Econometrics 709

Problem Set 7

Fall 2017

Due: Monday, October 23

1. Take the Pareto model $f(x) = \alpha x^{-1-\alpha}$ for $x \geq 1$. Compute the MLE $\widehat{\alpha}$ using Newton optimation

   (a) Find the log-likelihood, gradient (first derivative) and Hessian (second derivative)

   (b) Find the Newton updating rule $\alpha_i$

   (c) Using the initial value $\alpha_1 = 1$ and supposing $n^{-1}\sum_{i=1}^{n}\log(X_i) = 1/2$ calculate the iterations $\alpha_2$, $\alpha_3$ and $\alpha_4$

2. Take the Bernoulli model $P(X = 1) = p$ and $P(X = 0) = 1 - p$.

   (a) Calculate the information for $p$ by taking the variance of the score

   (b) Calculate the information for $p$ by taking the expectation of the second derivative. Did you obtain the same answer?

3. Take the Pareto model. Calculate the information for $\alpha$ using the second derivative rule.

4. Find the Cramer-Rao lower bound for $p$ in the Bernoulli model. Compare this to the variance of the MLE for $p$. (You calculated this in Problem Set #6, problem 4(c).)

5. Take the Pareto model. Recall the MLE $\widehat{\alpha}$ for $\alpha$ from Problem Set #6, problem 9(b).

   (a) Show that $\widehat{\alpha} \to_p \alpha$ by using the WLLN and continuous mapping theorem.

6. Take the model $f(x) = \theta\exp(-\theta x)$

   (a) Find the Cramer-Rao lower bound for $\theta$

   (b) Recall the MLE $\widehat{\theta}$ for $\theta$ (from Problem Set #6, problem 5(c)).

   (c) Find the asymptotic distribution for $\widehat{\theta}$ using the delta method applied to the sample mean.

   (d) Find the asymptotic distribution for $\widehat{\theta}$ using the asymptotic distriubtion for the MLE. Do you find the same answer as in part (c)?

7. Take the Gamma model

$$f(x \mid \alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta}$$

   Assume $\beta$ is known, so the only parameter to estimate is $\alpha$. Let $g(\alpha) = \log\Gamma(\alpha)$. Write your answers in terms of the derivatives $g'(\alpha)$ and $g''(\alpha)$.

(a) Calculate the information for $\alpha$.

(b) Using the general formula for the asymptotic distribution of the MLE, find the asymptotic distribution of $\sqrt{n}\,(\widehat{\alpha} - \alpha)$, where $\widehat{\alpha}$ is the MLE.

(c) Letting $V$ denote the asymptotic variance from part (c), propose an estimator $\widehat{V}$ for $V$.

8. In the Bernoulli model, you found the asymptotic distribution of the MLE in Problem Set #6, part (d)

   (a) Propose an estimator of $V$, the asymptotic variance

   (b) Show that this estimator is consistent for $V$ as $n \to \infty$

   (c) Propose a standard error $s(\widehat{p})$ for the MLE $\widehat{p}$

9. Take the model $X \sim N(\mu, \sigma^2)$ . Propose a test for $H_0 : \mu = 1$ against $H_1 : \mu \neq 1$

10. Take the Bernoulli model. Consider testing for $H_0 : p = 1/2$ against $H_1 : p \neq 1/2$

    (a) First, consider testing based on the t-statistic $(\widehat{p} - p)\,/s(\widehat{p})$ where $s(\widehat{p})$ is from question 8 (and $p = 1/2$). Describe the test

    (b) Second, consider testing based on the statistic $(\widehat{p} - p)\,/s(p)$ where $s(p)$ is the standard error formula, but evaluated at $p = 1/2$ rather than at the MLE. Describe this test. What is the difference with part (b)?

11. Take the model $X \sim N(\mu, 1)$. Consider testing $H_0 : \mu = 0$ against $H_1 : \mu > 0$

    (a) Calculate the general log-likelihood function $\ell_n(\mu)$ and at the null hypothesis $\ell_n(0)$

    (b) Calculate the likelihood ratio statistic $LR_n(\mu_1)$ for $H_0 : \mu = 0$ against the simple alternative $H_1 : \mu = \mu_1$ for $\mu_1$ known.

    (c) Show that the LR test in part (b) (Reject $H_0$ if $LR_n(\mu_1) > c$ for some $c$) is the same as "Reject $H_0$ if $\overline{X}_n > k$ for some $k$"

    (d) Find $k$ for part (c) such that the test has size $\alpha$. Does this $k$ depend on $\mu_1$?

    (e) The Neyman-Pearson Lemma shows that the LR test is the most powerful test for simple hypotheses. Conclude that the test "Reject $H_0$ if $\overline{X}_n > k$ for $k$ from part (d)" is the most powerful test against $H_1 : \mu = \mu_1$ for $\mu_1$ known.

    (f) Since the test "Reject $H_0$ if $\overline{X}_n > k$ for $k$ from part (d)" does not depend on $\mu_1$, conclude that this test is the most powerful test of $H_0 : \mu = 0$ against $H_1 : \mu > 0$, even though the alternative is not simple.