Econometrics 710
Final Exam, Spring 2015
Sample Answers

1. Consumer Surplus

   (a) The plug-in estimator is $\hat{A} = -\hat{\alpha}^2/2\hat{\beta}$.

   (b) Notice

$$a = \frac{\partial}{\partial\theta}A(\theta) = \begin{pmatrix} \frac{\partial}{\partial\alpha}\left(-\alpha^2/2\beta\right) \\ \\ \frac{\partial}{\partial\beta}\left(-\alpha^2/2\beta\right) \end{pmatrix} = \begin{pmatrix} -\alpha/\beta \\ \\ \alpha^2/2\beta^2 \end{pmatrix}$$

   Thus by the Delta method, since $\beta \neq 0$

$$\sqrt{n}\left(\hat{\theta} - \theta\right) \rightarrow_d N\left(0, a'V_\theta a\right)$$

   An estimate of the asymptotic variance is $\hat{a}'\hat{V}_\theta\hat{a}$ where

$$\hat{a} = \begin{pmatrix} -\hat{\alpha}/\hat{\beta} \\ \\ \hat{\alpha}^2/2\hat{\beta}^2 \end{pmatrix}$$

   Thus a 95% confidence interval for A is

$$\hat{A} \pm 1.96 * \sqrt{\hat{a}'\hat{V}_\theta\hat{a}/n}$$

   (c) For a bootstrap percentile interval: For $b = 1, ..., B$, where $B = 1000$ or higher

      i. Sample n iid observations from the empirical sample
      ii. Construct the OLS estimates $\hat{\theta}_b^* = (\hat{\alpha}_b^*, \hat{\beta}_b^*)$
      iii. Form $\hat{A}_b^* = -\hat{\alpha}_b^{*2}/2\hat{\beta}_b^*$

   Given $\{\hat{A}_1^*, ..., \hat{A}_B^*\}$, calculate the $\alpha/2$ and $1-\alpha/2$ empirical quantiles $q_{\alpha/2}$ and $q_{1-\alpha/2}$
   The percentile interval is $[q_{\alpha/2}, q_{1-\alpha/2}]$.

2. Structural quadratic equation

   (a) x_i^2 is endogenous. (If x_i and e_i are correlated, we should in general expect functions of x_i and e_i to be possibly correlated.)

   (b) The number of rhs structural variables is 3, the number of instruments are 3, so this is just-identified (and thus sufficient).

   (c) Given equation (2) and squaring

$$\begin{aligned} x_i^2 &= \gamma_0^2 + \gamma_1^2 z_i^2 + u_i^2 + 2\gamma_0\gamma_1 z_i + 2\gamma_0 u_i + 2\gamma_1 z_i u_i \\ &= (\gamma_0^2 + Eu_i^2) + 2\gamma_0\gamma_1 z_i + \gamma_1^2 z_i^2 \\ &\quad + \left(\left(u_i^2 - Eu_i^2\right) + 2\gamma_0 u_i + 2\gamma_1 z_i u_i\right) \\ &= \gamma_0^* + 2\gamma_0\gamma_1 z_i + \gamma_1^2 z_i^2 + u_{2i} \end{aligned}$$

say, where the error u_{2i} is mean zero (conditionally on z_i). This is the reduced form equation for x_i^2. A minimal condition for identification is that the coefficients on the instruments must be non-zero, which requires that $\gamma_1 \neq 0$. (If $\gamma_1 = 0$ then both coefficients are zero.) Furthermore, since the equation for x_i is linear in z_i it will be furthermore necessary for z_i^2 to be relevant in this equation, e.g. that the coefficient γ_1^2 is non-zero. However, this occurs if $\gamma_1 \neq 0$ so this is not an additional requirement.
In summary, $\gamma_1 \neq 0$ is sufficient for identification.

3. Two-step model

The question should have stated that $E(z_i e_i) = 0$ and $E(z_i u_i) = 0$.

(a) The first-step is $\hat{\gamma} = \sum z_i x_i / \sum z_i^2$. The second step is

$$\begin{aligned}\hat{\beta} &= \frac{\sum \hat{x}_{i2} y_i}{\sum \hat{x}_i^4} \\ &= \frac{\sum (\hat{\gamma} z_i)^2 y_i}{\sum (\hat{\gamma} z_i)^4} \\ &= \frac{\sum z_i^2 y_i}{\hat{\gamma}^2 \sum z_i^4}\end{aligned}$$

(b) By the WLLN,

$$\hat{\gamma} = \frac{\frac{1}{n}\sum z_i x_i}{\frac{1}{n}\sum z_i^2} \to_p \gamma = \frac{E z_i x_i}{E z_i^2}.$$

Using $y_i = \beta x_i^2 + e_i$, and then the WLLN

$$\begin{aligned}\frac{\sum z_i^2 y_i}{\sum z_i^4} &= \beta \frac{\frac{1}{n}\sum z_i^2 x_i^2}{\frac{1}{n}\sum z_i^4} + \frac{\frac{1}{n}\sum z_i^2 e_i}{\frac{1}{n}\sum z_i^4} \\ &\to_p \beta \frac{E\left(z_i^2 x_i^2\right)}{E z_i^4} + \frac{E\left(z_i^2 e_i\right)}{E z_i^4}.\end{aligned}$$

since $E z_i^2 e_i = 0$ under $E(e_i \mid z_i) = 0$. (If we do not assume $E(e_i \mid z_i) = 0$ then there will be an extra term as well.)
One could stop here and state that

$$\hat{\beta} \to_p \frac{\beta}{\gamma^2} \frac{E\left(z_i^2 x_i^2\right)}{E z_i^4} + \frac{E\left(z_i^2 e_i\right)}{\gamma^2 E z_i^4} \tag{1}$$

However, we can try and say a little more. Since $x_i = \gamma z_i + u_i$ then $x_i^2 = \gamma^2 z_i^2 + 2\gamma z_i u_i + u_i^2$ so

$$\frac{E\left(z_i^2 x_i^2\right)}{\gamma^2 E z_i^4} = 1 + 2\frac{E\left(z_i^3 u_i\right)}{\gamma E z_i^4} + \frac{E\left(z_i^2 u_i^2\right)}{\gamma^2 E z_i^4}$$

Thus we have

$$\hat{\beta} \to_p \beta\left(1 + \frac{E\left(z_i^3 u_i\right)}{\gamma E z_i^4} + \frac{E\left(z_i^2 u_i^2\right)}{\gamma^2 E z_i^4}\right) + \frac{E\left(z_i^2 e_i\right)}{\gamma^2 E z_i^4}$$

Two of the terms drop out under slightly stronger assumptions. Suppose the linear reduced form for x_i is the conditional mean, then $E(u_i \mid z_i) = 0$ and thus $E\left(z_i^3 u_i\right) = 0$.

Furthermore if z_i is exogenous in the strong sense that $E(e_i \mid z_i) = 0$ then $E\left(z_i^2 e_i\right) = 0$ as well. Under these two plausible restrictions we find

$$\hat{\beta} \to_p \beta \left(1 + \frac{E\left(z_i^2 u_i^2\right)}{\gamma^2 E z_i^4}\right).$$

Finally if we make the even stronger assumption that the error u_i^2 is conditionally homoskedastic $E\left(u_i^2 \mid z_i\right) = \sigma_u^2$ then the limit simplifies to

$$\hat{\beta} \to_p \beta \left(1 + \frac{\sigma_u^2 E z_i^2}{\gamma^2 E z_i^4}\right) = \beta \left(1 + \frac{\sigma_u^2 E z_i^2}{\gamma^2 E z_i^4}\right).$$

$$= \gamma^2 + \frac{E z_i^2 u_i^2}{E z_i^4} = \gamma^2 + \frac{E z_i^2 E u_i^2}{E z_i^4}$$

where the second equality holds if we assume $E(u_i \mid z_i) = 0$, and the third holds if we further assume that u_i is independent of z_i. We thus have that

$$\hat{\beta} \to_p \beta^* = \beta + \frac{\beta}{\gamma^2} \frac{E z_i^2 E u_i^2}{E z_i^4} \tag{2}$$

(c) In general, $\hat{\beta}$ is not consistent for β, as the limit obtained in the previous sub-question is not simply β. However, we found that if $E(e_i \mid z_i) = 0$, $E(u_i \mid z_i) = 0$, and $E\left(u_i^2 \mid z_i\right) = \sigma_u^2$ then we found the simplfied limit (2). Are there conditions under which this simplifies to β? For $\beta \neq 0$ the answer is no. In fact, the plim β^* is biased away from 0, in the sense that if $\beta > 0$ then $\beta^* > \beta$ and if $\beta < 0$ then $\beta^* < \beta$. So there is no condition under which the estimator is generally consistent.

There is, however, one special condition. If $\beta = 0$ then the limit in (2) is also 0, so $\hat{\beta} \to_p 0$ and the estimator is consistent. If you go back to the more general case (1) we see that $E(e_i \mid z_i) = 0$ and $\beta = 0$ are sufficient for $\hat{\beta} \to_p 0$ and hence consistency.

It might seem somewhat odd to say "The estimator is consistent when the true coefficient is zero" but this is a general finding for two-step estimators.

4. The relation between the structural and reduced forms is $\beta = \Gamma\lambda$. Partitioning,

$$\begin{aligned} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} &= \begin{pmatrix} \Gamma_{11} & \Gamma_{21} \\ \Gamma_{21} & \Gamma_{22} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \\ &= \begin{pmatrix} I & \Gamma_{21} \\ 0 & \Gamma_{22} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \end{aligned}$$

since x_{1i} is exogenous and in both the structural and reduced form equations. Thus

$$\begin{aligned} \beta_1 &= \lambda_1 + \Gamma_{21}\lambda_2 \\ \beta_2 &= \Gamma_{22}\lambda_2 \end{aligned}$$

The hypothesis is $\beta_2 = 0$, which holds if and only if $\lambda_2 = 0$ (if Γ_{22} is full rank which holds if the model is identified). Thus the hypothesis is equivalent to the restriction $\lambda_2 = 0$, which is a restriction on the reduced form equation (3).

The hypothesis $\lambda = 0$ can be tested in equation (3) by OLS estimation of (3) and a (heteroskedasticityrobust) Wald test for $\lambda_2 = 0$

$$W = \hat{\lambda}_2' V_{\hat{\lambda}_2}^{-1} \hat{\lambda}_2$$

Under $H_0 : \lambda_2 = 0$ the statistic converges to a $\chi^2_{\ell_2}$ distribution. We reject H_0 in favor of $H_1 : \lambda_2 \neq 0$ at the level α if $W > c$ where c satisfies $P(\chi^2_{\ell_2} > c) = \alpha$.