
7	128.857	48	185.455
8	128.686	60	188.559

However, something more can be said. Recall that under the stationarity hypothesis (13) the periodogram ordinates $\rho_n(\xi)$ converge in distribution, pointwise in $\xi \in (0,\pi)$, to $|\eta(e^{i\xi})|^2\chi_2^2(\xi)$, where $\{\chi_2^2(\xi), \xi \in (0,\pi)\}$ is a collection of independent χ_2^2 distributed random variables. Therefore, if y_t is an AR(p) process: $\theta_p(L)y_t = \mu_* + \sigma\varepsilon_t$, with $\theta_p(L) = 1 - \theta_1 L - \cdots - \theta_p L^p$, and $\mu_* = \theta_p(1)\mu$, then $\eta(L) = \sigma\theta_p(L)^{-1}$; hence $\sigma^{-2}|\theta_p(e^{i\xi})|^2\rho_n(\xi) \to \chi_2^2(\xi)$ in distribution, pointwise in $\xi \in (0,\pi)$. Consequently, for given values $0 < \phi_1 < \phi_2 < \cdots < \phi_k < \pi$ we have $\sigma^{-2}\sum_{j=1}^k|\theta_p(\exp(i\phi_j))|^2\rho_n(\phi_j) \to \chi_{2k}^2$ in distribution. On the other hand, if one or more of the values ϕ_j correspond to complex-conjugate unit roots, then $\sigma^{-2}\sum_{j=1}^k|\theta_p(\exp(i\phi_j))|^2\rho_n(\phi_j) \to \infty$. These results carry over if we replace σ^2 and the parameters in $\theta_p(L)$ by ordinary least squares (OLS) estimates. Thus, denoting the estimated lag polynomial by $\hat{\theta}_p(L)$, and the usual OLS estimate of σ^2 by $\hat{\sigma}^2$, we can test the stationary AR(p) hypothesis against the complex unit root hypothesis by using the test statistic

$$\hat{A}_{k,p} = \hat{\sigma}^{-2}\sum_{j=1}^{k}|\hat{\theta}_p(\exp(i\phi_j))|^2\rho_n(\phi_j).$$

For the $k = 6$ frequencies corresponding to the cycles in Table 2, and a variety of values of p, the test statistics $\hat{A}_{k,p}$ take the values shown in Table 3.

Because the null distribution is χ_{12}^2, with 10% and 5% critical values 18.549 and 21.026, respectively, these null hypotheses are strongly rejected.

These results provide evidence that the NBER business cycles are indeed due to complex-conjugate unit roots. Whether this evidence is compelling depends on how one weighs the pretesting problem mentioned before.

NOTES

1. The empirical application involved has been conducted with the author's free software package *EasyReg 2000*, which is downloadable from web page http://econ.la.psu.edu/~hbierens/

EASYREG.HTM. The monthly unemployment time series involved is included in the *EasyReg* database.

2. This Appendix is included in the working paper version, which is downloadable as a PDF file from web page http://econ.la.psu.edu/~hbierens/PAPERS.HTM.

3. As a result of the presence of the deterministic term d_t in (2), we can avoid the assumption in Gregoir (1999c) that $u_t = 0$ for $t < 1$.

4. Throughout this paper we adopt the convention that for $t < s$ the sum $\sum_{j=s}^{t}(\bullet)$ is zero.

5. Chan and Wei (1988) assume that the errors u_t are martingale differences, which is more general than the i.i.d. assumption. The latter assumption is made for the sake of transparency of the arguments. All our results carry over under the martingale difference assumption in Chan and Wei (1988).

6. Following Billingsley (1968), throughout this paper the double arrow $\Rightarrow$ indicates weak convergence of random functions, or convergence in distribution in the case of random variables. The single arrow $\to$ indicates convergence in probability, unless otherwise stated.

7. Because $\theta_1(L)$ is real valued, all complex-valued roots come in conjugate pairs. Hence $\theta_1(e^{i\phi}) \neq 0$ implies $\theta_1(e^{-i\phi}) \neq 0$ and vice versa.

8. Under this null hypothesis the statistic min $\hat{\rho}(\phi_{0,j})/n$ has an asymptotic upper bound equal to B_k. However, under stationarity min $\hat{\rho}(\phi_{0,j})/n \to 0$, so that a test based on min $\hat{\rho}(\phi_{0,j})/n$ has no asymptotic power against stationarity.

9. These critical values have been computed by Monte Carlo simulation, on the basis of 10,000 replications of 10 independent Gaussian random walks z_t, $t = 1,\ldots,n = 5{,}000$, $z_0 = 0$, and the well-known convergence results $(1/n)\sum_{t=1}^{n} z_t/\sqrt{n} \Rightarrow \int_0^1 W(x)dx$, $(1/n^2)\sum_{t=1}^{n} z_t^2 \Rightarrow \int_0^1 W(x)^2 dx$, where W is a standard Wiener process.

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APPENDIX: PROOF OF THEOREMS 1–3

Proof of Theorem 1. Except for part (17), which is proved in the separate Appendix, Theorem 1 follows from the following lemma.

LEMMA A.1. *Under Assumption 1,*

$$\hat{\sigma}_y^2/n \Rightarrow \frac{\sigma^2}{4\sin^2(\phi)}\left(\int_0^1 W_1(x)^2 dx + \int_0^1 W_2(x)^2 dx\right). \quad \textbf{(A.1)}$$

Moreover,

$$\frac{1}{n\sqrt{n}}\sum_{t=1}^n y_t \begin{pmatrix}\cos(\phi t)\\ \sin(\phi t)\end{pmatrix} \Rightarrow \frac{\sigma}{2\sqrt{2}\sin(\phi)}\begin{pmatrix}\int_0^1 W_1(x)dx\\ \int_0^1 W_2(x)dx\end{pmatrix}. \quad \textbf{(A.2)}$$

Furthermore, for fixed $\xi \in (0,\phi) \cup (\phi,\pi)$*,*

$$\frac{1}{n\sqrt{n}}\sum_{t=1}^n y_t \begin{pmatrix}\cos(\xi t)\\ \sin(\xi t)\end{pmatrix} = O_p(1/\sqrt{n}). \quad \textbf{(A.3)}$$

It follows from (12) and (A.2) that

$$\rho_n(\phi)/n^2 = 2\left[\left(\frac{1}{n\sqrt{n}}\sum_{t=1}^n y_t\cos(\phi t)\right)^2 + \left(\frac{1}{n\sqrt{n}}\sum_{t=1}^n y_t\sin(\phi t)\right)^2\right]$$
$$\Rightarrow \frac{\sigma^2}{4\sin^2(\phi)}\left[\left(\int_0^1 W_1(x)dx\right)^2 + \left(\int_0^1 W_2(x)dx\right)^2\right],$$

which together with (A.1) implies that (15) holds. Moreover, it follows from (A.2) that for fixed $\xi \in (0,\phi) \cup (\phi,\pi)$, $\rho_n(\xi)/n^2 = O_p(1/n)$, which together with (A.1) implies that (16) holds.

Proof of (A.1). Part (A.1) of Lemma A.1 follows from

$$\bar{y} = \frac{1}{n}\sum_{t=1}^n y_t = O_p(1) \quad \textbf{(A.4)}$$

and

$$\frac{1}{n^2}\sum_{t=1}^n y_t^2 \Rightarrow \frac{\sigma^2}{4\sin^2(\phi)}\left(\int_0^1 W_1(x)^2 dx + \int_0^1 W_2(x)^2 dx\right). \quad \textbf{(A.5)}$$

■

Proof of (A.4). First, observe that there exist functions $a(\xi), b(\xi), c(\xi)$, and $d(\xi)$, not depending on t, such that for $t = 1, 2, \dots,$

$$\cos(\xi t) = a(\xi)\int_t^{t+1}\cos(\xi x)dx + b(\xi)\int_t^{t+1}\sin(\xi x)dx,$$

$$\sin(\xi t) = c(\xi)\int_t^{t+1}\cos(\xi x)dx + d(\xi)\int_t^{t+1}\sin(\xi x)dx. \qquad \textbf{(A.6)}$$

Therefore, it follows from (8) that

$$\begin{aligned}
&\frac{\sqrt{2}\sin(\phi)}{\sigma n\sqrt{n}}\sum_{t=1}^{n} y_t \\
&\quad = \frac{1}{n}\sum_{t=1}^{n}\cos(\phi t)W_{1,n}(t/n) + \frac{1}{n}\sum_{t=1}^{n}\sin(\phi t)W_{2,n}(t/n) + O_p(1/\sqrt{n}) \\
&\quad = a(\phi)\int_0^1\cos(n\phi x)W_{1,n}(x)dx + b(\phi)\int_0^1\sin(n\phi x)W_{1,n}(x)dx \\
&\qquad \times c(\phi)\int_0^1\cos(n\phi x)W_{2,n}(x)dx + d(\phi)\int_0^1\sin(n\phi x)W_{2,n}(x)dx \\
&\qquad + O_p(1/\sqrt{n}). \qquad \textbf{(A.7)}
\end{aligned}$$

Moreover, it is not hard to verify that

$$E[W_{1,n}(x)W_{1,n}(y)] = \min(x, y) + O(1/n).$$

Therefore

$$\begin{aligned}
&E\left(\int_0^1\cos(n\phi x)W_{1,n}(x)dx\right)^2 \\
&\quad = \int_0^1\int_0^1\cos(n\phi x)\cos(n\phi y)\min(x, y)dxdy + O(1/n) \\
&\quad = O(1/n),
\end{aligned}$$

where the last equality is an elementary calculus result. Thus,

$$\int_0^1\cos(n\phi x)W_{1,n}(x)dx = O_p(1/\sqrt{n}).$$

Along the same lines it can be shown that the other terms in (A.7) are $O_p(1/\sqrt{n})$. ■

Proof of (A.5). It follows from (8) that

$$\frac{1}{n^2}\sum_{t=1}^{n} y_t^2 = \frac{\sigma^2}{2\sin^2(\phi)}\frac{1}{n}\sum_{t=1}^{n}(\cos^2(\phi t)W_{1,n}(t/n)^2 + \sin^2(\phi t)W_{2.n}(t/n)^2$$
$$+ 2\cos(\phi t)\sin(\phi t)W_{1,n}(t/n)W_{2,n}(t/n)) + O_p(1/\sqrt{n})$$
$$= \frac{\sigma^2}{4\sin^2(\phi)}\left(\frac{1}{n}\sum_{t=1}^{n} W_{1,n}(t/n)^2 + \frac{1}{n}\sum_{t=1}^{n} W_{2,n}(t/n)^2\right.$$
$$+ \frac{1}{n}\sum_{t=1}^{n}\cos(2\phi t)W_{1,n}(t/n)^2 - \frac{1}{n}\sum_{t=1}^{n}\cos(2\phi t)W_{2,n}(t/n)^2$$
$$\left. + 2\frac{1}{n}\sum_{t=1}^{n}\sin(2\phi t)W_{1,n}(t/n)W_{2,t}(t/n)\right).$$

It is easy to show that

$$\frac{1}{n}\sum_{t=1}^{n} W_{1,n}(t/n)^2 = \int_0^1 W_{1,n}(x)^2dx + O_p(1/n),$$
$$\frac{1}{n}\sum_{t=1}^{n} W_{2,n}(t/n)^2 = \int_0^1 W_{2,n}(x)^2dx + O_p(1/n).$$

Hence by the continuous mapping theorem (see Billingsley, 1968),

$$\frac{1}{n}\sum_{t=1}^{n} W_{1,n}(t/n)^2 + \frac{1}{n}\sum_{t=1}^{n} W_{2,n}(t/n)^2 \Rightarrow \int_0^1 W_1(x)^2dx + \int_0^1 W_2(x)^2dx.$$

Moreover, it follows similarly to (A.7) that

$$\frac{1}{n}\sum_{t=1}^{n}\cos(2\phi t)W_{1,n}(t/n)^2$$
$$= a(2\phi)\int_0^1 \cos(2n\phi x)W_{1,n}(x)^2dx$$
$$+ b(2\phi)\int_0^1 \sin(2n\phi x)W_{1,n}(x)^2dx + O_p(1/n), \quad \textbf{(A.8)}$$

$$\frac{1}{n}\sum_{t=1}^{n}\sin(2\phi t)W_{1,n}(t/n)W_{2,t}(t/n)$$
$$= c(2\phi)\int_0^1 \cos(2n\phi x)W_{1,n}(x)W_{2,n}(x)dx$$
$$+ d(2\phi)\int_0^1 \sin(2n\phi x)W_{1,n}(x)W_{2,n}(x)dx + O_p(1/n). \quad \textbf{(A.9)}$$

In analyzing the asymptotic properties of continuous functions of $W_{1,n}$ and/or $W_{2,n}$, it often suffices to analyze the properties of the same functions of the independent standard Wiener processes W_1, W_2, because of the Skorohod (1956), Dudley (1968), and Wichura (1970) representation theorem. See also Gaenssler (1983, p. 83). Loosely speaking, this representation theorem states that there exist versions $\overline{W}_n = (\overline{W}_{1,n}, \overline{W}_{2,n})'$ and $\overline{W} = (\overline{W}_1, \overline{W}_2)'$ of $W_n = (W_{1,n}, W_{2,n})'$ and $W = (W_1, W_2)'$, respectively, such that $\overline{W}_n$ has the same distribution as W_n, $\overline{W}$ has the same distribution as W (namely, a bivariate standard Wiener process), and $\overline{W}_n \to \overline{W}$ a.s. (More precisely,

$$P\left[\lim_{n\to\infty} \rho(\overline{W}_n, \overline{W})\right] = 1,$$

where ρ is the Skorohod norm on the space $D^2[0,1]$ of right-continuous mappings from $[0,1]$ into $\mathbb{R}^2$. See Billingsley (1968).)

Due to the representation theorem, the limiting distribution of

$$\int_0^1 \cos(2n\phi x) W_{1,n}(x)^2 dx$$

is the same as the limiting distribution of

$$\int_0^1 \cos(2n\phi x) W_1(x)^2 dx.$$

The latter limited distribution is constant zero, because

$$\begin{aligned} E\left(\int_0^1 \cos(2n\phi x) W_1(x)^2 dx\right)^2 &= \int_0^1 \int_0^1 \cos(2n\phi x)\cos(2n\phi y) E[W_1(x)^2 W_1(y)^2]\, dxdy \\ &= \int_0^1 \int_0^1 \cos(2n\phi x)\cos(2n\phi y)(2(\min(x,y))^2 + xy)\, dxdy \\ &= O(1/n). \end{aligned}$$

The second equality is a standard Wiener measure calculus result, and the last equality is an easy calculus exercise. Thus by Chebyshev's inequality

$$\int_0^1 \cos(2n\phi x) W_{1,n}(x)^2 dx \to 0. \tag{A.10}$$

The same applies to the sinus case. Along the same lines it can be shown that

$$\int_0^1 \cos(2n\phi x) W_{1,n}(x) W_{2,n}(x) dx \to 0, \tag{A.11}$$

and the same applies to the sinus case. ■

Proof of (A.2). It follows from (8) that

$$\begin{aligned}\frac{1}{n\sqrt{n}}\sum_{t=1}^{n} y_t\cos(\phi t) &= \frac{\sigma}{\sqrt{2}\sin(\phi)}\frac{1}{n}\sum_{t=1}^{n}\cos^2(\phi t)W_{1,n}(t/n)\\ &\quad+\frac{\sigma}{\sqrt{2}\sin(\phi)}\frac{1}{n}\sum_{t=1}^{n}\cos(\phi t)\sin(\phi t)W_{2,n}(t/n)\\ &=\frac{\sigma}{2\sqrt{2}\sin(\phi)}\frac{1}{n}\sum_{t=1}^{n}W_{1,n}(t/n)\\ &\quad+\frac{\sigma}{2\sqrt{2}\sin(\phi)}\frac{1}{n}\sum_{t=1}^{n}\cos(2\phi t)W_{1,n}(t/n)\\ &\quad+\frac{\sigma}{2\sqrt{2}\sin(\phi)}\frac{1}{n}\sum_{t=1}^{n}\sin(2\phi t)W_{2,n}(t/n)\\ &=\frac{\sigma}{2\sqrt{2}\sin(\phi)}\int_0^1 W_{1,n}(x)dx+O_p(1/\sqrt{n}).\end{aligned}$$

The last step follows similarly to the proof of (A.4). Similarly,

$$\frac{1}{n\sqrt{n}}\sum_{t=1}^{n} y_t\sin(\phi t) = \frac{\sigma}{2\sqrt{2}\sin(\phi)}\int_0^1 W_{2,n}(x)dx+O_p(1/\sqrt{n}). \quad \textbf{(A.12)}$$

Part (A.2) of Lemma A.1 follows now from the continuous mapping theorem. ■

Proof of (A.3). This follows similarly to the proof of (A.2). ■

Proof of Theorem 2. It follows straightforwardly from Lemma 2 that under Assumption 2,

$$\begin{aligned}y_t/\sqrt{n} &= \sum_{j=1}^{k}\frac{\sigma_j}{\sqrt{2}\sin(\phi_j)}(\cos(\phi_j t)W_{1,j,n}(t/n)+\sin(\phi_j t)W_{2,j,n}(t/n))\\ &\quad+O_p(1/\sqrt{n}).\end{aligned}$$

Hence (A.4) still holds, and (A.5) becomes

$$\frac{1}{n^2}\sum_{t=1}^{n}y_t^2 \Rightarrow \frac{1}{4}\sum_{j=1}^{k}\frac{\sigma_j^2}{\sin^2(\phi_j)}\left(\int_0^1 W_{1,m}(x)^2dx+\int_0^1 W_{2,m}(x)^2dx\right),$$

where $\sigma_j = |\eta_j(\exp(i\phi_j))|$ and $(W_{1,j,n}, W_{2,j,n})' \Rightarrow W_j = (W_{1,j}, W_{2,j})'$ jointly, with $W_1,\dots,W_k$ independent bivariate standard Wiener processes. Note that without the assumption that the $\varepsilon_{j,t}$'s are contemporaneously independent the W_j's would be dependent, but that is the only difference. ■

Proof of Theorem 3. Denote

$$a_m = \left(\int_0^1 W_{1,m}(x)dx\right)^2 + \left(\int_0^1 W_{2,m}(x)dx\right)^2,$$

$$b_m = \int_0^1 W_{1,m}(x)^2 dx + \int_0^1 W_{2,m}(x)^2 dx. \quad \textbf{(A.13)}$$

Then

$$\psi_k(\phi_j) = \frac{\omega_j^2 a_j}{\sum_{m=1}^{k} \omega_m^2 b_m}. \quad \textbf{(A.14)}$$

Hence

$$\sum_{j=1}^{k} \psi_k(\phi_j) \frac{b_j}{a_j} = 1, \quad \textbf{(A.15)}$$

and consequently

$$\min_{m=1,\dots,k} \psi_k(\phi_m) \sum_{m=1}^{k} \frac{b_m}{a_m} \le 1 \le \max_{m=1,\dots,k} \psi_k(\phi_m) \sum_{m=1}^{k} \frac{b_m}{a_m}.$$

■