Econometrics 710
Midterm Exam
Sample Answers
Spring, 2000

1. It is worth first noting that

$$\tilde{\beta} = \frac{\sum_{i=1}^{n} (x_i\beta + e_i)}{\sum_{i=1}^{n} x_i} = \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} x_i}\beta + \frac{\sum_{i=1}^{n} e_i}{\sum_{i=1}^{n} x_i} = \beta + \frac{\sum_{i=1}^{n} e_i}{\sum_{i=1}^{n} x_i}.$$

(a)

$$E\left(\tilde{\beta} - \beta \mid X\right) = E\left(\frac{\sum_{i=1}^{n} e_i}{\sum_{i=1}^{n} x_i} \mid X\right) = \frac{\sum_{i=1}^{n} E\left(e_i \mid x_i\right)}{\sum_{i=1}^{n} x_i} = 0,$$

so $E\left(\tilde{\beta} \mid X\right) = \beta$ and $\tilde{\beta}$ is unbiased for β.

(b) As $E\left(\tilde{\beta} \mid X\right) = \beta$,

$$\begin{aligned} Var\left(\tilde{\beta} \mid X\right) &= E\left(\left(\tilde{\beta} - \beta\right)^2 \mid X\right) \\ &= E\left(\left(\frac{\sum_{i=1}^{n} e_i}{\sum_{i=1}^{n} x_i}\right)^2 \mid X\right) \\ &= \frac{\sum_{i=1}^{n} E\left(e_i^2 \mid X\right)}{\left(\sum_{i=1}^{n} x_i\right)^2} = \frac{\sum_{i=1}^{n} \sigma_i^2}{\left(\sum_{i=1}^{n} x_i\right)^2}, \end{aligned}$$

where $\sigma_i^2 = E\left(e_i^2 \mid x_i\right)$. Note: Under the stated assumptions, σ_i^2 may be *random*, not a constant.

(c) As $n \to \infty$, by the WLLN (since the data are iid)

$$\begin{aligned} \frac{1}{n}\sum_{i=1}^{n} e_i &\to {}_pE(e_i) = 0 \\ \frac{1}{n}\sum_{i=1}^{n} x_i &\to {}_pE(x_i) = \mu, \end{aligned}$$

say, so if $\mu \neq 0$, then

$$\tilde{\beta} - \beta = \frac{\frac{1}{n}\sum_{i=1}^{n} e_i}{\frac{1}{n}\sum_{i=1}^{n} x_i} \to_p \frac{0}{\mu} = 0.$$

This requires the *assumption* that $\mu \neq 0$.

(d) As $n \to \infty$, by the CLT (as e_i is iid)

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} e_i \to_p N\left(0, \sigma^2\right)$$

where $\sigma^2 = E(e_i^2)$. Thus (if again $\mu \neq 0$),

$$\sqrt{n}\left(\tilde{\beta} - \beta\right) = \frac{\frac{1}{\sqrt{n}} \sum_{i=1}^{n} e_i}{\frac{1}{n} \sum_{i=1}^{n} x_i} \to_d \frac{N\left(0, \sigma^2\right)}{\mu} = N\left(0, \frac{\sigma^2}{\mu^2}\right).$$

2. The following asymptotic confidence interval is based on the "delta method". Notationally, it is helpful to let V_{11} denote the first diagonal element in V, and similarly $\hat{V}_{11}$. Since

$$\sqrt{n}\left(\hat{\beta} - \beta\right) \to_d N(0, V),$$

then

$$\sqrt{n}\left(\hat{\beta}_1 - \beta_1\right) \to_d N(0, V_{11}),$$

where V_{11} is the first diagonal element in V. Letting $h(\beta_1) = 1/\beta_1$, then $\frac{\partial}{\partial\beta} h(\beta) = -\beta_1^{-2}$. By the delta method formula,

$$\sqrt{n}\left(1/\hat{\beta}_1 - 1/\beta_1\right) \to_d N\left(0, \frac{V_{11}}{\beta_1^4}\right).$$

Thus a standard error for $1/\hat{\beta}_1$ is $\hat{\beta}_1^{-2} \cdot \hat{V}_{11}^{1/2}$, where $\hat{V}_{11}$ is the first diagonal element in $\hat{V}$. We conclude that a 95% confidence interval for $1/\beta_1$ is

$$\left[\frac{1}{\hat{\beta}_1} - 2\frac{\hat{V}_{11}^{1/2}}{\hat{\beta}_1^2}, \quad \frac{1}{\hat{\beta}_1} + 2\frac{\hat{V}_{11}^{1/2}}{\hat{\beta}_1^2}\right].$$

3.

(a)

$$\begin{aligned} R\tilde{\beta} &= R\hat{\beta} - R\left(X'X\right)^{-1} R' \left[R\left(X'X\right)^{-1} R'\right]^{-1} R\hat{\beta} \\ &= R\hat{\beta} - R\hat{\beta} = 0 \end{aligned}$$

(b) Since $E\left(\hat{\beta} \mid X\right) = \beta$,

$$\begin{aligned} E\left(\tilde{\beta} \mid X\right) &= E\left(\hat{\beta} - \left(X'X\right)^{-1} R' \left[R\left(X'X\right)^{-1} R'\right]^{-1} R\hat{\beta} \mid X\right) \\ &= E\left(\hat{\beta} \mid X\right) - \left(X'X\right)^{-1} R' \left[R\left(X'X\right)^{-1} R'\right]^{-1} RE\left(\hat{\beta} \mid X\right) \\ &= \beta - \left(X'X\right)^{-1} R' \left[R\left(X'X\right)^{-1} R'\right]^{-1} R\beta \\ &= \beta \end{aligned}$$

since $R\beta = 0$. So $\tilde{\beta}$ is unbiased for β.

(c)

$$\begin{aligned}\tilde{\beta} &= \hat{\beta} - (X'X)^{-1} R' \left[R(X'X)^{-1} R'\right]^{-1} R\hat{\beta} \\ &= \left(I - (X'X)^{-1} R' \left[R(X'X)^{-1} R'\right]^{-1} R\right)\hat{\beta} \\ &= A\hat{\beta},\end{aligned}$$

say where

$$A = I - (X'X)^{-1} R' \left[R(X'X)^{-1} R'\right]^{-1} R.$$

Then

$$\begin{aligned}Var\left(\tilde{\beta} \mid X\right) &= Var\left(A\hat{\beta} \mid X\right) \\ &= AVar\left(\hat{\beta} \mid X\right) A' \\ &= A(X'X)^{-1}(X'DX)(X'X)^{-1} A'\end{aligned}$$

where $D = diag\{\sigma_1^2, ..., \sigma_n^2\}$.

(d) Setting $\hat{D} = diag\{\hat{e}_1^2, ..., \hat{e}_n^2\}$, the White estimator for $Var\left(\tilde{\beta} \mid X\right)$ is

$$\hat{V} = A(X'X)^{-1}\left(X'\hat{D}X\right)(X'X)^{-1} A'.$$

The standard errors are the square roots of the diagonal elements of $\hat{V}$.

4. Note that $g = g(x) = x'\beta$. The estimate of g is $\hat{g} = x'\hat{\beta}$ which has standard error $s(\hat{g}) = \left(x'\hat{V}x\right)^{1/2}$. The t-ratio for g is

$$T_n = \frac{\hat{g} - g}{s(\hat{g})} = \frac{x'\hat{\beta} - x'\beta}{\left(x'\hat{V}x\right)^{1/2}}.$$

As $\hat{g}$ is a linear function of $\hat{\beta}$, $T_n \rightarrow_d N(0,1)$. This is a context where the use of the percentile-t bootstrap makes sense.

For the bootstrap, we draw independently and with replacement from the sample, to create a bootstrap sample with n observations, and on this sample, run the OLS regression, to obtain $\hat{\beta}^*$ and $\hat{V}^*$. The bootstrap t-ratio is

$$T_n^* = \frac{\hat{g}^* - \hat{g}}{s(\hat{g}^*)} = \frac{x'\hat{\beta}^* - x'\hat{\beta}}{\left(x'\hat{V}^*x\right)^{1/2}} = \frac{x'\left(\hat{\beta}^* - \hat{\beta}\right)}{\left(x'\hat{V}^*x\right)^{1/2}}.$$

Calculating a large number B of independent draws of the random variable T_n^*, we find the $\alpha/2\%$ quantile $q_n^*(\alpha/2)$ and the $1-\alpha/2\%$ quantile $q_n^*(1-\alpha/2)$ of this distribution. (Numerically, we sort the T_n^* and find the $\alpha/2\%$ and $1-\alpha/2\%$ order statistics.) Then the $(1-\alpha)\%$ equal-tailed percentile-t interval for g is

$$\begin{aligned}&[\hat{g} - q_n^*(1-\alpha/2)\cdot s(\hat{g}),\quad \hat{g} - q_n^*(\alpha/2)\cdot s(\hat{g})] \\ =\ &\left[x'\hat{\beta} - q_n^*(1-\alpha/2)\cdot\left(x'\hat{V}x\right)^{1/2},\quad x'\hat{\beta} - q_n^*(\alpha/2)\cdot\left(x'\hat{V}x\right)^{1/2}\right].\end{aligned}$$