Volatility

• Many economic series, and most financial series, display conditional volatility
  – The conditional variance changes over time
  – There are periods of high volatility
    • When large changes frequently occur
  – And periods of low volatility
    • When large changes are less frequent
Weekly Stock Prices
Levels and Returns

![Graphs showing weekly stock prices and returns from 1950 to 2020. The left graph displays the index levels, while the right graph shows the return distribution over time.](image)
Conditional Mean

• The conditional mean of $y$ is

$$E(y_t \mid \Omega_{t-1})$$

• The regression error is mean zero and unforecastable

$$E(e_t \mid \Omega_{t-1}) = 0$$
Conditional Variance

- The conditional variance of $y$ is

$$\text{var}(y_t \mid \Omega_{t-1}) = E\left(\left( y_t - E(y_t \mid \Omega_{t-1}) \right)^2 \mid \Omega_{t-1} \right)$$

$$= E(e_t^2 \mid \Omega_{t-1})$$

- The squared regression error can be forecastable
Forecastable Conditional Variance

• If the squared error is forecastable, then the conditional variance is time-varying and correlated.
  – The magnitude of changes is predictable
  – The sign is not predictable
Stock returns are unpredictable

```
. reg return L(1/4).return, r

Linear regression
Number of obs = 3380
F(  4,  3375) = 1.56
Prob > F = 0.1819
R-squared = 0.0039
Root MSE = 2.0696

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<td>.0400493</td>
<td>4.19</td>
<td>0.000</td>
<td>.0892721     .2463187</td>
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. testparm L(1/4).return

( 1)  L1.return = 0
( 2)  L2.return = 0
( 3)  L3.return = 0
( 4)  L4.return = 0

F(  4,  3375) = 1.56
Prob > F = 0.1819
```

Barlett's formula for MA(q) 95% confidence bands
Squared Returns are predictable

```
. gen y = (return-0.1629)^2
. reg y L(1/4).y, r

Linear regression                     Number of obs =    3380
F(  4,  3375) =    8.73
Prob > F =    0.0000
R-squared =  0.1181
Root MSE =  10.626

Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]

---                  ---                  ---                  ---                  ---                  ---                  ---                  ---                  ---                  ---
  y                  _cons
L1.                 0.2228415  .1012677    2.20  0.028     .0242893    .4213937
L2.                 0.0689282  .032054    2.15  0.032     .006081    .1317754
L3.                 0.1423761  .044068    3.23  0.001     .0559733    .2287789
L4.                 0.0649141  .0540807   1.20  0.230    -.0411202    .1709484
  _cons              2.150521  .3351092    6.42  0.000     1.493483    2.807558
```

```
. testparm L(1/4).y

( 1)  L.y = 0
( 2)  L2.y = 0
( 3)  L3.y = 0
( 4)  L4.y = 0

F(  4,  3375) =    8.73
Prob > F =    0.0000
```
Squared Returns

Graph 1: Squared return over time from 1950 to 2020

Graph 2: Autocorrelations of squared returns with 95% confidence bands for MA(q)
ARCH

• Robert Engle (1982) proposed a model for the conditional variance
  – AutoRegressive Conditional Heteroskedasticity
  – “ARCH” now describes volatility models
• Nobel Prize 2003
**ARCH(1) Model**

\[ y_t = \mu + e_t \]

\[ \sigma_t^2 = \text{var}(e_t \mid \Omega_{t-1}) = \omega + \alpha e_{t-1}^2 \]

\[ \omega > 0 \]

\[ \alpha \geq 0 \]

- \(\alpha > 0\) means that the conditional variance is high when the lagged squared error is high.
- Large errors (either sign) today mean high expected errors (in magnitude) tomorrow.
- Small magnitude errors forecast next period small magnitude errors.
Unconditional variance

• A property of expectations is that expected (average) conditional expectations are unconditional expectations.

• So the average conditional variance is the average variance – the variance of the regression error.

\[
\sigma^2 = E\left(\sigma_t^2\right) = \omega + \alpha E\left(e_{t-1}^2\right) = \omega + \alpha \sigma^2
\]

• Solving for the variance:

\[
\sigma^2 = \frac{\omega}{1 - \alpha}
\]
• Rewriting, this implies
  \[ \omega = \sigma^2 (1 - \alpha) \]

• Substituting into ARCH(1) equation
  \[ \sigma_t^2 = (1 - \alpha)\sigma^2 + \alpha e_{t-1}^2 \]
  or
  \[ \sigma_t^2 = \sigma^2 + \alpha(e_{t-1}^2 - \sigma^2) \]

• This shows that the conditional variance is a combination of the unconditional variance, and the deviation of the squared error from its average value.
ARCH(1) as AR(1) in squares

• The model

\[
\text{var}(e_t | \Omega_{t-1}) = E(e_t^2 | \Omega_{t-1}) = \omega + \alpha e_{t-1}^2
\]

implies the regression

\[
e_t^2 = \omega + \alpha e_{t-1}^2 + u_t
\]

where \( u \) is white noise

• Thus e-squared is an AR(1)
Estimation

- `.arch return, arch(1)`

Sample: 1950w2 - 2015w5  
Number of obs = 3384
Distribution: Gaussian  
Wald chi2(.) = .
Log likelihood = -7065.525  
Prob > chi2 = .

|               | Coef.   | Std. Err. | z     | P>|z|   | [95% Conf. Interval] |
|---------------|---------|-----------|-------|-------|---------------------|
| **return**    |         |           |       |       |                     |
|    _cons      | 0.2311745 | 0.0299371 | 7.72  | 0.000 | 0.1724988  \ 0.2898503 |
| **ARCH**      |         |           |       |       |                     |
|    arch L1.   | 0.3089586 | 0.0215146 | 14.36 | 0.000 | 0.2667907  \ 0.3511264 |
|    _cons      | 2.89934  | 0.067296  | 43.08 | 0.000 | 2.767443  \ 3.031238  |
Variance Forecast

• Given the parameter estimates, the estimated conditional variance for period $t$ is

$$\hat{\sigma}_t^2 = \hat{\omega} + \hat{\alpha} \hat{\epsilon}_{t-1}^2 = \hat{\omega} + \hat{\alpha}(y_{t-1} - \hat{\mu})^2$$

• The forecasted out-of-sample variance is

$$\hat{\sigma}_{n+1}^2 = \hat{\omega} + \hat{\alpha}(y_n - \hat{\mu})^2$$
Forecast Interval for the mean

- You can use the estimated conditional standard deviation to obtain forecast intervals for the mean
  \[ \hat{y}_{n+1|n} \pm Z_{\alpha/2} \hat{\sigma}_{n+1} \]

- These forecast intervals will vary in width depending on the estimated conditional variance.
  - Wider in periods of high volatility
  - More narrow in periods of low volatility
ARCH(p) model

- Allow p lags of squared errors

\[ y_t = \mu + e_t \]
\[ \sigma_t^2 = \omega + \alpha_1 e_{t-1}^2 + \alpha_2 e_{t-2}^2 + \cdots + \alpha_p e_{t-p}^2 \]

- Similar to AR(p) in squares
- Estimation: ARCH(8)
  - `.arch return, arch(1/8)`
  - ARCH model with lags 1 through 8
## ARCH(8) Estimates

- **.arch return, arch(1/8)**

|       | Coef.   | Std. Err. |     z  |   P>|z|     | [95% Conf. Interval] |
|-------|---------|-----------|-------|---------|---------------------|
| **return** |         |           |       |         |                     |
| _cons | .2329821| .0278051  | 8.38  | 0.000   | .1784852            |
| **ARCH** |         |           |       |         |                     |
| arch  |         |           |       |         |                     |
| L1.   | .1864161| .0169841  | 10.98 | 0.000   | .1531278            |
| L2.   | .1108989| .0201296  | 5.51  | 0.000   | .0714457            |
| L3.   | .1458022| .0215004  | 6.78  | 0.000   | .1036622            |
| L4.   | .0889981| .0188903  | 4.71  | 0.000   | .0519737            |
| L5.   | .0255236| .0201296  | 1.58  | 0.115   | -.0062224           |
| L6.   | .0793973| .0215004  | 3.68  | 0.000   | .0543748            |
| L7.   | .0434073| .0188903  | 2.30  | 0.021   | .0155155            |
| L8.   | .0646394| .0201296  | 3.21  | 0.001   | .0285752            |
| _cons | 1.182066| .0979648  | 12.07 | 0.000   | .9900582            |

Sample: 1950w2 - 2015w5  
Number of obs      =      3384  
Distribution: Gaussian  
Wald chi2(.)       =         .  
Log likelihood = -6898.673  
Prob > chi2        =         .
ARCH needs many lags

• Notice that we included 8 lags, and all appeared significant.

• This is commonly observed in estimated ARCH models
  – The conditional variance appears to be a function of many lagged past squares
GARCH Model

- Tim Bollerslev (1986)
  - A student of Engle
  - Professor at Duke University

proposed the GARCH model to simplify this problem

$$\sigma_t^2 = \omega + \beta \sigma_{t-1}^2 + \alpha e_{t-1}^2$$

- $\beta > 0$
- $\omega > 0$
- $\alpha \geq 0$
GARCH(1,1)

- This makes the variance a function of all past lags:
  \[ \sigma_t^2 = \omega + \beta \sigma_{t-1}^2 + \alpha e_{t-1}^2 \]
  \[ = \sum_{j=0}^{\infty} \beta^j \left( \omega + \alpha e_{t-1-j}^2 \right) \]

- It is also smoother than an ARCH model with a small number of lags
GARCH(p,q)

- $p$ lags of squared error
- $q$ lags of conditional variance

\[
\sigma_t^2 = \omega + \beta_1 \sigma_{t-1}^2 + \cdots + \beta_q \sigma_{t-q}^2 + \alpha_1 e_{t-1}^2 + \cdots + \alpha_p e_{t-p}^2
\]

- GARCH(1,1):
  - `.arch r, arch(1) garch(1)`
- GARCH(3,2):
  - `.arch r, arch(1/3) garch(1/2)`
**GARCH(1,1)**

Sample: 1950w2 - 2015w5  
Distribution: Gaussian  
Log likelihood = -6889.998  
Number of obs = 3384  
Wald chi2(.) = .  
Prob > chi2 = .

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- **Common GARCH features**
  - Lagged variance (garch) has large coefficient
  - Sum of two coefficients very close to (but less than) one
GARCH(2,2) for Stock Returns

Sample: 1950w2 - 2015w5  
Distribution: Gaussian  
Log likelihood = -6887.561  
Number of obs = 3384  
Wald chi2(.) = .  
Prob > chi2 = .

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<td>L1.</td>
<td>0.5775524</td>
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<td>0.076</td>
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<td>0.2733315</td>
<td>0.90</td>
<td>0.368</td>
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<td>2.85</td>
<td>0.004</td>
<td>0.0442329 0.2381978</td>
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</table>
The GARCH(1,1) often fits well, and is a useful benchmark.

- Daily, weekly, or monthly asset returns, exchange rates, or interest rates
Extensions

• There are many extensions of the basic GARCH model, developed to handle a variety of situations
  – Asymmetric Response
  – Garch-in-mean
  – Explanatory variables in variance
  – Non-normal errors
Asymmetric GARCH

• Threshold GARCH

\[ \sigma_t^2 = \omega + \beta \sigma_{t-1}^2 + \alpha e_{t-1}^2 + \gamma e_{t-1}^2 1(e_{t-1} > 0) \]

• The last term is dummy variable for positive lagged errors

• This model specifies that the ARCH effect depends on whether the error was positive or negative
  – If the error is negative, the effect is \( \alpha \)
  – If the error is positive, the full effect is \( \alpha + \gamma \)
### TARCH estimation

- **.arch return, arch(1) tarch(1) garch(1)**
- Negative errors have coefficient of 0.21
- Positive errors have coefficient of 0.04
- Negative returns increase volatility much more than positive returns

---

**Sample:** 1950w2 - 2015w5

**Distribution:** Gaussian

**Log likelihood:** -6853.192

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Leverage Effect

- This model describes what is called the “leverage effect”
  - A negative shock to equity increases the ratio debt/equity of investors
  - This increases the leverage of their portfolios
  - This increases risk, and the conditional variance
  - Negative shocks have stronger effect on variance than positive shocks
GARCH-in-mean

• If investors are risk averse, risky assets will earn higher returns (a risk premium) in market equilibrium

• If assets have varying volatility (risk), their expected return will vary with this volatility
  – Expected return should be positively correlated with volatility
GARCH-M model

\[ y = \beta_1 + \beta_1 \sigma_{t-1}^2 + e_t \]

\[ \sigma_t^2 = \omega + \beta \sigma_{t-1}^2 + \alpha e_{t-1}^2 \]

• .arch return, arch(1) garch(1) archm
GARCH-M for Stock Returns

- Marginally positive effect

|                | Coef.    | Std. Err. | z     | P>|z|   | [95% Conf. Interval] |
|----------------|----------|-----------|-------|-------|----------------------|
| return _cons   | 0.1238608| 0.0486927 | 2.54  | 0.011 | 0.0284247 to 0.2192968|
| ARCHM sigma2   | 0.0327126| 0.0136338 | 2.40  | 0.016 | 0.0059908 to 0.0594344|
| ARCH arch L1.  | 0.1327171| 0.010073  | 13.18 | 0.000 | 0.1129743 to 0.1524599|
|                | 0.8401155| 0.012407  | 67.71 | 0.000 | 0.8157983 to 0.8644327|
|                | 0.130583 | 0.0234566 | 5.57  | 0.000 | 0.0846088 to 0.1765571|

Sample: 1950w2 - 2015w5
Distribution: Gaussian
Log likelihood = -6886.719
Number of obs = 3384
Wald chi2(1) = 5.76
Prob > chi2 = 0.0164
TARCH and GARCH-M

- `.arch return, arch(1) tarch(1) garch(1) archm`
- `archm` term appears insignificant

| Coef.   | Std. Err. | z    | P>|z| | [95% Conf. Interval] |
|---------|-----------|------|------|----------------------|
| **return** _cons | 0.1323894 | 0.0478167 | 2.77 | 0.006 | 0.0386703, 0.2261085 |
| **ARCHM** sigma2 | 0.0123845 | 0.0131658 | 0.94 | 0.347 | -0.01342, 0.0381891 |
| **ARCH** arch L1. | 0.2054669 | 0.0173142 | 11.87 | 0.000 | 0.1715315, 0.2394022 |
| tarch L1. | -0.1634648 | 0.0176488 | -9.26 | 0.000 | -0.1980558, -0.1288738 |
| garch L1. | 0.8300779 | 0.0144876 | 57.30 | 0.000 | 0.8016827, 0.8584730 |
| _cons | 0.1807148 | 0.0249614 | 7.24 | 0.000 | 0.1317914, 0.2296383 |

Sample: 1950w2 - 2015w5
Number of obs = 3384
Distribution: Gaussian
Wald chi2(1) = 0.88
Log likelihood = -6852.694
Prob > chi2 = 0.3469
Estimated standard deviation

- Estimated TARCH model
- \texttt{.predict v, variance}
- \texttt{.gen s=sqrt(v)}
- Average volatility=1.94
S&P, returns, and estimated volatility
2006-2014