1. The answer is $\hat{V}^* = C^{-1}\hat{V}C^{-1'}$. Note that since $C$ is $k \times k$ and full rank,

$$
\hat{\beta}^* = (X'^*X'^*)^{-1}(X'^*Y)
$$
$$
= (C'X'XC)^{-1}(C'X'Y)
$$
$$
= C^{-1}(X'X)^{-1}(C')^{-1}C'X'Y
$$
$$
= C^{-1}(X'X)^{-1}X'Y
$$
$$
= C^{-1}\hat{\beta}
$$

Note also that

$$
\hat{e}^* = Y - X^*\hat{\beta}^*
$$
$$
= Y - XCC^{-1}\hat{\beta}
$$
$$
= Y - X\hat{\beta}
$$
$$
= \hat{e}
$$

which implies $\hat{e}^*_i = \hat{e}_i$. Then

$$
\hat{\Omega}^* = \frac{1}{n}\sum_{i=1}^{n} x'^*_i x'^*_i \hat{e}^2_i
$$
$$
= C'^{-1}C\frac{1}{n}\sum_{i=1}^{n} x_i x'_i \hat{e}^2_i C
$$
$$
= C'\hat{\Omega}C,
$$

$$
\hat{Q}^* = \frac{1}{n}X'^*X'^*
$$
$$
= C'^{-1}X'XC
$$
$$
= C'\hat{Q}C,
$$

and

$$
\hat{Q}^{*-1} = (C'\hat{Q}C)^{-1} = C^{-1}\hat{Q}^{-1}C^{-1'}.
$$
Thus
\[
\hat{V}^* = \hat{Q}^{-1} \hat{\Omega} \hat{Q}^{-1} = C^{-1} \hat{Q}^{-1} C^{-1} \hat{\Omega} CC^{-1} \hat{Q}^{-1} C^{-1} = C^{-1} \hat{V} C^{-1}
\]

2. Since \( E(y_i \mid x_i) = x_i \beta_1 + x_i^2 \beta_2 \), then \( E(y_i \mid x_i = 40) = 40 \beta_1 + 40 \beta_2 \). The hypothesis is thus
\[
H_0: 40 \beta_1 + 40 \beta_2 = 20
\]
which is a linear restriction. If desired, this can be rewritten as
\[
H_0: 2 \beta_1 + 80 \beta_2 = 1
\]

Let \((\hat{\beta}_1, \hat{\beta}_2)\) be the OLS estimates of the coefficients, and let \(\hat{V}\) denote the estimated asymptotic covariance matrix. The Wald statistic for this hypothesis is
\[
W_n = n \left( 2 \hat{\beta}_1 + 80 \hat{\beta}_2 - 1 \right)^2 / R' \hat{V} R
\]
where
\[
R = \begin{pmatrix} 2 \\ 80 \end{pmatrix}
\]
It has an asymptotic \(\chi^2_1\) distribution under \(H_0\). A 5% size test is to reject \(H_0\) if \(W_n\) exceeds the 5% \(\chi^2_1\) critical value of 3.84. Otherwise, \(H_0\) is not rejected.

Alternatively, the 10% or 1% level could be used, or a t-statistic used instead of the Wald statistic. Furthermore, since the model is a regression, the FGLS estimator could be used instead of the OLS estimator.
3. We calculate that

\[
\tilde{\beta} = \left( \sum_{i=1}^{n} w_i x_i' x_i \right)^{-1} \left( \sum_{i=1}^{n} w_i x_i y_i \right)
\]

\[
= \beta + \left( \sum_{i=1}^{n} w_i x_i x_i' \right)^{-1} \left( \sum_{i=1}^{n} w_i x_i e_i \right)
\]

\[
= \beta + \left( \frac{1}{n} \sum_{i=1}^{n} w_i x_i x_i' \right)^{-1} \left( \frac{1}{n} \sum_{i=1}^{n} w_i x_i e_i \right)
\]

\[
\rightarrow_p \beta + \left( E(w_i x_i x_i') \right)^{-1} E(w_i x_i e_i)
\]

by the WLLN. This implicitly assumes that the \( k \times k \) matrix \( E(w_i x_i x_i') \) is invertible. The probability limit in general is not \( \beta \), thus \( \tilde{\beta} \) is inconsistent for \( \beta \).

The question asks to find a assumption under which \( \tilde{\beta} \) is consistent for \( \beta \). A sufficient condition is \( E(w_i x_i e_i) = 0 \), we need to find a reasonable assumption which implies this. One assumption is that the regression model \( E(e_i | x_i) = 0 \), for then \( E(w_i x_i e_i) = E(w_i x_i E(e_i | x_i)) = 0 \). Another assumption is that \( w(x_i) = w \) is a constant, but that is not a very interesting assumption given the context of the question.

4. We know that \( \tilde{\beta} - \beta = (X'X)^{-1} (X'e) \) and \( \tilde{\beta} - \beta = (X'D^{-1}X)^{-1} (X'D^{-1}e) \). Thus

\[
E\left( (\tilde{\beta} - \beta) (\tilde{\beta} - \beta)' \mid X \right) = E\left( (X'X)^{-1} X' e e' X (X'D^{-1}X)^{-1} \mid X \right)
\]

\[
= (X'X)^{-1} X' E(e e' \mid X) D^{-1} X (X'D^{-1}X)^{-1}
\]

\[
= (X'X)^{-1} X' D D^{-1} X (X'D^{-1}X)^{-1}
\]

\[
= (X'X)^{-1} X' X (X'D^{-1}X)^{-1}
\]

\[
= (X'D^{-1}X)^{-1}.
\]

Furthermore, we know that

\[
E\left( (\tilde{\beta} - \beta) (\tilde{\beta} - \beta)' \mid X \right) = (X'X)^{-1} X' D X (X'X)^{-1}
\]

and

\[
E\left( (\tilde{\beta} - \beta) (\tilde{\beta} - \beta)' \mid X \right) = (X'D^{-1}X)^{-1}
\]
Thus

\[
E \left( (\hat{\beta} - \tilde{\beta}) (\tilde{\beta} - \bar{\beta})' | X \right) = E \left( ((\hat{\beta} - \beta) - (\tilde{\beta} - \beta)) \left( (\hat{\beta} - \beta) - (\tilde{\beta} - \beta) \right)' | X \right)
\]

\[
= E \left( (\hat{\beta} - \beta) (\tilde{\beta} - \beta)' | X \right) \\
+ E \left( (\hat{\beta} - \beta) (\tilde{\beta} - \beta)' | X \right)
\]

\[
- E \left( (\hat{\beta} - \beta) (\tilde{\beta} - \beta)' | X \right) \\
- E \left( (\hat{\beta} - \beta) (\tilde{\beta} - \beta)' | X \right)
\]

\[
= (X'X)^{-1} X'DX (X'X)^{-1} \\
+ (X'D^{-1}X)^{-1} - (X'D^{-1}X)^{-1} - (X'D^{-1}X)^{-1}
\]

\[
= (X'X)^{-1} X'DX (X'X)^{-1} - (X'D^{-1}X)^{-1}
\]

\[
= \text{Var}(\hat{\beta}) - \text{Var}(\tilde{\beta})
\]