Econometrics 710
Final Exam, Spring 2010

$$\begin{aligned} y_i &= \mathbf{x}_i'\boldsymbol{\beta} + e_i \\ \mathbf{x}_i &= \boldsymbol{\Pi}'\mathbf{z}_i + \boldsymbol{u}_i \\ E(\mathbf{z}_i e_i) &= 0 \\ E(\mathbf{z}_i \boldsymbol{u}_i') &= 0 \end{aligned}$$

The dimensions are: $\mathbf{x}_i$, $\boldsymbol{u}_i$, and $\boldsymbol{\beta}$ are $k \times 1$, $\mathbf{z}_i$ is $\ell \times 1$ where $\ell \geq k > 1$, $\boldsymbol{\Pi}$ is $\ell \times k$ and y_i and e_i are 1×1.

The difficulty in the problem is that $(y_i, \mathbf{x}_i, \mathbf{z}_i)$ are not jointly observed.
Instead, we have two independent samples from the marginal distributions of $(y, \mathbf{z})$ and $(\mathbf{x}, \mathbf{z})$:

- Sample 1: iid observations of $(y_i, \mathbf{z}_i)$, $i = 1, ..., n4$
- Sample 2: iid observations of $(\mathbf{x}_j, \mathbf{z}_j)$, $j = 1, ..., J$

You can imagine that you have two independent samples from the same joint distribution, but in the first sample $\mathbf{x}_i$ is missing, and in the second sample y_j is missing.

1. Write out the reduced form equations:
   (a) Write the reduced form equation for y_i as a function of $\mathbf{z}_i$, $\boldsymbol{\beta}$, and $\boldsymbol{\Pi}$.
   (b) Explicitly write the error in this reduced form as a function of the errors e_i and $\boldsymbol{u}_i$ and parameters.
   (c) Write the population parameter $\boldsymbol{\beta}$ as a function of population moments of $(y_i, \mathbf{x}_i, \mathbf{z}_i, \boldsymbol{\Pi})$
   (d) Write the population parameter $\boldsymbol{\Pi}$ as a function of population moments of $(y_i, \mathbf{x}_i, \mathbf{z}_i)$
   (e) What is the condition for identification of $\boldsymbol{\beta}$?

2. Define $\boldsymbol{Q} = E(\mathbf{z}_i\mathbf{z}_i')$.
   (a) Write out estimators $\widetilde{\boldsymbol{Q}}$ and $\widehat{\boldsymbol{Q}}$ for $\boldsymbol{Q}$ using Sample 1 and Sample 2
   (b) Find the probability limit of $\widetilde{\boldsymbol{Q}}$ as $n \to \infty$
   (c) Find the probability limit of $\widehat{\boldsymbol{Q}}$ as $J \to \infty$
   (d) Are the probabiltiy limits in (b) and (c) the same?
   (e) Which estimator is more efficient?

3. Suppose you know $\boldsymbol{\Pi}$. Find an estimator $\widetilde{\boldsymbol{\beta}}$ for $\boldsymbol{\beta}$.
   Hint: Use the reduced form equation for y_i
   (a) Write out this estimator.
   (b) Which sample is used?
   (c) Show that $\widetilde{\boldsymbol{\beta}} \to_p \boldsymbol{\beta}$. Which sample size ($n$ or J) goes to infinity for this convergence?

4. Find an estimator $\widehat{\boldsymbol{\Pi}}$ for $\boldsymbol{\Pi}$
   (a) Write out the estimator.
   (b) Which sample is used?
   (c) Show that $\widehat{\boldsymbol{\Pi}} \to_p \boldsymbol{\Pi}$. Which sample size ($n$ or J) goes to infinity for this convergence?

5. Put your answers to 2 and 3 together to find an estimator $\widehat{\boldsymbol{\beta}}$ for $\boldsymbol{\beta}$ when $\boldsymbol{\Pi}$ is unknwon.
   (a) Write down the estimator.
   (b) Show that $\widehat{\boldsymbol{\beta}} \to_p \boldsymbol{\beta}$. What assumptions on n and J are required?