Econometrics 710
Midterm Exam, Spring 2003
Sample Answers

1. Substituting in $Y = X\beta + e$ and $\lambda = cn$ we see

$$\begin{aligned}\hat{\beta} &= \left(X'X + cnI_k\right)^{-1}\left(X'Y\right) \\ &= \left(\frac{1}{n}X'X + cI_k\right)^{-1}\left(\frac{1}{n}X'X\beta + \frac{1}{n}X'e\right) \\ &\to_p \left(Q + cI_k\right)^{-1} Q\beta \\ &= \beta - c\left(Q + cI_k\right)^{-1}\beta \neq \beta\end{aligned}$$

2. The moment conditions are

$$g\left(w_i, \beta, \Omega\right) = \left(\begin{array}{c} x_i\left(y_i - x_i'\beta\right) \\ \left(x_i x_i'\left(y_i - x_i'\beta\right)^2\right) - \Omega \end{array}\right).$$

The sample average is

$$\overline{g}_n\left(\beta, \Omega\right) = \frac{1}{n}\sum_{i=1}^{n} g\left(w_i, \beta, \Omega\right)$$

The MME $(\hat{\beta}, \hat{\Omega})$ sets $\overline{g}_n(\hat{\beta}, \hat{\Omega}) = 0$. Thus

$$\frac{1}{n}\sum_{i=1}^{n} x_i\left(y_i - x_i'\hat{\beta}\right) = 0 \quad (1)$$

$$\frac{1}{n}\sum_{i=1}^{n}\left(\left(x_i x_i'\left(y_i - x_i'\hat{\beta}\right)^2\right) - \hat{\Omega}\right) = 0 \quad (2)$$

From (1) we see $\hat{\beta} = \left(X'X\right)^{-1}\left(X'Y\right)$. From (2) we see

$$\hat{\Omega} = \frac{1}{n}\sum_{i=1}^{n}\left(x_i x_i'\left(y_i - x_i'\hat{\beta}\right)^2\right) = \frac{1}{n}\sum_{i=1}^{n} x_i x_i' \hat{e}_i^2$$

where $\hat{e}_i = y_i - x_i'\hat{\beta}$. In this model, these estimators are efficient in the sense of asymptotic semi-parametric efficiency – no other consistent estimator has a lower asymptotic variance. We know this is true because in just-identified moment models, where the is no information other than the stated moments, the MME is semi-parametrically efficient.

3. By the delta method, the asymptotic distribution of $\hat{\theta}$ is

$$\sqrt{n}\left(\hat{\theta} - \theta\right) \to^d N\left(0, V_\theta\right)$$

where

$$V_\theta = \left(\frac{d}{d\beta}\left(\beta^2\right)\right)^2 V = \left(2\beta\right)^2 V = 4\beta^2 V$$

Thus a valid asymptotic standard error for $\hat{\theta}$ is $s(\hat{\theta}) = 2\left|\hat{\beta}\right|\left(\hat{V}/n\right)^{1/2}$. Let $Z_{\alpha/2}$ denote the $\alpha/2$ quantile of the $N(0,1)$ distribution. An asymptotic $(1-\alpha)\%$ confidence interval for θ is

$$\hat{\theta} \pm Z_{\alpha/2} s(\hat{\theta}) = \hat{\beta}^2 \pm Z_{\alpha/2} 2\left|\hat{\beta}\right|\left(\frac{\hat{V}}{n}\right)^{1/2}.$$

If $\beta = 0$, the asymptotic distribution for $\hat{\theta}$ is degenerate. Since $V_\theta = 0$, it follows that $\sqrt{n}\left(\hat{\theta} - \theta\right) \to^d N(0,0) = 0$ almost surely. In consequence, the asymptotic confidence interval fails to have asymptotic coverage $(1-\alpha)\%$.

Instead, when $\beta = 0$, you can observe that $\sqrt{n}\hat{\beta} \to^d N(0,V)$ implies

$$\frac{n\hat{\theta}}{V} = \frac{\left(\sqrt{n}\hat{\beta}\right)^2}{V} \to^d \frac{N(0,V)^2}{V} = \chi_1^2.$$

The discontinuity in the asymptotic distribution at $\beta = 0$ is the source of the problem.

In this simple context, a better confidence interval for θ can be found by squaring the endpoints of the confidence interval for β :

$$\left(\hat{\beta} - Z_{\alpha/2}\left(\frac{\hat{V}}{n}\right)^{1/2}\right)^2, \qquad \left(\hat{\beta} + Z_{\alpha/2}\left(\frac{\hat{V}}{n}\right)^{1/2}\right)^2$$

4. We know that $n^{1/2}\left(\hat{\beta} - \beta\right) \to_d N(0,V)$ where $V = Q^{-1}\Omega Q^{-1}$, $Q = E\left(x_i x_i'\right)$, $\Omega = E\left(x_i x_i' e_i^2\right)$, so $\hat{\beta}$ is consistent. We also know that in the context of the regression model, $\hat{\beta}$ is generally inefficient. It is efficient (in the semi-parametric sense) under homoskedasticity: $E\left(e_i^2 \mid x_i\right) = \sigma^2$.

   Now consider $\tilde{\beta}$. We see that if $E\left|\frac{e_i}{x_i}\right| < \infty$, then by the law of iterated expectations,

$$E\left(\frac{e_i}{x_i}\right) = E\left(E\left(\frac{e_i}{x_i} \mid x_i\right)\right) = E\left(\frac{1}{x_i}E\left(e_i \mid x_i\right)\right) = 0.$$

   Thus by the WLLN,

$$\tilde{\beta} = \frac{1}{n}\sum_{i=1}^n \frac{y_i}{x_i} = \frac{1}{n}\sum_{i=1}^n \frac{x_i\beta + e_i}{x_i} = \beta + \frac{1}{n}\sum_{i=1}^n \frac{e_i}{x_i} \to_p \beta + E\left(\frac{e_i}{x_i}\right) = \beta$$

   Hence $\tilde{\beta}$ is consistent.

   Now observe that we can obtain the estimator $\tilde{\beta}$ by dividing $y_i = x_i\beta + e_i$ by x_i so that

$$\frac{y_i}{x_i} = \beta + \frac{e_i}{x_i}$$

   and then estimating β by OLS (which is same as taking the average of y_i/x_i. Therefore $\tilde{\beta}$ is WLS using weights $w_i = x_i^{-1}$. WLS equals GLS when $w_i = \sigma_i^{-1}$. Since in the regression model GLS is efficient, $\tilde{\beta}$ is efficient when $\sigma_i^2 = x_i^2$.