Econometrics 710
Midterm Exam
March 21, 2002
Sample Answers

1. The non-parametric bootstrap estimates the distribution of $w_i = (y_i, x_i)$ by the empirical distribution function $F_n(w)$. Since the data w_i is iid, the EDF is consistent and the non-parametric bootstrap is valid. Heteroskedasticity, e.g. random $E\left(e_i^2 \mid x_i\right)$ is irrelevant. One way to see this is that the EDF is drawing the pairs (y_i, x_i) jointly, so the full conditional distribution of y_i given x_i matches the data. The only important cavaet is to observe that heteroskedasticity affects the validity of standard errors, and this is important for the percentile-t bootstrap. The presence of heteroskedasticity means that the standard errors should be calculated using the White formula. Doing so, the t-ratio is asymptotically standard normal (and hence pivotal) which is necessary to achieve an asymptotic refinement.

2. We have

$$\begin{aligned}
\hat{\beta} &= \left(\hat{Z}'\hat{Z}\right)^{-1}\hat{Z}'Y \\
&= \left(\hat{Z}'\hat{Z}\right)^{-1}\hat{Z}'(Z\beta + e) \\
&= \left(\hat{Z}'\hat{Z}\right)^{-1}\hat{Z}'Z\beta + \left(\hat{Z}'\hat{Z}\right)^{-1}\hat{Z}'e
\end{aligned}$$

Since $\beta = 0$ this simplifies to

$$\sqrt{n}\hat{\beta} = \sqrt{n}\left(\hat{Z}'\hat{Z}\right)^{-1}\hat{Z}'e = \left(\hat{\Gamma}'\frac{1}{n}X'X\hat{\Gamma}\right)^{-1}\hat{\Gamma}'\frac{1}{\sqrt{n}}X'e.$$

By the WLLN,

$$\frac{1}{n}X'X \to_p Q = E\left(x_i x_i'\right)$$

and the CLT

$$\frac{1}{\sqrt{n}}X'e \to_d N(0, \Omega)$$

where

$$\Omega = E\left(x_i x_i' e_i^2\right).$$

Combined with $\hat{\Gamma} \to_p \Gamma$ and the CMT, we find

$$\begin{aligned}
\sqrt{n}\hat{\beta} &= \left(\hat{\Gamma}'\frac{1}{n}X'X\hat{\Gamma}\right)^{-1}\hat{\Gamma}'\frac{1}{\sqrt{n}}X'e \\
&\to_d \left(\Gamma'W\Gamma\right)^{-1}\Gamma'N(0, \Omega) \\
&= N\left(0, \left(\Gamma'W\Gamma\right)^{-1}\Gamma'\Omega\Gamma\left(\Gamma'W\Gamma\right)^{-1}\right).
\end{aligned}$$

This is a complete answer.

Note: The following are examples of *inappropriate* statements, because the right-hand sides are functions of the sample

$$\hat{\Gamma}' X' X \hat{\Gamma} \rightarrow_p \Gamma' X' X \Gamma$$
$$\hat{\Gamma}' n^{-1} X' X \hat{\Gamma} \rightarrow_p \hat{\Gamma}' Q \hat{\Gamma}$$
$$\hat{\Gamma}' n^{-1/2} X' e \rightarrow_d \hat{\Gamma}' N(0, \Omega)$$

The following is also incorrect

$$E\left(\hat{\beta} \mid X\right) = \left(\hat{\Gamma}' X' X \hat{\Gamma}\right)^{-1} \hat{\Gamma}' X' E\left(e \mid X\right) = 0.$$

It is incorrect because $\hat{\Gamma}$ is not a function of X alone. (The question does not specify how $\hat{\Gamma}$ is constructed but in any event it would be highly unlikely it would be only a function of X.)

Finally, it is *very* incorrect to write Γ^{-1}, $\hat{\Gamma}^{-1}$, X^{-1} or $(X\Gamma)^{-1}$, as none of these matrices are square, so clearly do not have inverses!

3. By definition,

$$\hat{V}_n = \left(X'X\right)^{-1} X' \hat{D} X \left(X'X\right)^{-1}$$

where $D = diag\{\hat{e}_i^2\}$. Thus

$$E\left(\hat{V}_n \mid X\right) = \left(X'X\right)^{-1} X' E\left(\hat{D} \mid X\right) X \left(X'X\right)^{-1}$$

where

$$E\left(\hat{D} \mid X\right) = diag\{E\left(\hat{e}_i^2 \mid X\right)\}.$$

Thus the main problem is to find $E\left(\hat{e}_i^2 \mid X\right)$. There are two different ways to do this. First, you can observe that $\hat{e}_i^2$ are the diagonal elements of the $n \times n$ matrix

$$\hat{e}\hat{e}' = Mee'M$$

where

$$M = I_n - X\left(X'X\right)^{-1} X'$$

Under the homoskedasticity assumption,

$$E\left(\hat{e}\hat{e}' \mid X\right) = ME\left(ee' \mid X\right) M = MI_n\sigma^2 M = M\sigma^2.$$

Hence

$$E\left(\hat{e}_i^2 \mid X\right) = [M]_{ii}\,\sigma^2 \equiv \lambda_i \sigma^2,$$

where $[M]_{ii}$ is the $i'th$ diagonal element of M. This is

$$\lambda_i = [M]_{ii} = 1 - x_i' \left(X'X\right)^{-1} x_i.$$

The second way to calculate this is to observe that

$$\hat{e}_i = y_i - x_i'\hat{\beta} = e_i - x_i'\left(\hat{\beta} - \beta\right)$$

so

$$\hat{e}_i^2 = e_i^2 - 2e_i x_i' \left(\hat{\beta} - \beta\right) + x_i' \left(\hat{\beta} - \beta\right) \left(\hat{\beta} - \beta\right)' x_i.$$

Hence

$$E\left(\hat{e}_i^2 \mid X\right) = E\left(e_i^2 \mid X\right) - 2E\left(e_i x_i' \left(\hat{\beta} - \beta\right) \mid X\right) + E\left(x_i' \left(\hat{\beta} - \beta\right) \left(\hat{\beta} - \beta\right)' x_i \mid X\right).$$

Note that

$$E\left(e_i^2 \mid X\right) = \sigma^2,$$

and

$$\begin{aligned} E\left(e_i x_i' \left(\hat{\beta} - \beta\right) \mid X\right) &= E\left(e_i \left(\hat{\beta} - \beta\right)' x_i \mid X\right) \\ &= E\left(e_i e' X \left(X'X\right)^{-1} x_i \mid X\right) \\ &= \sigma^2 x_i' \left(X'X\right)^{-1} x_i, \end{aligned}$$

and

$$\begin{aligned} E\left(x_i' \left(\hat{\beta} - \beta\right) \left(\hat{\beta} - \beta\right)' x_i \mid X\right) &= x_i' E\left(\left(\hat{\beta} - \beta\right) \left(\hat{\beta} - \beta\right)' \mid X\right) x_i \\ &= x_i' \left(X'X\right)^{-1} \sigma^2 x_i \end{aligned}$$

Together

$$\begin{aligned} E\left(\hat{e}_i^2 \mid X\right) &= \sigma^2 - 2\sigma^2 x_i' \left(X'X\right)^{-1} x_i + \sigma^2 x_i' \left(X'X\right)^{-1} x_i \\ &= \sigma^2 \left(1 - x_i' \left(X'X\right)^{-1} x_i\right) \\ &= \sigma^2 \lambda_i \end{aligned}$$

as derived above.

Okay, now we know $E\left(\hat{e}_i^2 \mid X\right)$. Let $\Lambda = diag\{\lambda_i\}$. Then $E\left(\hat{D} \mid X\right) = \Lambda\sigma^2$, and we find

$$\begin{aligned} E\left(\hat{V}_n \mid X\right) &= \left(X'X\right)^{-1} X'\Lambda\sigma^2 X \left(X'X\right)^{-1} \\ &= \sigma^2 \left(X'X\right)^{-1} \sum_{i=1}^{n} x_i x_i' \lambda_i \left(X'X\right)^{-1} \\ &= \sigma^2 \left(X'X\right)^{-1} \sum_{i=1}^{n} x_i x_i' \left(1 - x_i' \left(X'X\right)^{-1} x_i\right) \left(X'X\right)^{-1}. \end{aligned}$$

It is intereting to observe that $E\left(\hat{V}_n \mid X\right)$ does not equal $V_n^0 = \sigma^2 \left(X'X\right)^{-1}$, the actual conditional covariance matrix under the stated conditions. Thus $\hat{V}_n$ is biased. What this calculation shows is that even under the strong assumption of homoskedasticity, the White covariance matrix estimator is biased.

Note: A very incorrect answer is to write

$$E\left(e_i^2 \mid x_i\right) = \sigma^2 \text{ implies that } \hat{V}_n = s^2\left(X'X\right)^{-1}$$

This is incorrect. An estimator, such as $\hat{V}_n$, is not affected by the distribution of the data. An estimator is a particular function of the data. The distribution of the data affects the distribution of the estimator, but not its definition. So $E\left(e_i^2 \mid x_i\right) = \sigma^2$ affects the distribution of $\hat{V}_n$ (including its expectation) but not the way it is constructed.