

Econometrics 710
Answers to Midterm Exam
March 6, 2001

1. We know that both $\hat{\beta}$ and $\tilde{\beta}$ are unbiased for β , $\hat{\beta} - \beta = (X'X)^{-1}(X'e)$ and $\tilde{\beta} - \beta = (X'D^{-1}X)^{-1}(X'D^{-1}e)$

(a) Note that

$$\begin{aligned}\hat{\beta} - \tilde{\beta} &= (\hat{\beta} - \beta) - (\tilde{\beta} - \beta) \\ &= (X'X)^{-1}(X'e) - (X'D^{-1}X)^{-1}(X'D^{-1}e) \\ &= \left[(X'X)^{-1}X' - (X'D^{-1}X)^{-1}X'D^{-1} \right] e\end{aligned}$$

and $E(\hat{\beta} - \tilde{\beta} | X) = 0$. Thus

$$\begin{aligned}& Cov(\hat{\beta} - \tilde{\beta}, \tilde{\beta} | X) \\ &= E\left[(\hat{\beta} - \tilde{\beta})(\tilde{\beta} - \beta)' | X \right] \\ &= E\left\{ \left[(X'X)^{-1}X' - (X'D^{-1}X)^{-1}X'D^{-1} \right] e (e'D^{-1}X)(X'D^{-1}X)^{-1} | X \right\} \\ &= \left[(X'X)^{-1}X' - (X'D^{-1}X)^{-1}X'D^{-1} \right] E(ee' | X) D^{-1}X (X'D^{-1}X)^{-1} \\ &= \left[(X'X)^{-1}X' - (X'D^{-1}X)^{-1}X'D^{-1} \right] DD^{-1}X (X'D^{-1}X)^{-1} \\ &= \left[(X'X)^{-1}X' - (X'D^{-1}X)^{-1}X'D^{-1} \right] X (X'D^{-1}X)^{-1} \\ &= \left[(X'X)^{-1}X'X - (X'D^{-1}X)^{-1}X'D^{-1}X \right] (X'D^{-1}X)^{-1} \\ &= [I_k - I_k] (X'D^{-1}X)^{-1} \\ &= 0\end{aligned}$$

(b) Since

$$\begin{aligned}0 &= Cov(\hat{\beta} - \tilde{\beta}, \tilde{\beta} | X) \\ &= Cov(\hat{\beta}, \tilde{\beta} | X) - Cov(\tilde{\beta}, \tilde{\beta} | X) \\ &= Cov(\hat{\beta}, \tilde{\beta} | X) - Var(\tilde{\beta} | X),\end{aligned}$$

it follows from our result in part (a) that $Cov(\hat{\beta}, \tilde{\beta} | X) = Var(\tilde{\beta} | X)$.

(c) Using part (b),

$$\begin{aligned}
\text{Var}(\hat{\beta} - \tilde{\beta} | X) &= \text{Var}(\hat{\beta} | X) + \text{Var}(\tilde{\beta} | X) + \text{Cov}(\hat{\beta}, \tilde{\beta} | X) + \text{Cov}(\hat{\beta}, \tilde{\beta} | X)' \\
&= \text{Var}(\hat{\beta} | X) + \text{Var}(\tilde{\beta} | X) - \text{Var}(\tilde{\beta} | X) - \text{Var}(\tilde{\beta} | X) \\
&= \text{Var}(\hat{\beta} | X) - \text{Var}(\tilde{\beta} | X)
\end{aligned}$$

(d) Using part (b) and the known variances of $\hat{\beta}$ and $\tilde{\beta}$,

$$\begin{aligned}
V_n &= \text{Var}(\hat{\beta} - \tilde{\beta} | X) \\
&= \text{Var}(\hat{\beta} | X) - \text{Var}(\tilde{\beta} | X) \\
&= (X'X)^{-1}(X'DX)(X'X)^{-1} - (X'D^{-1}X)^{-1}.
\end{aligned}$$

Note 1: Perhaps an easier method of proof would be to first prove part (b), and use this to prove part (a). The first argument works as follows:

$$\begin{aligned}
\text{Cov}(\hat{\beta}, \tilde{\beta} | X) &= E\left[(\hat{\beta} - \beta)(\tilde{\beta} - \beta)' | X\right] \\
&= E\left[(X'X)^{-1}(X'e)(e'D^{-1}X)(X'D^{-1}X)^{-1} | X\right] \\
&= (X'X)^{-1}X'E[ee' | X]D^{-1}X(X'D^{-1}X)^{-1} \\
&= (X'X)^{-1}X'DD^{-1}X(X'D^{-1}X)^{-1} \\
&= (X'X)^{-1}X'X(X'D^{-1}X)^{-1} \\
&= (X'D^{-1}X)^{-1} \\
&= \text{Var}(\tilde{\beta} | X).
\end{aligned}$$

Note 2: This result holds quite generally when one estimator is efficient. In general, for any unbiased estimators $\hat{\beta}$ and $\tilde{\beta}$, where $\tilde{\beta}$ is efficient, then $\text{Var}(\hat{\beta} - \tilde{\beta} | X) = \text{Var}(\hat{\beta} | X) - \text{Var}(\tilde{\beta} | X)$.

2. To solve this question, you need to recognize the following. First, you can write the estimator as

$$\tilde{\beta} - \beta = \left(\sum_{i=1}^n x_i x_i' 1(|x_i| \leq c)\right)^{-1} \left(\sum_{i=1}^n x_i e_i 1(|x_i| \leq c)\right).$$

Second, $x_i x_i' 1(|x_i| \leq c)$ is an iid random variable with mean

$$E(x_i x_i' 1(|x_i| \leq c)) \equiv Q_c.$$

Third, $x_i e_i 1(|x_i| \leq c)$ is an iid random variable with mean zero:

$$E(x_i e_i 1(|x_i| \leq c)) = E(x_i 1(|x_i| \leq c) E(e_i | x_i)) = 0$$

(using the law of iterated expectations), and has variance

$$E\left[(x_i e_i 1(|x_i| \leq c))(x_i e_i 1(|x_i| \leq c))'\right] = E\left(x_i x_i' e_i^2 1(|x_i| \leq c)\right) \equiv \Omega_c.$$

Note that $Q_c \neq Q = E(x_i x_i')$ and $\Omega_c \neq \Omega = E(x_i x_i' e_i^2)$.

(a) By the WLLN,

$$\frac{1}{n} \sum_{i=1}^n x_i x_i' 1(|x_i| \leq c) \rightarrow_p E(x_i x_i' 1(|x_i| \leq c)) = Q_c$$

and

$$\frac{1}{n} \sum_{i=1}^n x_i e_i 1(|x_i| \leq c) \rightarrow_p E(x_i e_i 1(|x_i| \leq c)) = 0.$$

Hence, if $Q_c > 0$, then by the CMT,

$$\tilde{\beta} - \beta = \left(\frac{1}{n} \sum_{i=1}^n x_i x_i' 1(|x_i| \leq c)\right)^{-1} \left(\frac{1}{n} \sum_{i=1}^n x_i e_i 1(|x_i| \leq c)\right) \rightarrow_p Q_c^{-1} \cdot 0 = 0.$$

(b) By the CLT,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n x_i e_i 1(|x_i| \leq c) \rightarrow_d N(0, \Omega_c).$$

Hence

$$\begin{aligned} \sqrt{n}(\tilde{\beta} - \beta) &= \left(\frac{1}{n} \sum_{i=1}^n x_i x_i' 1(|x_i| \leq c)\right)^{-1} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n x_i e_i 1(|x_i| \leq c)\right) \\ &\rightarrow {}_d Q_c^{-1} N(0, \Omega_c) = N\left(0, Q_c^{-1} \Omega_c Q_c^{-1}\right). \end{aligned}$$

(c) Bonus Question: The condition $E(x_i e_i) = 0$ is *not* sufficient for

$$E(x_i e_i 1(|x_i| \leq c)) = 0.$$

Thus $\tilde{\beta}$ will not necessarily be consistent for β . In fact,

$$\tilde{\beta} \rightarrow_p \beta + Q_c^{-1} E(x_i e_i 1(|x_i| \leq c)).$$

3.

- (a) These results follow from the fact that the sample mean is an unbiased estimator of the population mean, regardless of the distribution the data. Observe that

$$E(\bar{y}) = Ey_i = \mu.$$

Hence

$$\tau_n = E(\bar{y} - \mu) = \mu - \mu = 0.$$

Let y_i^* be a random variable with distribution F_n , and \bar{y}^* the sample mean of a random sample $\{y_1^*, \dots, y_n^*\}$. By linearity,

$$\begin{aligned} E\bar{y}^* &= Ey_i^* \\ &= \sum_{j=1}^n y_j P(y_i^* = y_j) \\ &= \frac{1}{n} \sum_{j=1}^n y_j = \bar{y} \end{aligned}$$

The bootstrap estimate of bias treats $\hat{\mu} = \bar{y}$ as the true value. Hence

$$\tau_n^* = E\bar{y}^* - \hat{\mu} = \bar{y} - \bar{y} = 0.$$

- (b) The bias is $E\hat{\mu}^2 - \mu^2$. This is the variance of $\hat{\mu} = \bar{y}$, which we know is $n^{-1}\sigma^2$. In detail,

$$\begin{aligned} \tau_n &= E\hat{\mu}^2 - \mu^2 \\ &= E(\hat{\mu} - \mu)^2 \\ &= E\left(\frac{1}{n} \sum_{i=1}^n (y_i - \mu)\right)^2 \\ &= \frac{1}{n^2} \sum_{i=1}^n E(y_i - \mu)^2 \\ &= \frac{\sigma^2}{n}. \end{aligned}$$

- (c) Bonus Question: The variance of the EDF is $\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n (y_i - \bar{y})^2$. Thus the bootstrap estimate τ_n^* of τ_n is

$$\tau_n^* = \frac{\hat{\sigma}^2}{n}.$$

In this case, you do not need to do a simulation to calculate τ_n^* .