

Econometrics 710
Midterm Exam, Spring 2005
Sample Answers

1. The answer is $\hat{V}^* = C^{-1}\hat{V}C^{-1'}$. Note that since C is $k \times k$ and full rank,

$$\begin{aligned}\hat{\beta}^* &= (X^{*'}X^*)^{-1}(X^{*'}Y) \\ &= (C'X'XC)^{-1}(C'X'Y) \\ &= C^{-1}(X'X)^{-1}(C')^{-1}C'X'Y \\ &= C^{-1}(X'X)^{-1}X'Y \\ &= C^{-1}\hat{\beta}\end{aligned}$$

Note also that

$$\begin{aligned}\hat{e}^* &= Y - X^*\hat{\beta}^* \\ &= Y - XCC^{-1}\hat{\beta} \\ &= Y - X\hat{\beta} \\ &= \hat{e}\end{aligned}$$

which implies $\hat{e}_i^* = \hat{e}_i$. Then

$$\begin{aligned}\hat{\Omega}^* &= \frac{1}{n} \sum_{i=1}^n x_i^* x_i^{*'} \hat{e}_i^{*2} \\ &= C' \frac{1}{n} \sum_{i=1}^n x_i x_i' \hat{e}_i^2 C \\ &= C' \hat{\Omega} C,\end{aligned}$$

$$\begin{aligned}\hat{Q}^* &= \frac{1}{n} X^{*'} X^* \\ &= C' \frac{1}{n} X' X C \\ &= C' \hat{Q} C,\end{aligned}$$

and

$$\hat{Q}^{*-1} = (C' \hat{Q} C)^{-1} = C^{-1} \hat{Q}^{-1} C'^{-1}.$$

Thus

$$\begin{aligned}
 \hat{V}^* &= \hat{Q}^{*-1} \hat{\Omega}^* \hat{Q}^{*-1} \\
 &= C^{-1} \hat{Q}^{-1} C'^{-1} C' \hat{\Omega} C C^{-1} \hat{Q}^{-1} C'^{-1} \\
 &= C^{-1} \hat{Q}^{-1} \hat{\Omega} \hat{Q}^{-1} C'^{-1} \\
 &= C^{-1} \hat{V} C'^{-1}
 \end{aligned}$$

2. Since $E(y_i | x_i) = x_i \beta_1 + x_i^2 \beta_2$, then $E(y_i | x_i = 40) = 40\beta_1 + 40^2\beta_2$. The hypothesis is thus

$$H_0 : 40\beta_1 + 40^2\beta_2 = 20$$

which is a linear restriction. If desired, this can be rewritten as

$$H_0 : 2\beta_1 + 80\beta_2 = 1$$

Let $(\hat{\beta}_1, \hat{\beta}_2)$ be the OLS estimates of the coefficients, and let \hat{V} denote the estimated asymptotic covariance matrix. The Wald statistic for this hypothesis is

$$W_n = \frac{n \left(2\hat{\beta}_1 + 80\hat{\beta}_2 - 1 \right)^2}{R' \hat{V} R}$$

where

$$R = \begin{pmatrix} 2 \\ 80 \end{pmatrix}$$

It has an asymptotic χ_1^2 distribution under H_0 . A 5% size test is to reject H_0 if W_n exceeds the 5% χ_1^2 critical value of 3.84. Otherwise, H_0 is not rejected.

Alternatively, the 10% or 1% level could be used, or a t-statistic used instead of the Wald statistic. Furthermore, since the model is a regression the FGLS estimator could be used instead of the OLS estimator.

3. We calculate that

$$\begin{aligned}
\tilde{\beta} &= \left(\sum_{i=1}^n w_i x_i x_i' \right)^{-1} \left(\sum_{i=1}^n w_i x_i y_i \right) \\
&= \beta + \left(\sum_{i=1}^n w_i x_i x_i' \right)^{-1} \left(\sum_{i=1}^n w_i x_i e_i \right) \\
&= \beta + \left(\frac{1}{n} \sum_{i=1}^n w_i x_i x_i' \right)^{-1} \left(\frac{1}{n} \sum_{i=1}^n w_i x_i e_i \right) \\
&\rightarrow_p \beta + \left(E(w_i x_i x_i') \right)^{-1} E(w_i x_i e_i)
\end{aligned}$$

by the WLLN. This implicitly assumes that the $k \times k$ matrix $E(w_i x_i x_i')$ is invertible. The probability limit in general is not β , thus $\tilde{\beta}$ is inconsistent for β .

The question asks to find a assumption under which $\tilde{\beta}$ is consistent for β . A sufficient condition is $E(w_i x_i e_i) = 0$, we need to find a reasonable assumption which implies this. One assumption is that the regression model $E(e_i | x_i) = 0$, for then $E(w_i x_i e_i) = E(w_i x_i E(e_i | x_i)) = 0$. Another assumption is that $w(x_i) = w$ is a constant, but that is not a very interesting assumption given the context of the question.

4. We know that $\hat{\beta} - \beta = (X'X)^{-1} (X'e)$ and $\tilde{\beta} - \beta = (X'D^{-1}X)^{-1} (X'D^{-1}e)$. Thus

$$\begin{aligned}
E\left(\left(\hat{\beta} - \beta\right)\left(\tilde{\beta} - \beta\right)' \mid X\right) &= E\left(\left(X'X\right)^{-1} X' e e' D^{-1} X \left(X'D^{-1}X\right)^{-1} \mid X\right) \\
&= \left(X'X\right)^{-1} X' E\left(e e' \mid X\right) D^{-1} X \left(X'D^{-1}X\right)^{-1} \\
&= \left(X'X\right)^{-1} X' D D^{-1} X \left(X'D^{-1}X\right)^{-1} \\
&= \left(X'X\right)^{-1} X' X \left(X'D^{-1}X\right)^{-1} \\
&= \left(X'D^{-1}X\right)^{-1}.
\end{aligned}$$

Furthermore, we know that

$$E\left(\left(\hat{\beta} - \beta\right)\left(\hat{\beta} - \beta\right)' \mid X\right) = \left(X'X\right)^{-1} X' D X \left(X'X\right)^{-1}$$

and

$$E\left(\left(\tilde{\beta} - \beta\right)\left(\tilde{\beta} - \beta\right)' \mid X\right) = \left(X'D^{-1}X\right)^{-1}$$

Thus

$$\begin{aligned} E\left(\left(\hat{\beta} - \tilde{\beta}\right)\left(\hat{\beta} - \tilde{\beta}\right)' \mid X\right) &= E\left(\left(\left(\hat{\beta} - \beta\right) - \left(\tilde{\beta} - \beta\right)\right)\left(\left(\hat{\beta} - \beta\right) - \left(\tilde{\beta} - \beta\right)\right)' \mid X\right) \\ &= E\left(\left(\hat{\beta} - \beta\right)\left(\hat{\beta} - \beta\right)' \mid X\right) \\ &\quad + E\left(\left(\tilde{\beta} - \beta\right)\left(\tilde{\beta} - \beta\right)' \mid X\right) \\ &\quad - E\left(\left(\hat{\beta} - \beta\right)\left(\tilde{\beta} - \beta\right)' \mid X\right) \\ &\quad - E\left(\left(\tilde{\beta} - \beta\right)\left(\hat{\beta} - \beta\right)' \mid X\right) \\ &= \left(X'X\right)^{-1} X'DX \left(X'X\right)^{-1} \\ &\quad + \left(X'D^{-1}X\right)^{-1} - \left(X'D^{-1}X\right)^{-1} - \left(X'D^{-1}X\right)^{-1} \\ &= \left(X'X\right)^{-1} X'DX \left(X'X\right)^{-1} - \left(X'D^{-1}X\right)^{-1} \\ &= \text{Var}(\hat{\beta}) - \text{Var}(\tilde{\beta}) \end{aligned}$$