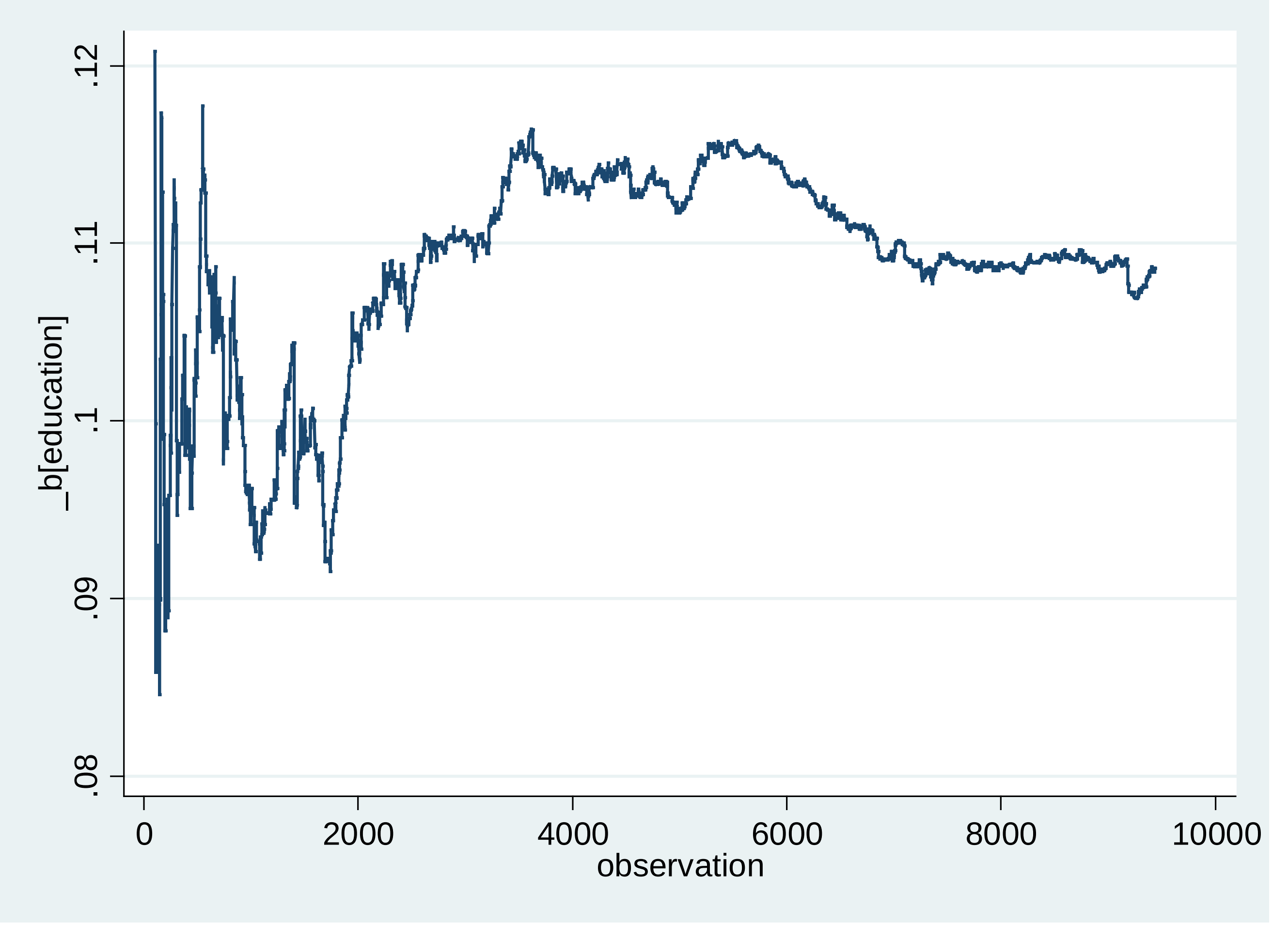
Distribution of Estimates

- From Econometrics (410)
- Linear Regression Model $y_t = \alpha + \beta x_t + e_t$
  - Assume (y_t,x_t) is iid and $E(x_t e_t)=0$
- Estimation Consistency
  - The estimates approach the true values as the sample size increases
  - Estimation variance decreases as the sample size increases

Illustration of Consistency

- Take random sample of U.S. white men
- Estimate linear regression of log(wages) on education
- Total sample = 2089
- Start with 100 observations, sequentially increase to 2089

Sequence of Slope Coefficients



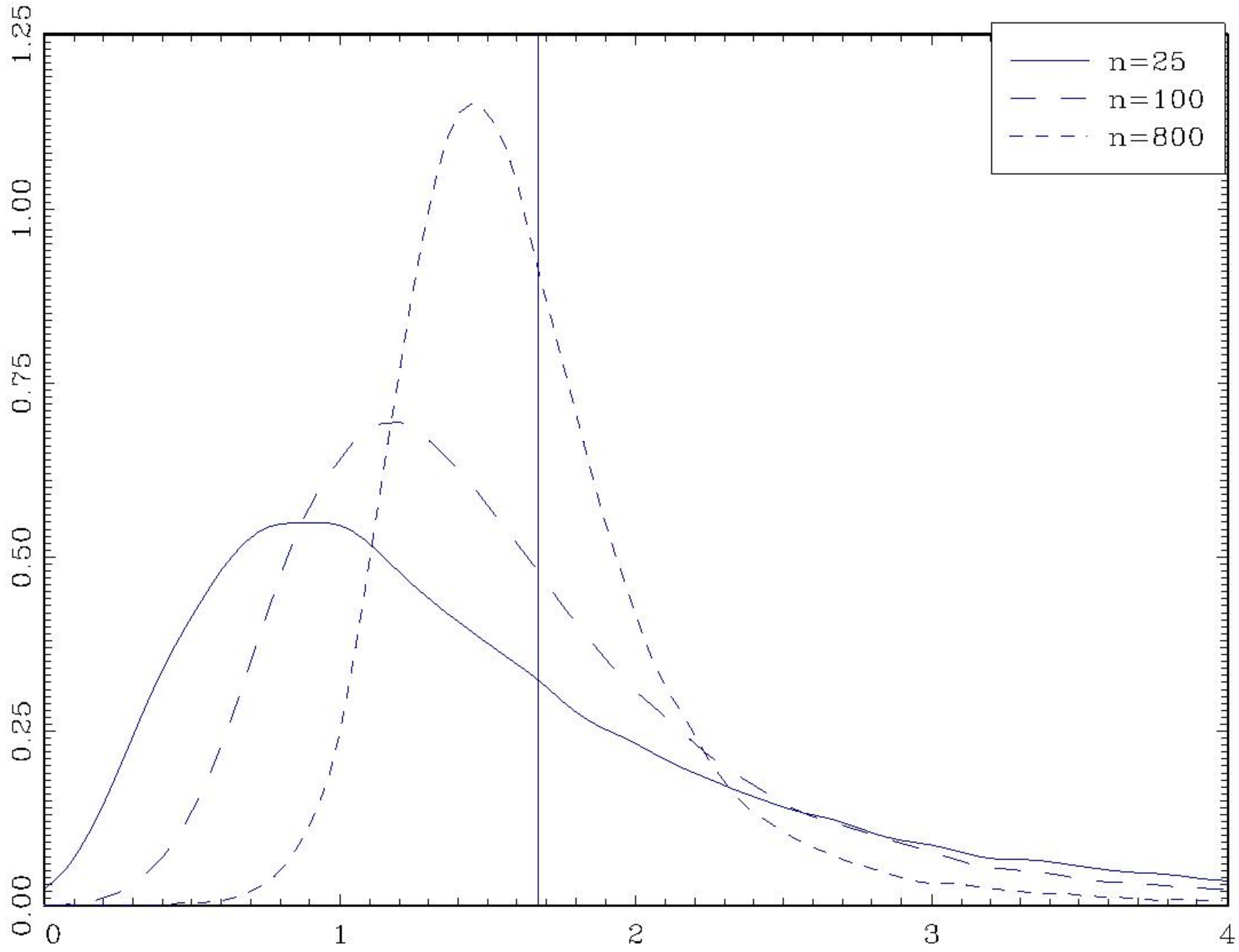
Asymptotic Normality

$$y_t = \alpha + \beta x_t + e_t$$

$$\hat{\beta} \overset{a}{\sim} N\left(\beta, \sigma^2_{\hat{\beta}}\right)$$

$$\sigma^2_{\hat{\beta}} = \frac{1}{T} \frac{\mathrm{var}\left(x_t e_t\right)}{\left[\mathrm{var}\left(x_t\right)\right]^2}$$

Illustration of Asymptotic Normality



Time Series

- Do these results apply to time-series data?
  - Consistency
  - Asymptotic normality
  - Variance formula
- Time-series models
  - AR models, i.e., $x_t = y_{t-1}$
  - Trend and seasonal models
  - One-step and multi-step forecasting

Derivation of Variance Formula

- For simplicity
  - Assume the variables have zero mean
  - The regression has no intercept
- Model with no intercept:

$$y_t = \beta x_t + e_t$$

- Model with no intercept

$$y_t = \beta x_t + e_t$$

- OLS minimizes the sum of squares

$$\sum_{t=1}^{T} \left(y_t - \beta x_t\right)^2 = \sum_{t=1}^{T} y_t^2 - 2\beta \sum_{t=1}^{T} x_t y_t + \beta^2 \sum_{t=1}^{T} x_t^2$$

- The first-order condition is

$$0 = -2\sum_{t=1}^{T} x_t y_t + 2\hat{\beta} \sum_{t=1}^{T} x_t^2$$

- Solution $$\hat{\beta} = \frac{\sum_{t=1}^{T} x_t y_t}{\sum_{t=1}^{T} x_t^2} = \frac{\frac{1}{T}\sum_{t=1}^{T} x_t y_t}{\frac{1}{T}\sum_{t=1}^{T} x_t^2}$$
- Now substitute $y_t = \beta x_t + e_t$

$$\hat{\beta} = \frac{\frac{1}{T}\sum_{t=1}^{T} x_t\left(x_t \beta + e_t\right)}{\frac{1}{T}\sum_{t=1}^{T} x_t^2} = \beta + \frac{\frac{1}{T}\sum_{t=1}^{T} x_t e_t}{\frac{1}{T}\sum_{t=1}^{T} x_t^2}$$

- We have

$$\hat{\beta} = \beta + \frac{\frac{1}{T}\sum_{t=1}^{T} x_t e_t}{\frac{1}{T}\sum_{t=1}^{T} x_t^2}$$

- The denominator is the sample variance (when x has mean zero), so

$$\frac{1}{T}\sum_{t=1}^{T} x_t^2 \overset{a}{\sim} \mathrm{var}(x_t)$$

- Then $$\hat{\beta} \overset{a}{\sim} \beta + \frac{\sum_{t=1}^{T} v_t}{T \operatorname{var}(x_t)}$$

  where $v_t = x_t e_t$
- Since $E(v_t) = E(x_t e_t) = 0$

  then

$$\operatorname{var}(\hat{\beta}) \overset{a}{\sim} \frac{\operatorname{var}\left(\sum_{t=1}^{T} v_t\right)}{[T \operatorname{var}(x_t)]^2}$$

- From the covariance formula

$$\operatorname{var}\left(\sum_{t=1}^{T} v_t\right) = \sum_{t=1}^{T} \operatorname{var}(v_t) + \sum_{j \neq t}^{T} \operatorname{cov}(v_t, v_j)$$

- When the observations are independent, the covariances are zero.
- And since $\operatorname{var}(v_t) = \operatorname{var}(x_t e_t)$

we obtain $\operatorname{var}\left(\sum_{t=1}^{T} v_t\right) = T \operatorname{var}(x_t e_t)$

- We have found

$$\operatorname{var}\left(\hat{\beta}\right) \overset{a}{\sim} \frac{T\operatorname{var}\left(x_t e_t\right)}{\left[T\operatorname{var}\left(x_t\right)\right]^2} = \frac{\operatorname{var}\left(x_t e_t\right)}{T\left[\operatorname{var}\left(x_t\right)\right]^2}$$

as stated at the beginning

Extension to Time-Series

- The only place in this argument where we used the assumption of the *independence* of observations was to show that $v_t=x_te_t$ has zero covariance with $v_j=x_je_j$
- This is saying that v_t is not autocorrelated.
- When does this happen in time-series?

Unforecastable one-step errors

- **Claim**: In one-step-ahead forecasting, if the regression error is **unforecastable** then v_t is not autocorrelated
- In this case, the variance formula for the least-squares estimate is the same as in regression

$$\mathrm{var}\left(\hat{\beta}\right) \overset{a}{\sim} \frac{\mathrm{var}\left(x_t e_t\right)}{T\left[\mathrm{var}\left(x_t\right)\right]^2}$$

- Why is the claim true?
- The error is unforecastable if $E(e_t \mid \Omega_{t-1})=0$
- For simplicity suppose $x_t=1$
- Then for $t \neq j$

$$\mathrm{cov}(v_t, v_j) = E\left(e_t e_j\right) = 0$$

Summary

- In one-step-ahead time-series models, if the error is unforecastable, then least-squares estimates satisfy the asymptotic (approximate) distribution

$$\hat{\beta} \overset{a}{\sim} N\left(\beta, \sigma_{\hat{\beta}}^2\right)$$

$$\sigma_{\hat{\beta}}^2 = \frac{1}{T} \frac{\operatorname{var}(x_t e_t)}{\left[\operatorname{var}(x_t)\right]^2}$$

- As the sample size T is in the denominator, the variance **decreases** as the sample size **increases**.
- This means that least-squares is consistent

Variance Formula

- The variance formula for the least-squares estimate takes the form

$$\sigma_{\hat{\beta}}^{2} = \frac{1}{T}\frac{\mathrm{var}(x_t e_t)}{\left[\mathrm{var}(x_t)\right]^2}$$

- This formula is valid in time-series regression when the error is unforecastable

Classical Variance Formula

If we make the simplifying assumption

$$\mathrm{var}(x_t e_t) = \mathrm{var}(x_t)\mathrm{var}(e_t)$$

then

$$\sigma^2_{\hat{\beta}} = \frac{1}{T}\frac{\mathrm{var}(x_t e_t)}{\left[\mathrm{var}(x_t)\right]^2} = \frac{1}{T}\frac{\mathrm{var}(e_t)}{\mathrm{var}(x_t)}$$

This can be a useful simplification

Homoskedasticity

- The variance simplification is valid under "conditional homoskedasticity"

$$E\left(e_t \mid \Omega_{t-1}\right) = 0$$
$$E\left(e_t^2 \mid \Omega_{t-1}\right) = \sigma^2$$

- This is a simplifying assumption made to make calculations easier, and is a conventional *assumption* in introductory econometrics courses
- It is not used in serious econometrics

Variance Formula : AR(1) Model

- Take the AR(1) model with unforecastable homoskedastic errors

$$y_t = \alpha + \beta y_{t-1} + e_t$$
$$E(e_t \mid \Omega_{t-1}) = 0$$
$$E(e_t^2 \mid \Omega_{t-1}) = \sigma^2$$

- Then the variance of the OLS estimate is

$$\sigma_{\hat{\beta}}^2 = \frac{1}{T}\frac{\mathrm{var}(e_t)}{\mathrm{var}(x_t)} = \frac{1}{T}\frac{\mathrm{var}(e_t)}{\mathrm{var}(y_{t-1})}$$

since $x_t = y_{t-1}$ in this model

AR(1) Asymptotic Variance

- We know that

$$\operatorname{var}(y_{t-1}) = \frac{\operatorname{var}(e_t)}{1-\beta^2}$$

- So

$$\sigma^2_{\hat{\beta}} = \frac{1}{T}\frac{\operatorname{var}(e_t)}{\operatorname{var}(y_{t-1})} = \frac{1-\beta^2}{T}$$

- The asymptotic distribution is very simple

$$\hat{\beta} \overset{a}{\sim} N\left(\beta, \frac{1-\beta^2}{T}\right)$$

$$\hat{\beta} \overset{a}{\sim} N\left(\beta, \frac{1-\beta^2}{T} \right)$$

- The variance is a function of the unknown true value of β
- As |β| increases, the variance decreases, so the OLS estimate is actually more precise

Distribution of Least-Squares

- In classic regression, if the errors are iid normal, and independent of the regressors, then the least-squares estimates have an **exact** normal distribution, not just asymptotic
- This is not true in most time-series regressions.

Non-Classical Distributions

- Estimates in autoregressive models
  - Biased downwards
  - Skewed
  - Thick tails
- Especially
  - When autoregressive coefficients are large
  - Sample sizes are small
- These issues diminish in large samples

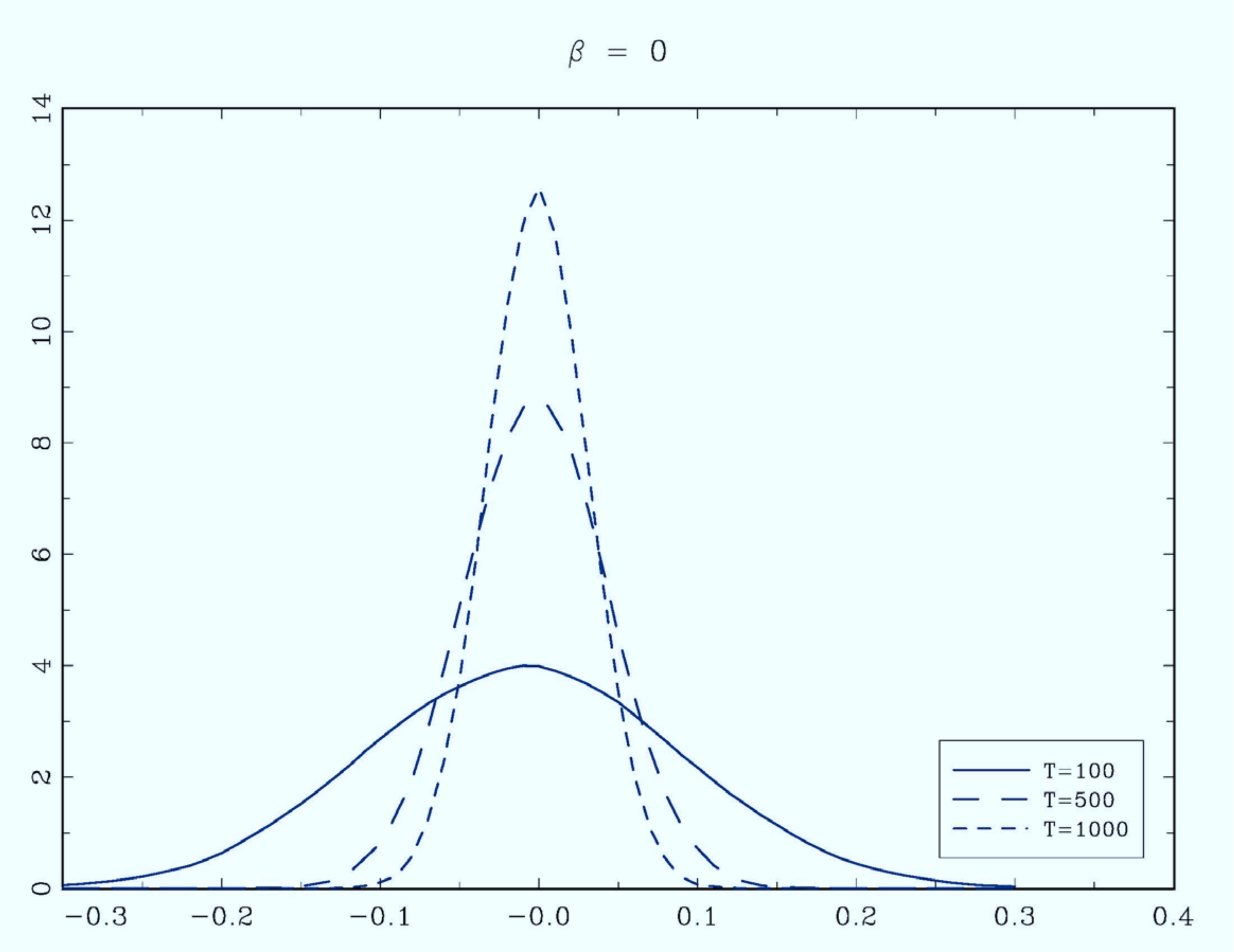
Example

- Take the AR(1) model with intercept

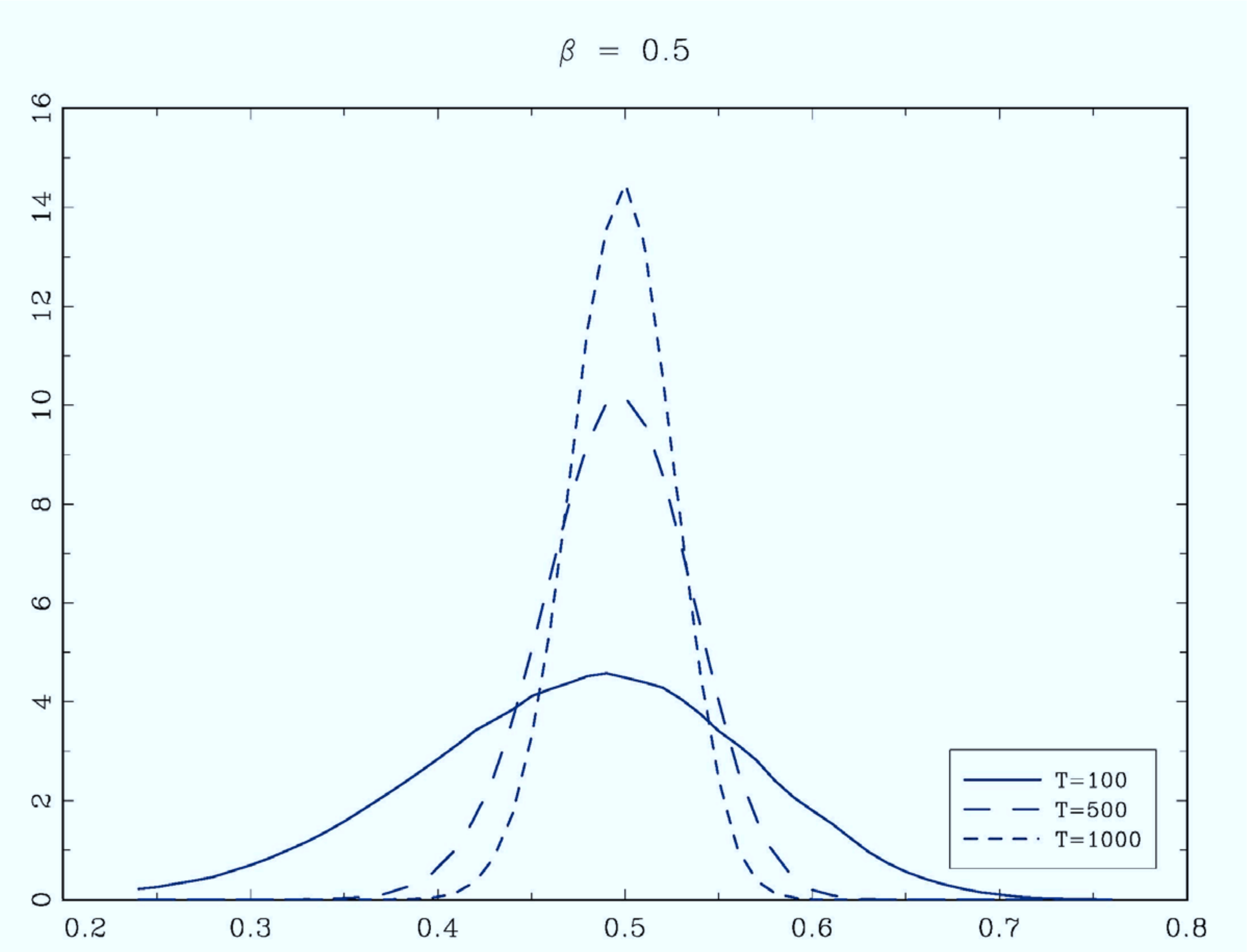
$$y_t = \alpha + \beta y_{t-1} + e_t$$

- e_t N(0,1)
- T=100, 500, 1000
- β=0.0, β=0.5, β=0.9,
- Numerically calculate distribution of least-squares estimate of β

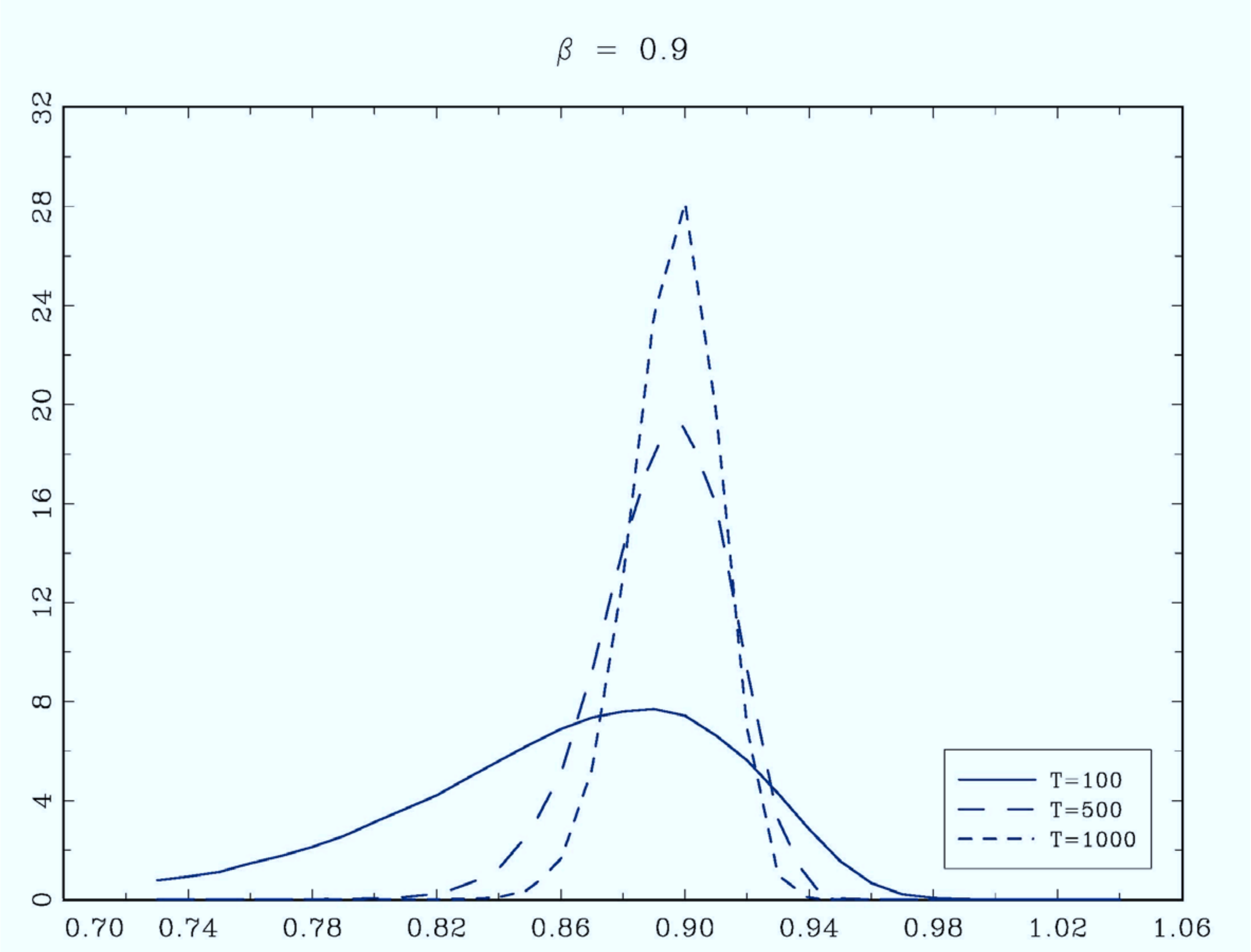
Distribution, β=0.0



Distribution, β=0.5



Distribution, β=0.9



Interpretation

- Estimates of autoregressive parameters are random
- Even if regression error is normal, the parameter estimates are not normally distributed
- Distributions are less normal when AR coefficient is large
- Distributions are more concentrated and normal when sample size is large

Asymptotic Standard Deviation

- The least-squares estimate is asymptotically (approximately) normally distributed
- In the simple model $y_t = \beta x_t + e_t$

  then

$$\hat{\beta} \overset{a}{\sim} N\left(\beta, \sigma^2_{\hat{\beta}}\right)$$

$$\sigma^2_{\hat{\beta}} = \frac{1}{T} \frac{\operatorname{var}(x_t e_t)}{\left[\operatorname{var}(x_t)\right]^2}$$

- The standard deviation measures the precision of the estimate, but it is unknown.

Standard Errors

- Estimates of the standard deviations are called **standard errors**, and are reported in regression output
- They are used to measure estimation precision.

Classical standard errors

A **classic standard error** is an estimate of the standard deviation from the formula

$$\sigma_{\hat{\beta}}^2 = \frac{1}{n} \frac{\mathrm{var}(e_t)}{\mathrm{var}(x_t)}$$

This formula is valid under conditional homoskedasticity

$$E(e_t \mid \Omega_{t-1}) = 0$$
$$E(e_t^2 \mid \Omega_{t-1}) = \sigma^2$$

Robust standard errors

- "Robust" standard errors are estimates of

$$\sigma_\beta = \sqrt{\frac{1}{n}\frac{\mathrm{var}(x_t e_t)}{[\mathrm{var}(x_t)]^2}}$$

- These are the conventional standard errors for regression analysis
- Also known as "White" standard errors

Halbert White

- Professor Hal White, UCSD (1950-2012)
- Leading contributor to econometric methods, especially time series analysis
- Introducted robust standard errors into econometrics (1980)
  - Most referenced paper in economics
- Founded Bates-White consulting firm, a leader in economic policy analysis

Have you seen robust standard errors?

- If you took an econometrics course other than 410, you may not be familiar with robust standard errors
- If you are currently taking 410, you won't cover robust standard errors until later in the course
  - Wooldridge uses the homoskedasticity assumption in the early part of his text
- Stock-Watson use robust standard errors throughout

Does the Choice Matter?

- Classic standard errors are for the assumption of conditional homoskedasticity

$$E\left(e_t^2 \mid \Omega_{t-1}\right) = \sigma^2$$

- This is **unforecastability in the variance**
  - This is not implied by conventional unforecastability
  - It may be a convenient approximation for macro data
  - It is a bad assumption (quite false) in financial data

Example: Stock Returns, AR(1)

return	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
return						
L1.	-.0329441	.0169119	-1.95	0.051	-.0661023	.0002142
_cons	.0016702	.0003498	4.78	0.000	.0009844	.002356

return	Coef.	Robust Std. Err.	t	P>|t|	[95% Conf.	Interval]
return						
L1.	-.0329441	.0286977	-1.15	0.251	-.0892101	.0233219
_cons	.0016702	.0003648	4.58	0.000	.0009549	.0023855

- The robust standard error on the AR(1) coefficient is almost twice as large as the conventional standard error

Computation

- In STATA, the default is conventional standard errors.
- They are automatically reported with the regress (reg) command
- For robust standard errors, use the “r” option

.reg return L.return, r

Example: Real GDP Growth

```
. reg gdp L(1/4).gdp

      Source |       SS           df       MS      Number of obs   =       275
-------------+----------------------------------   F(4, 270)       =     12.83
       Model |  670.393407         4  167.598352   Prob > F        =    0.0000
    Residual |  3527.86747       270  13.0661758   R-squared       =    0.1597
-------------+----------------------------------   Adj R-squared   =    0.1472
       Total |  4198.26087       274    15.32212   Root MSE        =    3.6147

------------------------------------------------------------------------------
         gdp |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         gdp |
         L1. |   .3363865   .0606839     5.54   0.000     .2169128    .4558602
         L2. |   .1422228   .0637859     2.23   0.027     .0166418    .2678038
         L3. |  -.0682526    .063699    -1.07   0.285    -.1936624    .0571573
         L4. |  -.0728112   .0604167    -1.21   0.229    -.1917589    .0461365
             |
       _cons |   2.131395   .3495548     6.10   0.000     1.443195    2.819594
------------------------------------------------------------------------------
```

With Robust st. errors

```
. reg gdp L(1/4).gdp, r

Linear regression                               Number of obs     =        275
                                                F(4, 270)         =       9.86
                                                Prob > F          =     0.0000
                                                R-squared         =     0.1597
                                                Root MSE          =     3.6147

------------------------------------------------------------------------------
             |               Robust
         gdp |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         gdp |
         L1. |   .3363865   .0743514     4.52   0.000     .1900044    .4827686
         L2. |   .1422228   .0834467     1.70   0.089    -.0220661    .3065118
         L3. |  -.0682526   .0721168    -0.95   0.345    -.2102354    .0737302
         L4. |  -.0728112   .0741333    -0.98   0.327     -.218764    .0731416
             |
       _cons |   2.131395   .4227888     5.04   0.000     1.299012    2.963777
------------------------------------------------------------------------------
```

Robust st. errors

- With the “r” option

  .reg y x, r

- You get robust
  - Standard errors
  - t statistics and p-values
  - test statistics

Annoyance

- In STATA, with the “r” option, STATA omits sum of squared error table
  - Yet this can be useful
- So both commands may be useful

.reg y x

.reg y x, r

Interpretation of standard errors

- The standard errors measure precision of the estimate
  - Forecasts use *estimated* coefficients.
- Small standard errors mean the estimate is precise
  - Good for forecasting
- Large standard errors mean the estimate is not precise
  - Bad for forecasting
  - Inaccurate estimates leads to inaccurate forecasts

gdp	Coef.	Robust Std. Err.	t	P>|t|	[95% Conf.	Interval]
gdp						
L1.	.3363865	.0743514	4.52	0.000	.1900044	.4827686
L2.	.1422228	.0834467	1.70	0.089	-.0220661	.3065118
L3.	-.0682526	.0721168	-0.95	0.345	-.2102354	.0737302
L4.	-.0728112	.0741333	-0.98	0.327	-.218764	.0731416
_cons	2.131395	.4227888	5.04	0.000	1.299012	2.963777

Interpretation of t-statistics

- “t” is the coefficient estimate divided by the standard error.
- It is used to *test* if the coefficient is zero
  - “P>|t|” is the p-value of the t-statistic
  - If p<.05 you “reject” the hypothesis of a zero coefficient
- Hypothesis tests are useful for assessing economic theories
  - But are less useful for picking good forecasting models

gdp	Coef.	Robust Std. Err.	t	P>|t|	[95% Conf.	Interval]
gdp						
L1.	.3363865	.0743514	4.52	0.000	.1900044	.4827686
L2.	.1422228	.0834467	1.70	0.089	-.0220661	.3065118
L3.	-.0682526	.0721168	-0.95	0.345	-.2102354	.0737302
L4.	-.0728112	.0741333	-0.98	0.327	-.218764	.0731416
_cons	2.131395	.4227888	5.04	0.000	1.299012	2.963777

Interpretation of Confidence Interval

- The 95% interval is the coefficient estimate plus and minus 1.96 times the standard error
- Helps gauge possible values for the true coefficient
- Useful tool

gdp	Coef.	Robust Std. Err.	t	P>|t|	[95% Conf.	Interval]
gdp						
L1.	.3363865	.0743514	4.52	0.000	.1900044	.4827686
L2.	.1422228	.0834467	1.70	0.089	-.0220661	.3065118
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_cons	2.131395	.4227888	5.04	0.000	1.299012	2.963777

Summary

- In one-step-ahead forecast regressions with unforecastable errors
  - Robust standard errors generally appropriate
  - Classical standard errors appropriate under conditional homoskedasticity

Assignments

- Read Wooldridge Chapter 12.1 and 12.5
  - An electronic copy is in files at Learn@UW
- Read Chapter 7 from *The Signal and the Noise*
  - Reading Reflection
  - Due Thursday (3/16)
- Forecasting Project
  - Project Description (3/28)